Doctoral Qualifying Exam

Differential Geometry (Spring, 2012)

1. (15 points) State the definition of an oriented differentiable manifold of dimension n.

2. (15 points) Let $M = \mathbb{R}^2$ with coordinates x and y, and $X = y\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y}$ be a smooth vector field on M. Find the maximal domain W and the local one-parameter group action $\theta : W \to M$ whose infinitesimal generator is X.

3. (15 points) Let V be a real vector space, and let $\varphi \in \Lambda^r(V)$, and $v \in V$. Define an element $\iota(v)\varphi$ of $\Lambda^{r-1}(V)$, called the *interior product of* φ *by* v, by

$$(\iota(v)\varphi)(v_1, \cdots, v_{r-1}) = \varphi(v, v_1, \cdots, v_{r-1}).$$

Show that for $\varphi \in \Lambda^r(V)$ and $\psi \in \Lambda^s(V)$

$$\iota(v)(\varphi \wedge \psi) = (\iota(v)\varphi) \wedge \psi + (-1)^r \varphi \wedge (\iota(v)\psi).$$

4. (20 points) Answer the following questions:

(a) Prove or disprove that, when $m = 3$, there are m nowhere vanishing smooth tangent vector fields on S^m which are linearly independent on the tangent space of each point of S^m.

(b) Do the same question (a) above for S^6 and the Möbius band.

5. (20 points) *Explicitly* find three linearly independent elements η_1, η_2, and η_3 in

$$H^1_{DR}(\mathbb{R}^2 - \{P, Q, R\}),$$

where P, Q, and R denote three distinct points in $\mathbb{R}^2$.

6. (15 points) Let ω_0 and ω_1 be two differential 2-forms on $\mathbb{R}^4$ with coordinates x_1, x_2, x_3, and x_4 given by

$$\omega_0 = dx_1 \wedge dx_3 + dx_2 \wedge dx_4, \quad \omega_1 = -dx_2 \wedge dx_1 - dx_3 \wedge dx_4.$$

Find, if any, a linear map $J : \mathbb{R}^4 \to \mathbb{R}^4$ such that $J^2 = -\mathrm{Id}_{\mathbb{R}^4}$ (here, $\mathrm{Id}_{\mathbb{R}^4}$ means the identity on $\mathbb{R}^4$) and such that

$$\omega_0(X, JX) > 0 \text{ and } \omega_1(X, JX) > 0 \text{ for all non-zero } X \in \mathbb{R}^4.$$

– The End –

Real Analysis (Winter 2012)

1. (15 points) Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous mapping. Prove that, if A is a Borel subset of $\mathbb{R}^n$, then $f^{-1}(A)$ is a Borel subset of $\mathbb{R}^m$.

2. (20 points)

   (a) State Fatou's Lemma.

   (b) State Lebesgue's monotone convergence theorem.

   (c) Prove Fatou's Lemma using Lebesgue's monotone convergence theorem.

3. (15 points) Let $f \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$ with $1 \le p < q < \infty$. Prove that $f \in L^r(\mathbb{R})$ for any r with $p < r < q$.

4. (15 points) Let $F : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$F(x) = \int_a^x f(y)dy$$

   for an integrable function f. Prove that F is absolutely continuous.

5. (15 points) Let ℓ be a bounded linear functional on a Hilbert space $\mathcal{H}$. Prove that there exists a unique $g \in \mathcal{H}$ such that

$$\ell(f) = \langle f, g \rangle$$

   for all $f \in \mathcal{H}$.

6. (20 points)

   (a) Prove that the sets of exterior measure zero are (Carathéodory) measurable.

   (b) State Radon-Nikodym Theorem.

Algebraic Topology I Qualifying Exam

(Homology is assumed to be the singular homology unless mentioned otherwise, which should not be assumed to be isomorphic to the simplicial or the cellular homology in this exam.)

1. (a) Prove that $\pi_1(S^1,*) \cong \mathbb{Z}$. [10 pts]
   (b) Prove that $\mathrm{H}_1(S^1) \cong \mathbb{Z}$. [10 pts]

2. A loop $[0,1] \to (X, x_0)$ can be viewed as a singular 1-simplex in X.

[5 pts] (a) Prove that this association gives a well defined map $F : \pi_1(X, x_0) \to \mathrm{H}_1(X)$.

[5 pts] (b) Prove that F is a group homomorphism.

[5 pts] (c) Prove that F is surjective.

[5 pts] (d) Prove that $F : \pi_1(S^1, *) \to \mathrm{H}_1(S^1)$ is an isomorphism.

3. Let M be the Mõbius band.

[5 pts] (a) Compute the singular homology $\mathrm{H}_*(M)$, and determine the homomorphism $\mathrm{H}_*(\partial M) \to \mathrm{H}_*(M)$ using degree;

[5 pts] (b) Compute the singular homology $\mathrm{H}_*(M, \partial M)$;

[10 pts] (c) Choose a CW-pair structure for $(M, \partial M)$, and compute the cellular homology $\mathrm{H}^{\mathrm{CW}}_*(M, \partial M) = \mathrm{H}_*(\mathrm{C}^{\mathrm{CW}}_*(M)/\mathrm{C}^{\mathrm{CW}}_*(\partial M), \partial^{\mathrm{CW}})$ by definition.

4. (a) Show that there is a double covering map of Klein bottle K onto itself. [5 pts]
   (b) Prove that Klein bottle K is homeomorphic to the connected sum of two real projective planes $\mathbb{RP}^2$. Namely, $K \cong \mathbb{RP}^2 \# \mathbb{RP}^2$. [5 pts]
   (c) Use Van Kampen theorem to compute $\pi_1(K)$, and show that its abelianization is $\cong \mathbb{Z} \oplus \mathbb{Z}_2$. [5 pts]
   (d) Compute $\mathrm{H}_*(K)$ using Mayer-Vietoris sequence. [5 pts]

5. For a pair (X, A) of topological spaces, denote $f : A \hookrightarrow X$ the embedding.

[5 pts] (a) Prove that (M_f, A) is a neighborhood deformation retraction, where M_f is the mapping cylinder;

[5 pts] (b) Prove that X is homotopy equivalent to M_f and the natural map $(M_f, A) \to (X, A)$ induces an isomorphism $\mathrm{H}_*(M_f, A) \cong \mathrm{H}_*(X, A)$;

[5 pts] (c) Prove that $\mathrm{H}_*(X, A) \cong \tilde{\mathrm{H}}_*(C_f)$, where C_f is the mapping cone;

[5 pts] (d) Derive the long exact sequence of (X, A) from the Mayer-Vietoris sequence applied to $C_f = X \cap CA$, where CA is the cone over A.

Qualifying Exam in Probability Theory (January 2012)

1. (10 pts) Suppose that $\{X_n, n \geq 1\}$ and X are random variables. X_n converges to X almost surely (a.s.) if and only if $\sup_{k \geq n} |X_k - X|$ converges to 0 in probability.

2. (15 pts) Let $(\Omega, \mathcal{B}, P)$ be a probability space, and let

$$\mathcal{G} := \{A \in \mathcal{B} : P\{A\} = 0 \text{ or } 1\}.$$

   (a) (7 pts) Show that $\mathcal{G}$ is a σ-field.

   (b) (8 pts) Let $X, Y : (\Omega, \mathcal{G}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be two random variables. Are X and Y independent?

3. (15 pts) Suppose that $\{X_n : n \geq 1\}$ are random variables on $(\Omega, \mathcal{B}, P)$ and define $S_0 := 0, S_n := \sum_{i=1}^{n} X_i, n \geq 1$. Let $\tau := \inf\{n > 0 : S_n > 0\}$. Assume that $\tau(\omega) < \infty$ for all $\omega \in \Omega$.

   (a) (7 pts) Prove that τ is a random variable.

   (b) (8 pts) Prove also that S_τ is a random variable. Here, S_τ is defined by $S_\tau(\omega) = S_{\tau(\omega)}(\omega)$ for $\omega \in \Omega$.

4. (10 pts) Let $\{X_n\}$ be a sequence of random variables. Let X be a random variable such that $E[(X_n - X)^2] < \infty$ and $\sum_{n=1}^{\infty} E[(X_n - X)^2] < \infty$. Show that $X_n \to X$ almost surely.

5. (10 pts) Let $(\Omega, \mathcal{B}, P)$ be a probability space and $X_n \to X$ a.s.. Is it true that $P\{X_n \in A\} \to P\{X \in A\}$ for all $A \in \mathcal{B}$? Prove it or give a counterexample.

6. (10 pts) Let $\{X_n\}$ be a sequence of random variables with $P\{X_n = \pm\frac{1}{n}\} = \frac{1}{2}$. Prove or disprove that $X_n \to 0$ a.s..

7. (10 pts) Let X_n have a uniform distribution on $(-n, n)$. Is there a proper, non-degenerate random variable X_0 such that $X_n \Rightarrow X_0$? Here, $\Rightarrow$ denotes the convergence in distribution.

8. (10 pts) If $X_n \Rightarrow X$ and $\{X_n\}$ are uniformly integrable, then X is integrable and $E[X_n] \to E[X]$.

9. (10 pts) Let $\{(X_n, \mathcal{G}_n), n \geq 0\}$ be a martingale. Suppose that N is a stopping time for the filteration $\{\mathcal{G}_n, n \geq 0\}$ such that $P\{N < \infty\} = 1$. Show that, if $X_{\min(N,n)}$ are uniformly bounded, then $E[X_N] = E[X_0]$.

KAIST DMS
QUALIFYING EXAM IN COMPLEX ANALYSIS
JANUARY 2012

Student ID: **Name:**

No document or electronic device allowed. Duration: two hours. Please return all material, including this sheet and draft paper used during the test. The test is long: it is not necessary to cover all problems. If you can not solve a problem, it is better to write clear and relevant ideas than irrelevant computations. Please do not claim that you solved a problem unless you believe you did. Good luck!

We denote by $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}$ the unit disk in $\mathbb{C}$.

(1) -10 pts- State the residue formula. Using contour integration of $f(z) = \frac{1}{1+z^2}$, compute $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$.

(2) -10 pts- Show that there exits no holomorphic bijection $f : A \to B$ between $A = \{ z \in \mathbb{C} \mid z \neq 0 \}$ and $B = \{ z \in \mathbb{C} \mid 0 < |z| < 2 \}$.

(3) -10 pts- Prove or disprove:
any holomorphic bijection $g : \mathbb{D} \to \mathbb{D}$ has a fixed point.

(4) -15 pts- Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function such that $f(0) = 0$. Prove (part of) Schwartz Lemma: for all $z \in \mathbb{D}$, $|f(z)| \leq |z|$. *Hint: consider* $\frac{f(z)}{z}$.

(5) (a) -15 pts- Prove the Casorati-Weierstrass theorem: if f is holomorphic in the punctured disc $\mathbb{D} - \{0\}$ and has an essential singularity at 0, then the image of $\mathbb{D} - \{0\}$ under f is dense in the complex plane. *Hint: assume that w does not belong to the closure of the image, and consider* $g(z) = \frac{1}{f(z)-w}$.

(b) -10 pts- Prove that an injective holomorphic function $f : \mathbb{C} \to \mathbb{C}$ takes the form $f(z) = az + b$ with $a, b \in \mathbb{C}$, and $a \neq 0$. *Hint: apply the Casorati-Weierstrass theorem to* $f(1/z)$.

(6) (a) -10 pts- Let f be a holomorphic function with a zero of order $m \geq 1$ at the origin. Show that on a small neighborhood D of the origin, there exists $g \in \mathcal{O}(D)$ such that $g'(0) \neq 0$ and $f(z) = \left(g(z)\right)^m$ on D. *Hint: use the logarithm.*

(b) -10 pts- Let $f : \Omega \to \mathbb{C}$ be a nonconstant holomorphic map. Show that f is open, i.e., the image of any open set is open. *Hint: you can use without proof the inverse function theorem.*

(c) -10 pts- Prove or disprove: If a holomorphic function $f : \Omega \to \mathbb{C}$ is one-to-one (i.e., injective), then its derivative never vanishes.

Numerical analysis, Qualifying Exam. 2012

Each problem is worth 10 points.

1. (a) Write down the explicit form for Newton's method to find zeros of the following system of equations:

$$\begin{aligned} (x+2)y + xy^2 - 1 &= 0 \\ e^x y + xy + y^2 - 2 &= 0 \end{aligned}$$

   Denote k-th iterate by (x_k, y_k) for $k = 0, 1, 2, \cdots$.

   (b) Find (x_1, y_1) when $(x_0, y_0) = (0, 1)$.

2. Data is given $(x_i, f(x_i))$, $\quad i = 0, \cdots, n$ as in the following table:

x	0	1	2	3
$f(x)$	6	5	2	1

	——	——————	——————	——————	——————	——————
x	$f(x)$					
0	6					
1	5					
2	2					
3	1					

Table 1: Newton's interpol.

   (a) Write down Lagrange interpolation formula

   (b) Write down Newton's interpolation formula(by filling in the Table above.)

   (c) Write down the error term in one of the above interpolation and prove it.

3. Let $A, C \in \mathbb{C}^{n,n}$ and assume A is invertible with $\|A^{-1}\| \leq \alpha$. Show: If $\|A - C\| \leq \beta$ and $\beta\alpha < 1$, then C is invertible and

$$\|C^{-1}\| \leq \frac{\alpha}{1 - \alpha\beta}.$$

4. Define a cubic spline with data $(x_i, f(x_i))$, $\quad i = 0, \cdots, n$, where $f \in C''$.

5. Let $s(x)$ be the natural cubic spline. Prove

$$\|f - s\|_\infty \leq h^{3/2} \left(\int_a^b |f''(x)|^2 dx \right)^{1/2} \tag{1}$$

where $a = x_0$ and $b = x_n$.

6. State Gaussian quadrature using n points for the approximation of integral $\int_a^b f(x)dx$ and prove it is exact for polynomials of degree $2n-1$.

7. Given any norm $\|\cdot\|$ on $\mathbb{R}^n$, and a matrix $A = \{a_{ij}\}_{i,j=1}^n$. Do the following:

   (a) Define the matrix norm of A subordinate to $\|\cdot\|$

   (b) What is $\|A\|_1$ in terms of its entries?

   (c) What is $\|A\|_\infty$ in terms of its entries?

   Here $\|\mathbf{u}\|_1 = \sum_i |u_i|$ and $\|\mathbf{u}\|_\infty = \max_i |u_i|$ are the usual ℓ^1 and ℓ_∞-norm.

8. We consider solving the system of equations $A\mathbf{x} = \mathbf{b}$ iteratively. Let $A = D - L - U$.

   (a) Define Jacobi method

   (b) Define Gauss-Seidel method

   (c) Prove that, under some conditions on A the Jacobi method converges

9. When we numerically solve a system of equations,

$$A\mathbf{x} = \mathbf{b}$$

   we usually get approximate solution $\mathbf{x} + \mathbf{h}$. We would like analyze the relative error $\frac{\|\mathbf{h}\|}{\|\mathbf{x}\|}$ and find its relation with matrix A. We can view $\mathbf{x} + \mathbf{h}$ as the solution of a perturbed problem

$$(A + E)(\mathbf{x} + \mathbf{h}) = \mathbf{b} + \mathbf{k}.$$

   Find a reasonable bound of $\frac{\|\mathbf{h}\|}{\|\mathbf{x}\|}$ in terms of the relative errors of $\|\mathbf{k}\|$ and $\|E\|$ under some assumption that the error $\|E\|$ is small enough.

10. State QR algorithm to find all the eigenvalues of A. Discuss detailed algorithm, computational complexity(i.e, the number of operations), effectiveness and efficiency, etc.

Qualifying Exam-1 2012 in Advanced Statistics

1. [5 pts] Let X be a discrete random variable. Write down the distribution function of the cdf of X.

2. [15 pts] Let $f(\mathbf{x}|\theta)$ be the pmf or pdf of a random sample $\mathbf{X}$. Suppose there exists a function $T(\mathbf{x})$ such that, for every two sample points $\mathbf{x}$ and $\mathbf{y}$, the ratio $f(\mathbf{x}|\theta)/f(\mathbf{y}|\theta)$ is constant as a function of θ if and only if $T(\mathbf{x}) = T(\mathbf{y})$. Show that $T(\mathbf{X})$ is a minimal sufficient statistic for θ.

3. Suppose that $X_1, \cdots, X_n$ are iid Poisson(λ).

   (a) [10 pts] Find the best unbiased estimator of $P(X_1 = 1)$.

   (b) [10 pts] For the best estimator obtained in (a) above, find the asymptotic variance of it.

4. [15 pts] Write down an algorithm of generating a random sample from a pdf $f(x)$ for which there is no direct method of generation and show that the algorithm actually generates the desired random sample.

5. [15 pts] Let W_1 and W_2 be unbiased estimators of θ for which the statistic T is sufficient. Is it true that $Var(E(W_1|T)) \leq Var(W_2)$? Why?

6. [15 pts] Let $X_1, \cdots, X_n$ be iid Poisson(λ). Find a UMA $1-\alpha$ confidence interval based on inverting the UMP level α test of $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda > \lambda_0$.

7. [15 pts] Let $X_1, \cdots, X_n$ be a random sample from a pdf $f(x; \theta)$. Suppose that T is a sufficient statistic for θ. If T is also complete, does it imply that the pdf f belongs to an exponential family? Why?