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A TWO-LEVEL NONOVERLAPPING SCHWARZ ALGORITHM FOR THE STOKES PROBLEM: NUMERICAL STUDY *

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Abstract. A general framework of a two-level nonoverlapping Schwarz algorithm for the Stokes problem is developed. This framework allows both discontinuous and continuous pressure finite element spaces. The coarse problem is built by algebraic manipulation after selecting appropriate primal unknowns just like in BDDC algorithms. Performance of the suggested algorithm is presented depending on the selection of finite elements and primal unknowns. Under the same set of primal unknowns, the algorithm for the case with discontinuous pressure functions outperforms one with continuous pressure functions. For the two-dimensional Stokes problem, the algorithm with a set of primal unknowns consisting of velocity unknowns at corners, averages of velocity components over common edges, and pressure unknowns at corners presents good scalability when continuous pressure test functions are used. In both two- and three-dimensional Stokes problems, an improvement can be made for the case with continuous pressure test functions by applying the suggested algorithm to the interface problem, which is obtained by eliminating velocity unknowns and pressure unknowns interior to each subdomains.

Key words. Two-level Schwarz algorithm, Stokes problem, preconditioner, nonoverlapping partition

AMS subject classifications. 65N30, 65N55, 76D07

1. Introduction. There have been considerable researches on domain decomposition methods for the Stokes problem [3, 8, 9, 10, 12, 13, 14, 15, 18]. In the work based on the nonoverlapping subdomain partitions, Neumann-Neumann algorithms, BDDC and FETI-DP algorithms have been developed and presented good scalability [8, 9, 10, 14, 15, 18]. Those algorithms utilizing overlapping subdomains have been also considered in [3, 12, 13] for the Stokes problem and almost incompressible elasticity problems.

In the approaches using nonoverlapping partitions, quite complicated primal unknowns are necessary to achieve a good scalability so that the design and implementation of the algorithms become difficult especially in the three dimensional problems. On the other hand, in the approaches using overlapping partitions, an additional coarse triangulation is required to build a coarse problem of the methods.

In the recent works [9, 10, 7, 11] developed by the authors jointly with Park, a good scalability can be obtained using only the primal velocity unknowns differently to previously developed methods [12, 13, 14, 15, 18]. In [7], a two-level nonoverlapping preconditioner was

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built by using a close connection of the dual and primal formulations of the Stokes problem and convergence analysis for the GMRES iteration applied to the two-level nonoverlapping preconditioner was provided.

The algorithm developed in [10] is a FETI-DP algorithm with a lumped preconditioner. FETI-DP algorithms belong to a family of dual iterative substructuring methods, which are known to be one of the most scalable domain decomposition methods for solving numerical partial differential equations, see [4, 5]. A pair of inf-sup stable velocity and pressure finite element spaces is introduced to a given triangulation of the domain and the continuity of the finite element spaces is relaxed by decomposing the given domain into many subdomains. Among the degrees of freedom on subdomain interfaces, some are selected as primal unknowns. A strong continuity is enforced to the primal unknowns and at the remaining part of unknowns on the interface the continuity is imposed weakly by using Lagrange multipliers. After elimination of the unknowns other than the Lagrange multipliers, a system on the dual unknowns, i.e., the Lagrange multipliers, is solved iteratively with a preconditioner, which accelerates the convergence of the iteration. Velocity at subdomain corners are selected as the primal unknowns in the two-dimensional case and additionally averages of the velocity over common faces are selected as primal unknowns in the three-dimensional case. In [9, 10] it was proved that such selection of the primal unknowns gives the condition number bound, $C(H/h)(1 + \log(H/h))$ in $2D$ and $C(H/h)$ in $3D$ for the FETI-DP algorithm with the lumped preconditioner.

As a primal counterpart of FETI-DP algorithms, BDDC (Balancing Domain Decomposition by Constraints) algorithms were introduced by Dohrmann [2] and further analyzed by Mandel and Dohrmann [16]. Their close connection to the FETI-DP algorithms was studied in [15, 1, 6, 17]. In our previous work [7], we developed a primal counterpart of the FETI-DP algorithm for solving the Stokes problem. The primal algorithm was derived similarly to the work in [15]. In the primal formulation, by using its close connection to the dual form of the Stokes problem, a preconditioner for the primal form was designed so that the resulting preconditioned linear system has all its eigenvalues as positive real numbers. The primal counterpart of the FETI-DP algorithm turned out to be a two-level nonoverlapping Schwarz algorithm.

The two-level nonoverlapping Schwarz algorithm can be applied to more general finite element spaces such as Taylor-Hood finite elements with continuous pressure functions. In this work, numerical experiments are carried out for the two-level nonoverlapping Schwarz algorithm with various choices of finite element spaces and primal unknowns. Between the cases with discontinuous and continuous pressure test functions under the same set of primal unknowns, the case with discontinuous pressure test functions shows much better performance. On the other hand, in the case of the continuous pressure test functions the Stokes problem can be reduced to an interface problem by eliminating velocity and pressure unknowns interior to each subdomains. The elimination process is called the static condensation, see [2]. The same algorithm can be applied to the interface problem and it shows much better performance. In the two-dimensional case we observe good scalability with the set of primal unknowns consisting of velocity unknowns at corners, edge averages of each velocity components, and pressure unknowns at corners, when the algorithm is applied to the interface problem. We also plot distribution of eigenvalues of the preconditioned system. For the three-dimensional case, we observe that the static condensation results in a better distribution of eigenvalues.

In Section 2, we introduce a model problem and in Section 3 we develop a general framework for a two-level nonoverlapping Schwarz method for the Stokes problem. Previous theories for the convergence are discussed in Section 4 and numerical experiments are carried out in Section 5. Throughout this paper, C denotes a generic positive constant which does not depend on any mesh parameters and the number of subdomains.

2. A model problem and finite element spaces. We consider the Stokes problem,

$$
\begin{aligned}
-\triangle \boldsymbol{u} + \nabla p &= \boldsymbol{f} \ \text{ in } \Omega, \\
\nabla \cdot \boldsymbol{u} &= 0 \ \text{ in } \Omega, \\
\boldsymbol{u} &= 0 \ \text{ on } \partial\Omega,
\end{aligned}
\tag{2.1}
$$

where Ω is a bounded polygonal (polyhedral) domain in $\mathbb{R}^2$ ($\mathbb{R}^3$) and $\boldsymbol{f} \in [L^2(\Omega)]^2$ ($[L^2(\Omega)]^3$). A triangulation is provided for the domain Ω and a pair of velocity and pressure spaces $(\widehat{V}, \widehat{P})$ is equipped for the triangulation.

The functions in the velocity space $\widehat{V}$ are continuous across the elements with zero trace on the boundary of Ω and those in the pressure space $\widehat{P}$ can be continuous as well. We note that our algorithm is not limited to discontinuous pressure spaces only. We enforce the average zero condition on the pressure space and denote the resulting pressure space by $\widehat{P}_0$, i.e.,

$$
\widehat{P}_0 = \widehat{P} \bigcap L_0^2(\Omega),
$$

where $L_0^2(\Omega)$ is the space of square integrable functions that have zero average in Ω. We assume that the pair $(\widehat{V}, \widehat{P}_0)$ is inf-sup stable and obtain a discrete problem for (2.1):

Find $(\widehat{\boldsymbol{u}}, \widehat{p}) \in (\widehat{V}, \widehat{P}_0)$ satisfying

$$
\begin{aligned}
&\int_\Omega \nabla \widehat{\boldsymbol{u}} \cdot \nabla \boldsymbol{v} \, dx - \int_\Omega \widehat{p} \nabla \cdot \boldsymbol{v} \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx, \ \forall \boldsymbol{v} \in \widehat{V}, \\
&- \int_\Omega \nabla \cdot \widehat{\boldsymbol{u}} \, q \, dx = 0, \ \forall q \in \widehat{P}_0.
\end{aligned}
\tag{2.2}
$$

In our approach, we consider an equivalent problem by employing the pressure space $\widehat{P}$ instead of $\widehat{P}_0$ and obtain the following algebraic system:

Find $(\widehat{\boldsymbol{u}}, \widehat{p}) \in (\widehat{V}, \widehat{P})$ such that

$$\begin{pmatrix} \widehat{K} & \widehat{B}^T \\ \widehat{B} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{u}} \\ \widehat{p} \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \tag{2.3}$$

where $\widehat{K}$ is the stiffness matrix corresponding to $\int_\Omega \nabla \widehat{\boldsymbol{u}} \cdot \nabla \boldsymbol{v}\, dx$ and $\widehat{B}$ is the matrix to the term $-\int_\Omega \nabla \cdot \widehat{\boldsymbol{u}}\, q\, dx$ with $q \in \widehat{P}$. The advantage of using $\widehat{P}$ is that we do not need to deal with the global pressure component so that we will not necessarily have primal pressure unknowns in our domain decomposition methods, see [9, 10, 7]. In the following section, we will develop a two-level nonoverlapping Schwarz algorithm applied to the above algebraic system (2.3) obtained from the pair $(\widehat{V}, \widehat{P})$.

3. A two-level nonoverlapping Schwarz algorithm. We now decompose Ω into a nonoverlapping subdomain partition $\{\Omega_i\}_{i=1}^N$ in such a way that subdomain boundaries align to the given triangulation in Ω. We introduce local finite element spaces,

$$V^{(i)} = \widehat{V}|_{\Omega_i} \text{ and } P^{(i)} = \widehat{P}|_{\Omega_i}.$$

In the product spaces V and P defined as

$$V = \prod_{i=1}^N V^{(i)} \text{ and } P = \prod_{i=1}^N P^{(i)},$$

the functions can be discontinuous across the subdomain boundaries. When the pressure functions in $\widehat{P}$ are discontinuous, P is essentially the same as $\widehat{P}$.

Among those unknowns in V, we select some unknowns on the subdomain interface as primal unknowns and enforce strong continuity to obtain $\widetilde{V}$, where functions can be discontinuous at the remaining part of the interface unknowns. We call the remaining interface unknowns dual unknowns. The notations, $\boldsymbol{u}_I^{(i)}$, $\boldsymbol{u}_\Delta^{(i)}$, and $\boldsymbol{u}_\Pi^{(i)}$ are used to denote the unknowns at the interior part of $\Omega^{(i)}$, the dual unknowns on $\partial\Omega^{(i)}$, and the primal unknowns on $\partial\Omega^{(i)}$, respectively. The spaces $V_I^{(i)}$, $V_\Delta^{(i)}$, and $V_\Pi^{(i)}$ consist of the corresponding velocity unknowns, $\boldsymbol{u}_I^{(i)}$, $\boldsymbol{u}_\Delta^{(i)}$, and $\boldsymbol{u}_\Pi^{(i)}$, respectively. For the pressure unknowns p in P, we may select the primal unknowns when $\widehat{P}$ consists of continuous pressure functions. For this case, we use $p_I^{(i)}$, $p_\Delta^{(i)}$, and $p_\Pi^{(i)}$ to denote interior unknowns, dual unknowns, and the primal unknowns, respectively. As in the velocity spaces, we introduce related pressure spaces $P_I^{(i)}$, $P_\Delta^{(i)}$, and $P_\Pi^{(i)}$.

We now introduce restriction operators to each subdomain Ω_i,

$$R_{p,i} \,:\, P \to P^{(i)} \;\text{ and }\; R_{v,i} \,:\, V \to V^{(i)},$$

and the corresponding extension operators

$$R_{p,i}^T \,:\, P^{(i)} \to P \;\text{ and }\; R_{v,i}^T \,:\, V^{(i)} \to V.$$

Let

$$\begin{pmatrix} K^{(i)} & B^{(i)} \\ B^{(i)T} & 0 \end{pmatrix}$$

be the matrix of the local Stokes problem discretized with the pair of finite element spaces $(V^{(i)}, P^{(i)})$.

We introduce extension operators

$$\widetilde{R}_v : \widehat{V} \to \widetilde{V} \text{ and } \widetilde{R}_p : \widehat{P} \to \widetilde{P}.$$

We note that by using these extension operators the system of the Stokes problem in (2.3) discretized by the pair $(\widehat{V}, \widehat{P})$ can be rewritten as

$$\begin{pmatrix} \widehat{K} & \widehat{B}^T \\ \widehat{B} & 0 \end{pmatrix} = \begin{pmatrix} \widetilde{R}_v^T & 0 \\ 0 & \widetilde{R}_p^T \end{pmatrix} \begin{pmatrix} \widetilde{K} & \widetilde{B}^T \\ \widetilde{B} & 0 \end{pmatrix} \begin{pmatrix} \widetilde{R}_v & 0 \\ 0 & \widetilde{R}_p \end{pmatrix}, \tag{3.1}$$

where

$$\begin{pmatrix} \widetilde{K} & \widetilde{B}^T \\ \widetilde{B} & 0 \end{pmatrix}$$

is the system of the Stokes problem discretized by the pair $(\widetilde{V}, \widetilde{P})$,

We suggest a preconditioner, which is similar to the one in [7] for the Stokes problem discretized with discontinuous pressure finite element spaces,

$$M^{-1} = \begin{pmatrix} \widetilde{R}_v^T D_v & 0 \\ 0 & R_p^T D_p \end{pmatrix} \begin{pmatrix} \widetilde{K} & \widetilde{B}^T \\ \widetilde{B} & 0 \end{pmatrix}^{-1} \begin{pmatrix} D_v \widetilde{R}_v & 0 \\ 0 & D_p \widetilde{R}_p \end{pmatrix}. \tag{3.2}$$

Here D_v and D_p are diagonal matrices with weight factors for velocity unknowns $\widetilde{u} \in \widetilde{V}$ and pressure unknowns $\widetilde{p} \in \widetilde{P}$. The weight factors in D_v are defined as

$$D_v(x_v) = \begin{cases} \frac{1}{\nu(x_v)} & \text{for the dual unknowns,} \\ 1 & \text{for the interior and primal unknowns,} \end{cases}$$

where the notation x_v is used to denote the node corresponding to each velocity unknown and $\nu(x_v)$ is the number of subdomains containing the node x_v. The weight factors in D_p are defined similarly.

We introduce local velocity and pressure spaces consisting of interior and dual unknowns,

$$V_r^{(i)} = V_I^{(i)} \times V_\Delta^{(i)} \text{ and } P_r^{(i)} = P_I^{(i)} \times P_\Delta^{(i)}.$$

From these local spaces, we obtain corresponding product spaces

$$V_r = \prod_{i=1}^{N} V_r^{(i)} \text{ and } P_r = \prod_{i=1}^{N} P_r^{(i)}.$$

Let

$$A_{rr}^{(i)} = \begin{pmatrix} K_{II}^{(i)} & K_{I\Delta}^{(i)} & B_{II}^{(i)T} & B_{\Delta I}^{(i)T} \\ K_{\Delta I}^{(i)} & K_{\Delta\Delta}^{(i)} & B_{I\Delta}^{(i)T} & B_{\Delta\Delta}^{(i)T} \\ B_{II}^{(i)} & B_{I\Delta}^{(i)} & 0 & 0 \\ B_{\Delta I}^{(i)} & B_{\Delta\Delta}^{(i)} & 0 & 0 \end{pmatrix},\ A_{r\Pi}^{(i)} = \begin{pmatrix} K_{I\Pi}^{(i)} & B_{\Pi I}^{(i)T} \\ K_{\Delta\Pi}^{(i)} & B_{\Pi\Delta}^{(i)T} \\ B_{I\Pi}^{(i)} & 0 \\ B_{\Delta\Pi}^{(i)} & 0 \end{pmatrix},$$

$$A_{\Pi\Pi}^{(i)} = \begin{pmatrix} K_{\Pi\Pi}^{(i)} & B_{\Pi\Pi}^{(i)T} \\ B_{\Pi\Pi}^{(i)} & 0 \end{pmatrix},\ A_{\Pi r}^{(i)} = A_{r\Pi}^{(i)T}.$$

Using these matrices, we obtain the assembled matrices

$$A_{rr} = \text{diag}_i(A_{rr}^{(i)}),\ A_{\Pi r} = \left(R_{\Pi}^{(1)T} A_{\Pi r}^{(1)} \quad \cdots \quad R_{\Pi}^{(N)T} A_{\Pi r}^{(N)} \right),$$

and

$$A_{\Pi\Pi} = \sum_{i=1}^{N} R_{\Pi}^{(i)T} A_{\Pi\Pi}^{(i)} R_{\Pi}^{(i)},$$

where $R_{\Pi}^{(i)} : \widehat{V}_{\Pi} \times \widehat{P}_{\Pi} \rightarrow V_{\Pi}^{(i)} \times P_{\Pi}^{(i)}$ is the restriction of the primal unknowns to the subdomain $\Omega^{(i)}$. We introduce the coarse problem matrix,

$$S_{\Pi\Pi} = A_{\Pi\Pi} - A_{\Pi r} A_{rr}^{-1} A_{\Pi r}^{T}.$$

Let $R_r : \widehat{V} \times \widehat{P} \rightarrow V_r \times P_r$ and $R_{\Pi} : \widehat{V} \times \widehat{P} \rightarrow \widehat{V}_{\Pi} \times \widehat{P}_{\Pi}$ be restriction operators and

$$D_r = \begin{pmatrix} D_{r,v} & 0 \\ 0 & D_{r,p} \end{pmatrix}$$

be the matrix of weight factors corresponding to the unknowns in $V_r \times P_r$. Here $D_{r,v}$ and $D_{r,p}$ are submatrices of D_v and D_p in (3.2), respectively. The preconditioner M^{-1} is then written as

$$M^{-1} = R_r^T D_r A_{rr}^{-1} D_r R_r + R_0^T S_{\Pi\Pi}^{-1} R_0, \tag{3.3}$$

where

$$R_0 = R_{\Pi} - A_{\Pi r} A_{rr}^{-1} D_r R_r.$$

We note that the preconditioner consists of solving independent local Stokes problem discretized by the pair $(V_r^{(i)}, P_r^{(i)})$ and solving one global coarse problem associated to the primal unknowns.

In our algorithm, the average free condition on pressure functions is relaxed to remove the global component in the pressure function, see (2.3). This results in one null space component

$(\mathbf{0},1) \in (\widehat{V},\widehat{P})$. We introduce the L^2-orthogonal projection $P_{\mathcal{S}}$ onto $\mathcal{S}$, a subspace of $(\widehat{V},\widehat{P})$ which is orthogonal to the null space $N = \{(\mathbf{0},1)\}$. Since the system in (2.3) is indefinite, the GMRES method is used as an iterative solver. Therefore the system is solved on the subspace $\mathcal{S}$ with the GMRES method, i.e., the GMRES method is applied to

$$P_{\mathcal{S}}^T \widehat{M}^{-1} P_{\mathcal{S}} P_{\mathcal{S}}^T \begin{pmatrix} \widehat{K} & \widehat{B}^T \\ \widehat{B} & 0 \end{pmatrix} P_{\mathcal{S}} \begin{pmatrix} \widehat{\boldsymbol{u}} \\ \widehat{p} \end{pmatrix} = P_{\mathcal{S}}^T \widehat{M}^{-1} P_{\mathcal{S}} P_{\mathcal{S}}^T \begin{pmatrix} f \\ 0 \end{pmatrix},$$

where $\widehat{M}^{-1}$ is the preconditioner in (3.3). Since $P_{\mathcal{S}}$ is the orthogonal projection and the residual vector r from (2.3) is in the subspace $\mathcal{S}$, we only need to enforce $P_{\mathcal{S}}^T$ on the current preconditioned residual vector z once at each iteration.

4. Previous theories for convergence and static condensation. In this section, we will discuss the convergence of the GMRES method applied to our algorithm. The convergence of the GMRES method was analyzed in our previous work [7] for the case with discontinuous pressure spaces. For this case, our algorithm selects primal velocity unknowns which are unknowns at subdomain corners or averages of each velocity components over common edges in two dimensions. To achieve a scalability in three dimensions, averages of each velocity components over common faces in addition to velocity unknowns at corners are required.

The two-level nonoverlapping Schwarz algorithm applied to the case with a discontinuous pressure space is a primal counterpart of the FETI-DP algorithm developed in [9, 10, 11]. Using this, it was shown that the preconditioned primal form and the FETI-DP algorithm have the same spectra except zero and one. Therefore the preconditioned primal form in the two-level nonoverlapping Schwarz algorithm has its eigenvalues all positive. In detail, the minimum eigenvalue is bounded below by a constant which is independent of the mesh size, and the number of subdomains and the maximum eigenvalue follows the growth $(H/h)(1+\log(H/h))$ with primal unknowns consisting of velocity unknowns at corners in the two dimensions. In the two-dimensional case, by using averages of velocity unknowns over edges rather than velocity unknowns at corners a better bound CH/h of the maximum eigenvalue was obtained. The same bound CH/h of the maximum eigenvalue was analyzed for the three-dimensional case with primal unknowns consisting of velocity unknowns at corners and averages of velocity unknowns over common faces.

However, in the case of continuous pressure spaces there does not exist the corresponding FETI-DP algorithm which results in a positive definite system. In this paper, we study performance of our method for the continuous pressure spaces by carrying out numerical experiments. To the best of our knowledge there has not been even numerical study on nonoverlapping domain decomposition methods applied to the continuous pressure spaces.

As we will see in our numerical results in Section 5, iteration counts in continuous pressure finite elements are quite large compared to those in discontinuous pressure finite elements. One advantage of using continuous pressure finite elements is that the algebraic system (3.1) can be reduced to the unknowns on the subdomain interface by eliminating velocity and pressure unknowns located at the interior of each subdomains. This requires solving local Stokes problems. We call this process the static condensation. After the elimination process, we are left with the system of the interface unknowns U_Γ,

$$S_{\Gamma\Gamma}U_\Gamma = g_\Gamma, \tag{4.1}$$

where

$$U_\Gamma = \begin{pmatrix} \boldsymbol{u}_\Gamma \\ p_\Gamma \end{pmatrix}$$

and $S_{\Gamma\Gamma}$ is the Schur complement matrix of (3.1). Here $\boldsymbol{u}_\Gamma$ and p_Γ denote velocity and pressure unknowns located on the subdomain interface, respectively.

The preconditioner (3.2) can also be modified according to the static condensation. Just like in the previous section, we then build the local and coarse problems in the preconditioner by algebraic manipulation on the system (4.1) after selecting an appropriate set of primal unknowns from U_Γ.

The resulting algorithm obtained from the static condensation is a BDDC method for the Stokes problem with continuous pressure finite elements. We also note that a BDDC algorithm for the Stokes problem was studied in [15] for the case with discontinuous pressure finite elements. However there has been no theoretical or numerical study carried out for the case with continuous pressure finite elements. In the work [15], the average zero condition is enforced on pressure functions and primal pressure unknowns, which are constant in each subdomains, appear in the construction of the preconditioner. The static condensation in [15] produced a system with velocity unknowns on the subdomain interface and the primal pressure unknowns while the algebraic system considered in this paper can be reduced to the interface problem with both velocity and pressure unknowns on the subdomain interface, see (4.1).

We stress that in our framework, we are more free to choose primal unknowns, for example, we may select the set of primal unknowns from only velocity unknowns without introducing any primal pressure unknowns. Such selection of primal unknowns can make implementation much simpler and produces a symmetric and positive definite coarse problem matrix in the preconditioner.

5. Numerical experiments. We perform numerical experiments of our method for the Stokes problem. In two dimensions, we consider a model problem in the unit square domain, $\Omega = [0,1]^2$, and divide it into a uniform rectangular partition. Here $N_d = 3^2$ means that the domain Ω is partitioned into 3×3 uniform square subdomains. We triangulate each subdomain with uniform triangles and then we use $(H/h)^2$ to denote the number of triangles in each subdomain. To each triangle, we associate a Taylor-Hood finite element space,

TABLE 1

Iteration counts in our method for the two-dimensional Stokes problem with the Taylor-Hood finite element space $P_2(h) - P_1(h)$. Iter: iteration count, WOS: results without the static condensation, WS: results with the static condensation, N_d: number of subdomains, H/h: local problem size factor, vc: the set of primal unknowns consisting of velocity unknowns at corners, $vc + ve$: the set of primal unknowns consisting of velocity unknowns at corners and averages of velocity unknowns over edges.

	Iter (WOS/WS) $N_d = 3^2$			Iter (WOS/WS) $H/h = 4$	
H/h	vc	$vc + ve$	N_d	vc	$vc + ve$
2	45/27	40/25	3^2	69/25	59/21
3	58/24	46/24	4^2	92/30	71/24
4	69/25	59/21	5^2	108/34	70/26
5	78/24	66/23	6^2	117/37	69/24
6	85/25	71/23	8^2	138/44	67/26
7	93/27	88/23	10^2	146/44	69/27
8	94/26	90/22	12^2	147/48	67/26

$P_2(h) - P_1(h)$, where both the velocity and pressure functions are continuous. Since our system (3.1) is indefinite, we apply the GMRES method for solving the preconditioned system. We report iteration counts required in the GMRES method to get the relative residual norm reduced by a factor of 10^6.

In Table 1, we report the results obtained from the two dimensional model problem. Here we selected the primal unknowns from only velocity unknowns and tested the algorithm for both systems with and without the static condensation. The performance in the case without the static condensation is not quite satisfactory for both sets of primal unknowns, the set vc consisting of velocity unknowns at corners and the set $vc + ve$ consisting of averages of velocity unknowns over edges in addition to the velocity unknowns at corners. In the case of the static condensation, we observe good performance with respect to the local problem size factor H/h as well as the number of subdomains N_d. Comparing the selection of primal unknowns, we observe that the additional primal unknowns result in better scalability for increasing the number of subdomains while there seems to be not much improvement related to the local problem size.

In Table 2, we present the performance of our method applied to the finite element space with discontinuous pressure functions. Here we used $P_1(h) - P_0(2h)$ finite element space. As we mentioned, we can not reduce the system (3.1) to the interface problem with both velocity and pressure unknowns, since the pressure functions can be discontinuous. We report the iteration count of the GMRES method applied to (3.1) with the preconditioner (3.2). Here we select velocity unknowns at corners as the primal unknowns. As proved in [10], the preconditioned system has all positive eigenvalues and its performance is similar to that obtained from the corresponding FETI-DP algorithm, which was shown to have the condition

TABLE 2

Iteration counts in our method for the two-dimensional Stokes problem with $P_1(h) - P_0(2h)$ and the set of primal unknowns consisting of velocity unknowns at corners. Iter: iteration count, N_d: number of subdomains, H/h: local problem size factor.

H/h	Iter ($N_d = 3^2$)	N_d	Iter ($H/h = 8$)
4	14	3^2	17
6	15	4^2	21
8	17	5^2	23
10	19	6^2	24
12	20	8^2	25
14	21	10^2	26
16	22	12^2	26

number bound $C(1+\log(H/h))(H/h)$. Comparing the results in Tables 1 and 2, we observe that our algorithm with the static condensation gives quite competitive results in the case of the continuous pressure function. We note that the factors $H/h = 4$ in Table 1 and $H/h = 8$ in Table 2 give the same number of velocity unknowns in each subdomain.

In the case of the continuous pressure function, we may include the pressure unknowns at subdomain corners as the primal unknowns. In Table 3 we tested our method for the two-dimensional Stokes problem by choosing velocity unknowns at subdomain corners, velocity averages over common edges, and pressure unknowns at subdomain corners as the primal unknowns. We observe good performance in both cases, increasing the local problem size factor H/h and the number of subdomains N_d. With the static condensation and with additional primal unknowns, the pressure unknowns at corners, we get the iteration counts which present better performance than the result in Table 2 with the discontinuous pressure functions.

In Figure 1, we plot eigenvalues of the preconditioned system in our method for various selection of primal unknowns and for the choice of pressure functions. Here we consider the two-dimensional Stokes problem. In the case of discontinuous pressure functions with the velocity unknowns at corners as the primal unknowns, the eigenvalues are all positive except one zero eigenvalue, which is due to the extension of the pressure space $\widehat{P}_0$ to $\widehat{P}$ by relaxing the average zero condition. This case is supported by the theory developed in our previous work [10].

For the case with continuous pressure functions, we also present plots of eigenvalues with different choices of primal unknowns in Figure 1. The results presented here is from the preconditioned system without the static condensation. In both cases that the primal unknowns consist of the velocity unknowns at corners only and consist of the velocity unknowns at corners and velocity averages over edges, we obtain complex eigenvalues and we observe that the real parts of all eigenvalues are positive. By adding velocity averages, we observe less

TABLE 3

Iteration counts in our method for the two-dimensional Stokes problem with the Taylor-Hood finite element space $P_2(h) - P_1(h)$ using the primal unknowns consisting of velocity unknowns at corners, averages of velocity unknowns over edges, and pressure unknowns at corners. Iter: iteration count, WOS: results without the static condensation, WS: results with the static condensation, N_d: number of subdomains, H/h: local problem size factor.

H/h	Iter (WOS/WS) $N_d = 3^2$	N_d	Iter (WOS/WS) $H/h = 4$
2	14/14	3^2	28/16
3	22/15	4^2	29/16
4	28/16	5^2	30/16
5	35/16	6^2	30/15
6	41/17	8^2	30/16
7	47/17	10^2	30/16
8	48/18	12^2	30/15

number of pure complex eigenvalues and better distribution of eigenvalues. When pressure unknowns at corners are included in the set of primal unknowns, we obtain all the eigenvalues positive except one zero eigenvalue; the similar property was proved for the preconditioned system with discontinuous pressure functions in [10, 7].

In Figure 2, we plot eigenvalues of the preconditioned system in two-dimensional Stokes problem for the cases without and with static condensation. We observe that the static condensation makes the distribution of eigenvalues more clustered near one and less clustered near zero, which results in faster convergence in the GMRES iteration. In the case of static condensation, we observe complex eigenvalues for all three sets of primal unknowns. In the algorithm with static condensation and with the largest set of primal unknowns, velocity unknowns at corners, averages of velocity unknowns over common edges, and pressure unknowns at corners, there is no nonzero eigenvalues clustered near zero and all nonzero eigenvalues are located in a region away from zero.

In Table 4, we present performance of our method for the three-dimensional Stokes problem with $Q_2(h) - Q_1(h)$ finite elements. We consider a model problem in the unit cubic domain $\Omega = [0,1]^3$ and we triangulate it with uniform cubes. Subdomain partition is aligned to the given triangulation and N^3 means the domain Ω is partitioned into $N \times N \times N$ uniform cubic subdomains. The algorithm is tested for various selection of primal unknowns. Results with the static condensation are also reported.

In the left columns, the performance regarding to the local problem size factor H/h is presented for both the cases without and with the static condensation. Here we fix the subdomain partition $N_d = 3^3$ and increase the local problem size. We observe good performance for the sets of primal unknowns, $vc + vf$ and $vc + vf + pc$, with the static condensation, where vc and pc denote velocity unknowns and pressure unknowns at corners, respectively,

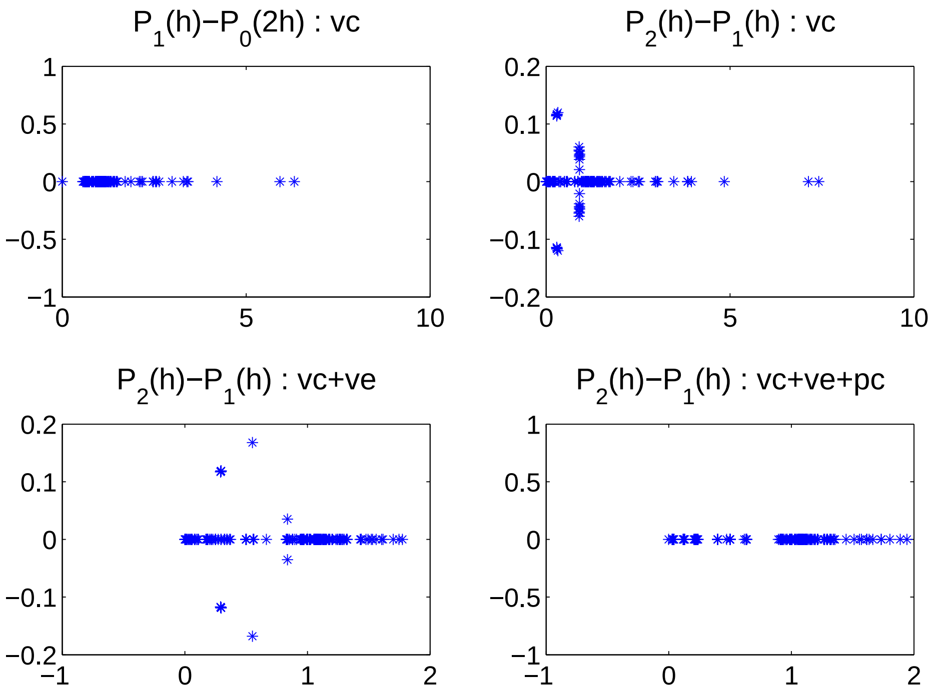

FIG. 1. *Plots of eigenvalues in the two-dimensional Stokes problem with* $P_1(h) - P_0(2h)$ $(H/h = 8,\ N_d = 3^2)$ *and with* $P_2(h) - P_1(h)$ $(H/h = 4,\ N_d = 3^3)$*:* vc *denotes the set of primal unknowns consisting of velocity unknowns at corners only,* $vc + ve$ *denotes the set of primal unknowns consisting of velocity unknowns at corners and averages of velocity unknowns over edges, and* $vc + ve + pc$ *denotes the set of primal unknowns consisting of velocity unknowns at corners, averages of velocity unknowns over edges, and pressure unknowns at corners.*

and vf denotes averages of velocity unknowns over common faces. The notation $vc + vf$ is used for the set of primal unknowns with vc and vf. We can see that the static condensation improves convergence a lot in the GMRES iteration. As observed already in [9, 7], more primal unknowns other than velocity unknowns at corners are required to achieve scalability in the three-dimensional problem.

In the right columns of Table 4, performance of our method is given for the number of subdomains N_d. Here we fix the number of elements in each subdomain with $H/h = 4$ and increase the number of subdomains. For the cases with $vc + vf$ and $vc + vf + pc$, we observe good scalability. The static condensation reduces the iteration count almost by half.

In the three-dimensional Stokes problem, the static condensation improves the scalability quite well while additional primal pressure unknowns at corners seem to give a little improvement in the scalability. For the three-dimensional Stokes problem, our method with the set of primal unknowns, velocity unknowns at corners and averages of velocity unknowns over faces, and with the static condensation, could provide quite good performance.

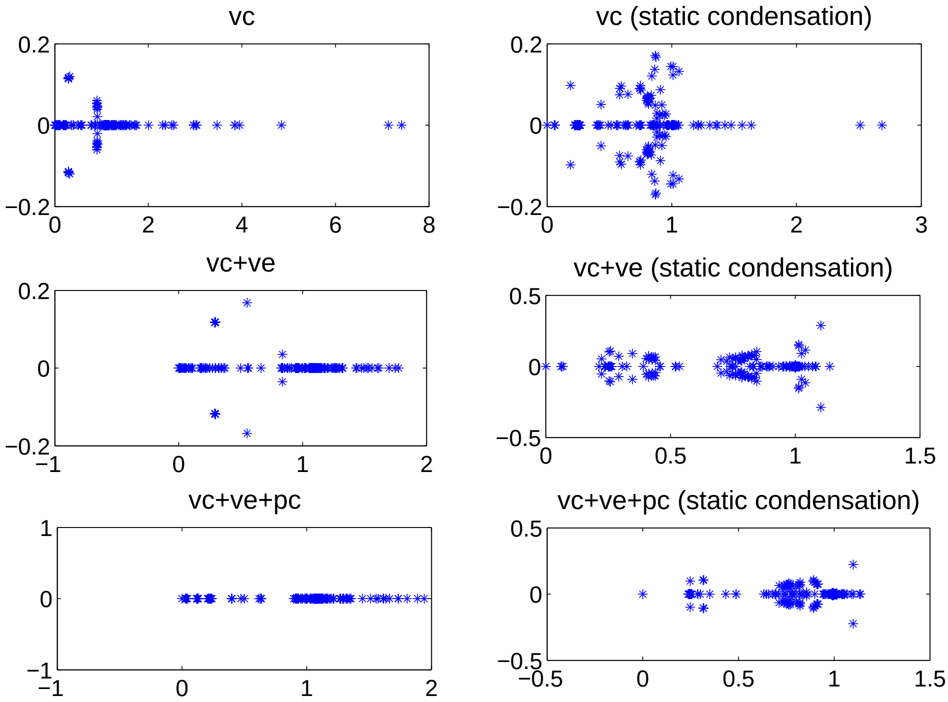

FIG. 2. *Plots of eigenvalues in the two-dimensional Stokes problem with* $P_2(h) - P_1(h)$ $(H/h = 4,\ N_d = 3^3)$*:* vc *denotes the set of primal unknowns consisting of velocity unknowns at corners,* $vc + ve$ *denotes the set of primal unknowns consisting of velocity unknowns at corners and averages of velocity unknowns over edges, and* $vc + ve + pc$ *denotes the set of primal unknowns consisting of velocity unknowns at corners, averages of velocity unknowns over edges, and pressure unknowns at corners.*

In Figure 3, we plot eigenvalues of the preconditioned system in our method for the three-dimensional Stokes problem. We compare the results from various selection of primal unknowns and for the cases with or without the static condensation. Differently to the two-dimensional case, we observe complex eigenvalues for all cases. When the static condensation is applied to the case with velocity unknowns at corners as the primal unknowns, it does not improve the distribution of eigenvalues, remaining many eigenvalues clustered near zero. For the other two cases with a larger set of primal unknowns including averages of velocity over faces or pressure unknowns at corners, we observe that the static condensation gives better distribution with more eigenvalues clustered near one and less eigenvalues near zero. Comparing the last two cases regarding to the sets of primal unknowns with and without pressure unknowns at corners, we can see that additional pressure unknowns seem to give a little improvement in achieving a better performance.

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TABLE 4

Iteration counts in our method applied to the three-dimensional Stokes problem with $Q_2(h) - Q_1(h)$ for various selection of primal unknowns: WOS denotes the method without the static condensation and WS denotes the method with the static condensation. Iter: iteration count, H/h: local problem size factor, and N_d: number of subdomains. vc denotes the set of primal unknowns consisting of velocity unknowns at corners, $vc + vf$ denotes the set of primal unknowns consisting of velocity unknowns at corners and averages of velocity over faces, and $vc + vf + pc$ denotes the set of primal unknowns consisting of velocity unknowns at corners, averages of velocity over faces, and pressure unknowns at corners.

	Iter (WOS/WS) $N_d = 3^3$				Iter (WOS/WS) $H/h = 4$		
H/h	vc	$vc + vf$	$vc + vf + pc$	N_d	vc	$vc + vf$	$vc + vf + pc$
2	16/73	56/55	40/35	3^3	79/75	70/55	60/40
4	98/76	77/51	73/43	6^3	203/147	79/51	68/41
6	134/73	120/53	117/44	9^3	301/205	93/52	87/44
8	149/77	171/55	167/47	12^3	288/223	93/52	87/43

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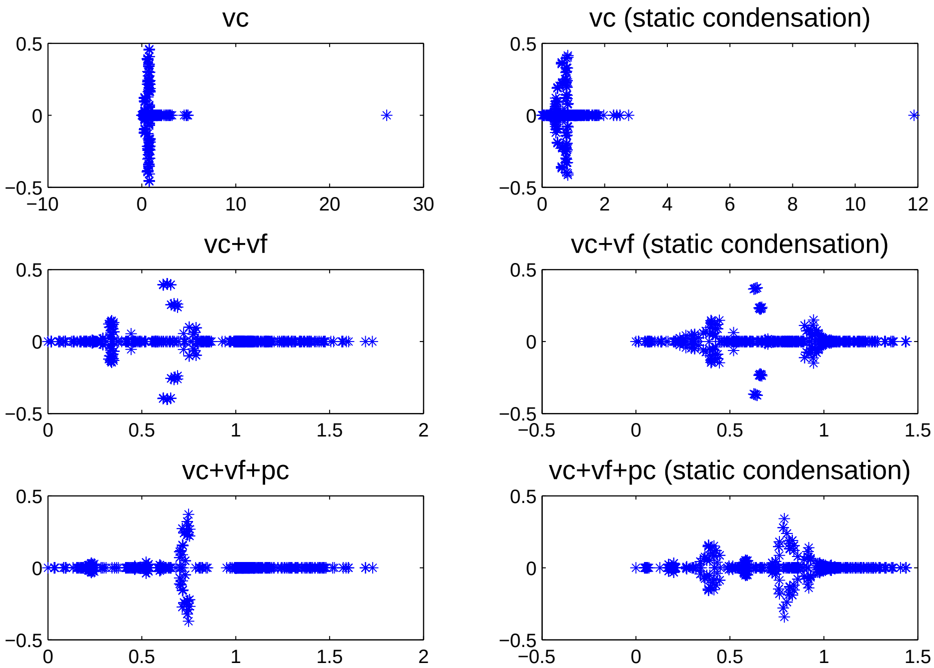

FIG. 3. *Plots of eigenvalues in the three-dimensional Stokes problem with $Q_2(h) - Q_1(h)$ ($H/h = 2$, $N_d = 3^3$): vc denotes the set of primal unknowns consisting of velocity unknowns at corners, $vc + vf$ denotes the set of primal unknowns consisting of velocity unknowns at corners and averages of velocity unknowns over faces, and $vc+vf+pc$ denotes the set of primal unknowns consisting of velocity unknowns at corners, averages of velocity unknowns over faces, and pressure unknowns at corners.*

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