Doctoral Qualifying Exam

Algebraic Topology I

July 31, 2012

Write your answer as detailed as you can. Problems 1, 2 and 3 are worth 20 points each and the rest 10 points each.

1. Let \( X = S^1 \times S^1 - \{p, q, r, s\} \) be the torus with four holes.
   
   (a) Compute the fundamental group of \( X \).
   
   (b) Compute the singular homology groups of \( X \).

2. Let \( X = S^1 \vee S^1 = S^1 \times \{s_0\} \cup \{s_0\} \times S^1 \subset S^1 \times S^1 \) and let \( f : X \rightarrow X \) be the restriction of the map \( (s, t) \mapsto (t, s) \) of \( S^1 \times S^1 \) to \( X \). Consider the quotient space \( Y = X \times [0, 1] / (x, 0) \sim (f(x), 1), \forall x \in X \)

   which can be visualized by the figure below.

   (a) Find the Euler characteristic of \( Y \).
   
   (b) Compute a presentation for \( \pi_1(Y) \).

3. Let \( A \) be the rectangular region in the plane \( \mathbb{R}^2 \) with two square holes as in the figure below.

   Use the Seifert-van Kampen theorem to compute the fundamental group of each space.

   (a) \( X = \partial(A \times [0, 1]) \).
   
   (b) \( Y = \mathbb{R}^3 - A \times [0, 1] \).

4. Let \( n \) be a positive integer and let \( P \in \mathbb{R}^n \). Compute \( H_k(\mathbb{R}^n, \mathbb{R}^n - P) \) for \( k \geq 0 \).

5. The following is a diagram of abelian groups and homomorphisms in which the rows are exact and the squares are commutative. Show that if \( \alpha \) and \( \gamma \) are isomorphisms then so is \( \beta \).

   \[
   \begin{array}{ccc}
   0 & \longrightarrow & A \\
   \downarrow^{\alpha} & \beta & \downarrow^{\gamma} \\
   0 & \longrightarrow & B \longrightarrow C \longrightarrow 0
   \end{array}
   \]

   \[
   \begin{array}{ccc}
   0 & \longrightarrow & D \\
   \downarrow^{h} & \beta & \downarrow^{\gamma} \\
   0 & \longrightarrow & E \longrightarrow F \longrightarrow 0
   \end{array}
   \]

6. The suspension of a space \( X \), denoted \( \Sigma X \), is defined as the quotient space \( \Sigma X = X \times [0, 1] / \sim \)

   where \( \sim \) indicates the identifications \( (x, 0) \sim (y, 0) \) and \( (x, 1) \sim (y, 1) \) for all \( x, y \in X \). Show that \( H_n(\Sigma X) \cong \tilde{H}_{n-1}X \) for \( n \geq 1 \).

7. Let \( A \) be a closed subset of a space \( X \). Show that \( H_k(X, A) \cong \tilde{H}_k(X/A) \), if \( A \) is a deformation retract of one of its neighborhood.

(gtj 2012/7/27 9:23)
1. [15] Let $X$ and $Y$ be nontrivial normed vector spaces such that the set of all bounded linear operators from $X^*$ to $Y$ is a Banach space. Show that $Y$ is complete. You may use Hahn-Banach Extension Theorem.

2. [15] Let $X$ and $Y$ be normed vector spaces. Let $\Gamma$ be a collection of bounded linear operators from $X$ to $Y$ such that the set of all points $x \in X$ with $\sup_{\Gamma} |Ax| < \infty$ is of the second category in $X$. Show that $\sup_{\Gamma} |A| < \infty$.

3. [15] Let $(X, \mathcal{A}, \mu)$ be a measure space and $\{f_n\}$ be a sequence of measurable, real-valued functions on $(X, \mathcal{A})$ with the property that, given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\mu(\{x \in X | |f_k(x) - f_l(x)| \geq \varepsilon\}) < \varepsilon$ for all $k, l \in \mathbb{N}$ with $k, l \geq N$. Show that there is a subsequence of $\{f_n\}$ that converges pointwise a.e. to a function $f$ on $X$. You may use Borel-Cantelli Lemma.

4. [15] Let $(X, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space and $\nu$ be a finite measure on $\mathcal{A}$. Show that $\nu \ll \mu$ if and only if there is no nonzero finite measure $\rho$ on $\mathcal{A}$ such that $\rho \ll \mu$ and $\rho(E) \leq \nu(E)$ for all $E \in \mathcal{A}$. You may use Radon-Nikodym Theorem.

5. [10] Let $(X, \mathcal{A}, \mu)$ be a measure space and $f$ be an integrable function on $(X, \mathcal{A}, \mu)$. Show that, given $\varepsilon > 0$, there is $E \in \mathcal{A}$ with $\mu(E) < \infty$ such that $f$ is bounded on $E$ and $\int_{X \setminus E} |f| d\mu < \varepsilon$. You may use Monotone Convergence Theorem.

6. [15] Let $(X, \mathcal{A}, \mu)$ be a measure space and $\Gamma$ be the set of integrable simple functions on $(X, \mathcal{A}, \mu)$. Show that if $\mu$ is semifinite, then $\int_X |f| d\mu = \sup_{\varphi \in \Gamma} \int_X (|f| \wedge \varphi) d\mu$ for all measurable functions $f$ on $(X, \mathcal{A})$. Decide whether the converse holds.

7. [15] Let $1 < p < \infty$. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f$ be a measurable function on $(X, \mathcal{A})$ such that $|f| > 0$ a.e. and $|\int_X f \varphi d\mu| \leq 1$ for all simple functions $\varphi$ on $(X, \mathcal{A})$ with $\int_X |\varphi|^p d\mu = 1$. Show that if $\mu$ is semifinite, then it is $\sigma$-finite.
We denote by $D = \{ z \in \mathbb{C} \mid |z| < 1 \}$ the unit disk in $\mathbb{C}$.

(1) -10 pts- Give the statement (not the proof) of Liouville's theorem on bounded entire functions.

(2) -10 pts- Give the definition of a meromorphic function on $\mathbb{C}$.

(3) -10 pts- Does there exist a holomorphic bijection between $D$ and $\{ z = x + iy \in \mathbb{C} \mid x > 0, y > 0, x + y < 1 \}$?

(Justify your answer in no more than two lines.)

(4) -10 pts- Does there exist a holomorphic bijection $f : \mathbb{C} \to D$?

(Justify your answer in no more than two lines.)

(5) -10 pts- Let $x \in \mathbb{R}$ with $x > 1$. Set $z = e^{i\theta}$, and use complex integration and the residue theorem to prove

$$\int_0^{2\pi} \frac{d\theta}{x + \cos \theta} = \frac{2\pi}{\sqrt{x^2 - 1}}.$$  

(Hint: $-x - \sqrt{x^2 - 1} < -1 < -x + \sqrt{x^2 - 1} < 0$.)

(6) -10 pts- Let $A$ be a family of functions holomorphic in $D$. Denote $A' = \{ f' \mid f \in A \}$. Assume $A'$ is normal and

$$\sup \{ |f(0)| \mid f \in A \} < \infty.$$  

Show that $A$ is normal.

(7) -10 pts- Let $(z_n)_{n \in \mathbb{N}}$ be a sequence of points in the complex plane converging to 0. Let $S = \{ 0 \} \cup \{ z_n \mid n \in \mathbb{N} \}$, and let $f$ be a holomorphic function on $\mathbb{C} - S$. Show that either $f$ extends to a meromorphic function on some disk centered at 0, or else for any $y \in \mathbb{C}$ there exists a sequence of complex numbers $(w_n)_{n \in \mathbb{N}}$ such that $w_n \to 0$ and $f(w_n) \to y$.

(8) -15 pts- Let $\Omega$ be a simply connected nonempty open subset of $\mathbb{C}$ such that $0 \notin \Omega$. Without using the Riemann Mapping Theorem, show that there is a holomorphic bijection $\varphi : \Omega \to \Omega'$ with $\Omega' \subset D$.

(9) -15 pts- Fix $\tau$ in the upper half-plane. We consider elliptic functions with respect to the lattice generated by 1 and $\tau$.

(a) (i) Describe the poles of the Weierstrass $p$ function.

(ii) Recall that the restriction to a fundamental parallelogram of the function $z \mapsto p(z) - p(a)$ has exactly one zero at $a$ when $a = 1/2, \tau/2$ or $(1 + \tau)/2$, and exactly two zeros at $a$ and $-a$ otherwise. What is the order of these zeros?

(b) Show that any elliptic function can be written as a rational expression in $p$. 
Problem 1. [10pt] Let $f(z)$ be a holomorphic function on a domain $D$. Prove that $\overline{f(\overline{z})}$ is a holomorphic function on $\{\overline{z} : z \in D\}$.

Problem 2. [20pt] Evaluate the following integrals

\begin{align*}
(1) & & \int_{-\infty}^{\infty} \frac{1}{x^4 + 1} \, dx \\
(2) & & \int_{0}^{2\pi} \frac{\cos \theta}{5 + 4 \cos \theta} \, d\theta
\end{align*}

Problem 3. [10pt] Establish the identity

$$\int_{-\infty}^{\infty} e^{\alpha x^2} e^{i\beta x} \, dx = \sqrt{\frac{\pi}{\alpha}} e^{-\beta^2/4\alpha}$$

where $\alpha, \beta$ are real numbers with $\alpha > 0$.

Problem 4. [20pt] Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree $n$.

1. Prove that for any $M > 0$, there exists a constant $R > 0$ such that $|p(z)| > M$ on $\{z \in \mathbb{C} : |z| > R\}$.
2. Prove that if $|p(z)|$ attains a local minimum at $z_0$, then $|p(z_0)| = 0$.
3. Deduce the fundamental theorem of algebra, saying that $p(z)$ has at least a zero in $\mathbb{C}$.

Problem 5. [15pt] Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function with compact support. Show that if the Fourier transform of $f$,

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) \, dx$$

also has a compact support, then $f \equiv 0$. (Hint. Use the identity theorem)

Problem 6. [15pt] Let $f(z)$ be a nonconstant entire function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Problem 7. [10pt] Let $f(z)$ be a holomorphic function on $\mathbb{C}$. Show that if $Re f \geq 0$, then $f$ is a constant.
2012 Algebra Qualifying Exam

(1) Let $G$ be an abelian group of order 72. Classify it in terms of both the elementary divisor decomposition and the invariant factor decomposition.

(2) Let $G$ be a finite group of order 56. By making use of Sylow's theorem determine whether it is a simple group or not.

(3) Let $R$ be a PID. Prove that $R$ is a UFD.

(4) Find the minimal polynomial for $\alpha = \sqrt{3} + \sqrt{5}$ over $\mathbb{Q}$.

(5) Let $K=\mathbb{Q}(\zeta)$ with $\zeta$ a primitive 5-th root of unity, and let $G = \text{Gal}(K/\mathbb{Q})$.
   (i) Find the group $G$.
   (ii) Let $H$ be a nontrivial proper subgroup of $G$, and $K^H$ be its fixed field. Find $K^H$.
   (iii) When $K^H = \mathbb{Q}(\alpha)$ in (ii), find the minimal polynomial of $\alpha$ over $\mathbb{Q}$.

(6) Let $R$ be a commutative ring with 1 and $P$ be an $R$-module. We say "$P$ is projective" if given any diagram of $R$-linear maps with the bottom row exact, there exists an $R$-linear map $h : P \rightarrow M$ such that $f = g \circ h$.

    $\exists h \xymatrix{ P \ar[dr]^f & \ar[d]^h \\
                          & M \ar[r]^g & N \ar[r] & 0}$

Prove that every $R$-module is a homomorphic image of a projective $R$-module.
1. (10 pts) Verify that
\[
P\{\lim_{n \to \infty} \inf A_n \} \leq \lim_{n \to \infty} \inf P\{A_n\} \leq \lim_{n \to \infty} \sup P\{A_n\} \leq P\{\lim_{n \to \infty} \sup A_n\}.
\]

2. (10 pts) Let \((\Omega, \mathcal{F})\) be a measurable space, and \(X\) and \(Y\) be two random variables. Define a new random variable \(Z := XY\). Consider \(\sigma(Z)\) and \(\sigma(X, Y)\). Prove or disprove that \(\sigma(Z) = \sigma(X, Y)\).

3. (10 pts) Let \(\{X_n\}\) be independent random variables and \(S_n = X_1 + \ldots + X_n\). Show that the event \(\{\omega : \sum_{n=1}^{\infty} X_n(\omega) \text{ converges}\}\) has probability 0 or 1.

4. (10 pts) A sequence \(\{X_n\}\) is said to converge completely to a random variable \(X\) if \(\sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < \infty\) for every \(\varepsilon > 0\). Show that, if \(X_n\) converges to \(X\) almost surely, then there exists a subsequence \(\{X_{n_j}\}\) such that \(X_{n_j}\) converges to \(X\) completely.

5. (10 pts) For random variables \(X_1, X_2\) and \(Y\) with \(Y \in L_1\), suppose that \(\sigma(Y, X_1)\) is independent of \(\sigma(X_2)\). Show almost surely \(E[Y|X_1, X_2] = E[Y|X_1]\).

6. (10 pts) Suppose that \(\{(X_n, B_n), n \geq 0\}\) is a martingale. Show the following: \(\{X_n\}\) is \(L_1\)-convergent if and only if there exists \(X \in L_1\) such that \(X_n = E[X|B_n], n \geq 0\).

7. (10 pts) Suppose \(X\) and \(Y\) are independent with common distribution function \(F(x)\) having mean zero and variance 1, and suppose further that

\[
\frac{X + Y}{\sqrt{2}} \overset{d}{=} X \overset{d}{=} Y.
\]

Show that both \(X\) and \(Y\) have a \(N(0, 1)\) distribution.

8. (10 pts) If \(X_n\) converges in distribution to \(X\) and \(Y_n\) converges in distribution to a constant \(c\), show that \(X_n Y_n\) converges in distribution to \(cX\). (Hint: Prove it first with \(c = 0\).)

9. (20 pts) True or false. You don't need to prove! 2 points for correct answers, but -2 points for incorrect answers.

(a) Suppose \(X\) is a non-negative random variable satisfying \(P\{0 \leq X < \infty\} = 1. \lim_{n \to \infty} nE[1_{X > n}] = 0\).

(b) If \(X_n \to X\) a.s., then \(E[X_n] \to E[X]\).

(c) If \(X_n \to X\) in \(L_p\) (\(p = 1, 2, \ldots\)), then \(E[X_n^p] \to E[X^p]\).

(d) If \(X_n \overset{L^2}{\to} 0\), \(S_n \overset{L^2}{\to} 0\) for \(p \geq 1\).

(e) Let \(\{X_n\}\) be a sequence of random variables with \(P\{X_n = \pm \frac{1}{n}\} = \frac{1}{2}\). \(X_n \to 0\) a.s.

(f) If \(X_n \overset{P}{\to} X\), there exists a subsequence \(\{X_{n_k}\}, k \geq 1\) which converges to \(X\) a.s.

(g) For i.i.d. random variables \(\{X_n\}\), if \(X_1\) is integrable, then \(\lim_{n \to \infty} \frac{|X_n|}{n} = 0\) a.s.

(h) If \(X_n\) converges in distribution to \(X\), then for every continuous function \(f : \mathbb{R} \to \mathbb{R}, E[f(X_n)] \to E[f(X)]\).

(i) Let \(X\) be an integrable random variable. If \(\sigma(X)\) and a \(\sigma\)-field \(\mathcal{F}\) are independent, then \(E[X|\mathcal{F}] = E[X]\) almost surely.

(j) If \(X_n \overset{L^2}{\to} X\), then \(E[X_n|\mathcal{B}] \overset{L^2}{\to} E[X|\mathcal{B}]\) for a \(\sigma\)-field \(\mathcal{B}\).
Qualifying Exam-2 2012 in Advanced Statistics

1. Let \((Y_1, \cdots, Y_k)\) be a multinomial random vector with cell probabilities, \(p_1, \cdots, p_k\), such that \(\sum_{i=1}^{k} p_i = 1\), and let \(\sum_{i=1}^{k} X_k = n\), where \(n\) is the total number of multinomial trials or the sample size.
   
   (a) [10 pts] For \(l < k\), find the distribution of \((Y_1, \cdots, Y_l)\).
   
   (b) [5 pts] For \(m < k\), find the conditional distribution of \((Y_{l+1}, \cdots, Y_m)\) given that \((Y_1, \cdots, Y_l) = (y_1, \cdots, y_l)\).

2. [15 pts] Let the statistic \(T(X_1, \cdots, X_n)\) be a minimal sufficient statistic (MSS), where \(X_1, \cdots, X_n\) are iid from a distribution. Is it true then that any complete sufficient statistic is also a MSS? Why?

3. Let \(X_1, \cdots, X_n\) be a random sample from the gamma distribution, \(\text{gamma}(\alpha, \beta)\). Let \(\bar{X}\) and \(S^2\) be the sample mean and sample variance, respectively.

   (a) [10 pts] Are \(\bar{X}\) and \(S^2\) independent of each other? Why?
   
   (b) [10 pts] Are \(\bar{X}\) and \(S^2/\bar{X}^2\) independent of each other? Why?

4. [15 pts] Let \(X_1, \cdots, X_n\) be a Normal random sample from \(N(\mu, 1)\) and define \(Y_n = \max\{X_1, \cdots, X_n\}\). Find a best unbiased estimator of \(P(Y_n > a)\) for a real value \(a\).

5. [15 pts] Find the 1-\(\alpha\) confidence set for \(a\) that is obtained based on a random sample \(X_1, \cdots, X_n\) from the normal distribution, \(N(\theta, a\theta)\), with \(\theta\) unknown. [Hint: Think of inverting an LRT.]

6. Let \(X_1, \cdots, X_n\) be a random sample from the uniform distribution, \(U(\theta, \theta+1)\).
For testing \(H_0 : \theta = 0\) vs \(H_1 : \theta > 0\), we use the test

\[
\text{reject } H_0 \text{ if } Y_n \geq 1 \text{ or } Y_1 \geq a,
\]

where \(a\) is a constant, \(Y_1 = \min\{X_1, \cdots, X_n\}\), and \(Y_n = \max\{X_1, \cdots, X_n\}\).

(a) [8 pts] Find \(a\) so that the test is of size \(\alpha\).

(b) [6 pts] Find an expression for the power function of the test in question (6a).

(c) [6 pts] Is the test UMP? Why?

THE END
1. [10 pts] Suppose $T_1$ and $T_2$ are sufficient and minimally sufficient for $\theta$, respectively, and that $U$ is an unbiased estimator of $\theta$. Let $U_i = E(U|T_i)$, $i = 1, 2$. Compare $\text{Var}(U_1)$ and $\text{Var}(U_2)$.

2. [20 pts] Given that $N = n$, the conditional distribution of $Y$ is $\chi^2_n$. Suppose that $N$ is a Poisson random variable with parameter $\theta$. Find the limit of the distribution of $(Y - E(Y))/\sqrt{\text{Var}(Y)}$ as $\theta \to \infty$.

3. Let $X_1, X_2, \cdots, X_n$ be a random sample from the pdf $f(x; \theta)$ and write $X = (X_1, \cdots, X_n)$. Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$, using a test with rejection region $R$ that satisfies

$$x \in R \text{ if } f(x; \theta_1)/f(x; \theta_0) > k,$$

and

$$x \in R^c \text{ if } f(x; \theta_1)/f(x; \theta_0) < k$$

for some $k \geq 0$, and

$$\alpha = P_{\theta_0}(X \in R).$$

Then show the following statements:

(a) [10 pts] Any test that satisfies (1) and (2) is a UMP level $\alpha$ test.

(b) [10 pts] If there exists a test satisfying (1) and (2), every UMP level $\alpha$ test satisfies (1) and (2) except a null set $A$ such that $P_{\theta_0}(X \in A) = P_{\theta_1}(X \in A) = 0$.

4. [15 pts] Let $X$ and $Y$ be independent exponential random variables, with

$$f(x; \lambda) = \frac{1}{\lambda} e^{-x/\lambda}, \ x > 0, \ f(y; \mu) = \frac{1}{\mu} e^{-y/\mu}, \ y > 0.$$

Suppose we have a random sample, $(Z_i, W_i)$, $i = 1, 2, \cdots, n$, where

$$Z_i = \min(X_i, Y_i) \quad \text{and} \quad W_i = \begin{cases} 1 & \text{if } Z_i = X_i, \\ 0 & \text{if } Z_i = Y_i. \end{cases}$$

Find the MLEs of $\lambda$ and $\mu$.

5. [20 pts] Let $Y$ and $V$ be random variables with $Y \sim f_Y$ and $V \sim f_V$, where $f_Y$ and $f_V$ share the same support with

$$M = \sup_y \frac{f_Y(y)}{f_V(y)} < \infty.$$

Suppose that we generate a random number $W$ as follows:

(a). Generate a uniform random number, $U$, i.e., $U \sim U(0, 1)$, and a random number $V$ from $f_Y$.

(b). If $U < \frac{1}{M} f_Y(V)/f_V(V)$, set $W = V$; otherwise, return to step (a).

Find the distribution of $W$.

6. [15 pts] Let $X_1, \cdots, X_n$ be a random sample from the normal distribution, $\mathcal{N}(\mu, 1)$ with unknown $\mu$. Find, if it exists, the best unbiased estimator of $\varphi = P(X_1 > c)$ for a constant $c$.

THE END
Numerical Analysis Qualifying Exam

July 2012

1. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a nonlinear smooth function. To determine a (local) minimum of \( f \) one can use a descent method of the form

\[
x_{k+1} = x_k + \alpha_k d_k
\]

where \( \alpha_k > 0 \) is a suitable parameter and \( d_k \) is a descent direction, i.e., it satisfies

\[
f(x_{k+1}) < f(x_k).
\]

(a) (7 points) Write the steepest descent (or gradient) method and show that there exist \( \alpha_k > 0 \) such that the resulting method satisfies (1).

(b) (8 points) Write the Newton method and examine whether or not there exist \( \alpha_k \) which yield (1). Establish conditions on the Hessian \( H(f(x)) \) of \( f(x) \) which guarantee the existence of \( \alpha_k \).

(c) (5 points) If we replace the Hessian by the matrix \( H(f(x)) + \gamma_k I \), where \( \gamma_k > 0 \) and \( I \) is the identity matrix, we obtain a quasi-Newton method. Find a condition on \( \gamma_k \) which leads to (1).

2. (a) (4 points) Consider the problem of interpolating the following data made up of \( n + 1 \) distinct points:

\[
(x_0, f_0), (x_1, f_1), (x_2, f_2), \ldots, (x_n, f_n).
\]

Prove that there exists a unique polynomial of degree at most \( n \) that interpolates the above data.

(b) (6 points) Consider the Chebyshev polynomials:

\[
T_0(x) = 1, T_1(x) = x, T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad n > 0.
\]

Prove all of the following:

i. \( T_n(x) \) is a polynomial of degree exactly \( n \) with \( n \) distinct real roots between \(-1 \leq x \leq 1\);
ii. \(-1 \leq T_n(x) \leq 1 \) for all \(-1 \leq x \leq 1\);
iii. The coefficient of \( x^n \) in \( T_n(x) \) is exactly \( 2^{n-1} \) for \( n > 0 \).

(c) (8 points) Consider the problem of interpolating the \( C^{n+1} \) function \( f(x) \) at the \( n + 1 \) distinct points \( x_0, x_1, \ldots, x_n \), where \(-1 \leq x_0 < x_1 < x_2 < \cdots < x_n \leq 1 \), with the polynomial \( p_n(x) \). Show that the max-norm error:

\[
\|f(x) - p_n(x)\|_\infty := \max_{-1 \leq x \leq 1} |f(x) - p_n(x)|
\]

is nearly minimized over all possible choices of the points \( x_0, x_1, \ldots, x_n \) if these points are the \( n + 1 \) roots of \( T_{n+1}(x) \).

(d) (2 points) How does \( \|f(x) - p_n(x)\|_\infty \) decay with increasing \( n \) for the \( C^\infty \) function \( f(x) \).

3. Consider the Runge-Kutta methods for the ODE \( u' = f(u) \):

\[
\begin{align*}
f_1 &= f(u^n) \\
\text{SCHEME 1:} \quad f_2 &= f(u^n + b_1 k f_1) \\
&= u^n + k(c_1 f_1 + c_2 f_2) \\
&= u^n + k(c_1 f_1 + c_2 f_2 + c_3 f_3)
\end{align*}
\]

\[
\begin{align*}
f_1 &= f(u^n) \\
\text{SCHEME 2:} \quad f_2 &= f(u^n + b_1 k f_1) \\
f_3 &= f(u^n + b_2 k f_2) \\
&= u^n + k(c_1 f_1 + c_2 f_2 + c_3 f_3)
\end{align*}
\]
4. Consider the linear system $Ax = b$ with an $n \times n$ nonsingular matrix $A$.

(a) (6 points) Write down the Jacobi iteration for solving $Ax = b$, in the way that it would be programmed on a computer.

(b) (5 points) Suppose $A$ has $m$ nonzero elements with $m << n^2$. How many operations per iteration does the Jacobi iteration take?

(c) (9 points) Assume that $A$ is strictly diagonally dominant. i.e., for $i = 1, \ldots, n$,

$$|a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ij}|.$$

Show that the Jacobi iteration converges for any initial guess $x^{(0)}$.

5. Let $A$ be a real square matrix whose eigenvalues are all algebraically simple. We intend to find an eigenvector of $A$ by using the power method. The power method is considered ‘convergent’ if, for every positive tolerance $\epsilon$, we obtain, after sufficiently many iteration, $x^{(n)}$ so that, for some eigenvector $v$ of $A$, $\|x^{(n)} - v\| \leq \epsilon \|v\|$.

(a) (5 points) Explain why the above definition of convergence is preferred over the simpler one, where we merely require that $\|x^{(n)} - v\| \leq \epsilon$.

(b) (8 points) Is the following statement true? “The power method will converge regardless of the choice of the initial (non-zero) vector, provided that $A$ is symmetric positive definite.” Explain your answer.

(c) (7 points) Will you change your answer to (b) if $A$ is not necessarily positive definite?
1. Consider the following iteration for calculating $\gamma^{1/3}$:

$$x_{n+1} = ax_n + \frac{b}{x_n^2} + \frac{c}{x_n^3}$$

(a) Choose $a, b, c$ so that the iteration will converge to $\gamma^{1/3}$ (for $x_0$ sufficiently close to $\gamma^{1/3}$ with the highest order possible). Show your work.

(b) Write an equation that relates the error at step $n$ to the error at step $n+1$. What is the limiting form of this expression?

2. Given the function $f(x) = e^x$,

(a) Form the divided difference table necessary to construct the cubic Hermite polynomial interpolating $f(x)$ on the nodes $x_0 = 0, x_1 = 1$.

(b) Use the table to construct the Hermite interpolating polynomial, $p_3(x)$, on this mesh.

(c) Write the formula for the error, $E(x) = f(x) - p_3(x)$, involving derivatives of $f(x)$.

(d) Derive an upper bound for the error: $\max_{x \in [0,1]} |f(x) - p_3(x)|$. This should be a number. Use the approximation $e \leq 3$.

3. Let $p_3(x) = x^3$. Find $q^*_2(x)$, the minimax approximation of $p_3(x)$ on the interval $[-1, 1]$.

4. (a) Derive Simpson's rule from the Trapezoid rule and the Midpoint rule.

(b) Which quadrature rule for $\int_a^b f(x) \, dx$ do you obtain by the following process? You approximate that integral by $\int_a^b p_2(x) \, dx$, with $p_2$ the unique polynomial of degree $< 3$ that agrees with $f$ at the three points $a, (a+b)/2$, and $b$.

(c) Discuss how you would go about obtaining, in an efficient way, an accurate value for

$$\int_{-1}^{1} \frac{1 + |x^2 - 1/4|}{\sqrt{1 - x^2}} \, dx$$

Justify your approach, by citing relevant facts about the quadrature rule(s) you are proposing to use.

5. Let $A$ be real, symmetric, positive definite, and of order $n$. Consider solving $Ax = b$ using Gaussian elimination without pivoting.

(a) Show that all of the diagonal elements satisfy $a_{ii} > 0$.

(b) After elimination of $x_1$ from equations 2 through $n$, let the resulting matrix $A^{(2)}$ be written as

$$A^{(2)} = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  0 & a_{22} & \cdots & a_{2n} \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & \cdots & 0 & \tilde{A}^{(2)}
\end{bmatrix}$$

Show that $\tilde{A}^{(2)}$ is symmetric and positive definite.