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by

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On the classification of positive quaternionic Kähler manifolds with $b_4 = 1$

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Abstract

Let M be a positive quaternionic Kähler manifold of dimension $4m$. We already showed that if the symmetry rank is greater than or equal to $\left[\frac{m}{2}\right]+2$ and the fourth Betti number b_4 is equal to one, then M is isometric to $\mathbb{H}P^m$. The goal of this paper is to report that we can improve the lower bound of the symmetry rank by one for higher even dimensional positive quaternionic Kähler manifolds. Namely, it is shown in this paper that if the symmetry rank of M with $b_4(M) = 1$ is greater than or equal to $\frac{m}{2}+1$ for $m \geq 10$, then M is isometric to $\mathbb{H}P^m$. One of the main strategies of this paper is to apply a more delicate argument of Frankel type to positive quaternionic Kähler manifolds with certain symmetry rank.

1 Introduction and Main Results

A compact quaternionic Kähler manifold M is a Riemannian manifold of real dimension $4m$ whose holonomy group is contained in the Lie group $\mathrm{Sp}(m)\mathrm{Sp}(1) = \mathrm{Sp}(m) \times \mathrm{Sp}(1)/\mathbf{Z}_2$ in $\mathrm{SO}(4m)$ for $m \geq 2$. Such a manifold is called *positive* if it has the positive scalar curvature. It is known that every quaternionic Kähler manifold is simply connected and Einstein. Moreover, it is easy to see from a well-known theorem of Meyers that every positive quaternionic Kähler manifold is compact. It is common to define a 4-dimensional quaternionic Kähler manifold to be both Einstein with non-zero scalar curvature and self-dual. While many complete, non-compact, non-symmetric quaternionic Kähler manifolds with negative scalar curvature are known to exist, so far the only known examples of positive (compact) quaternionic Kähler manifolds are the so-called

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Wolf spaces: Wolf proved that for each compact simple Lie group G, there is a symmetric space G/H. For examples, the classical ones are

$$\begin{aligned} \mathbb{H}P^m &= \frac{\mathrm{Sp}(m+1)}{\mathrm{Sp}(m)\times \mathrm{Sp}(1)}, \\ Gr_2(\mathbb{C}^{m+2}) &= \frac{\mathrm{SU}(m+2)}{S(\mathrm{U}(m)\times \mathrm{U}(2))}, \\ \widetilde{Gr_4(\mathbb{R}^{m+4})} &= \frac{\mathrm{SO}(m+4)}{S(\mathrm{O}(m)\times \mathrm{O}(4))}, \end{aligned}$$

and there are five more exceptional cases. Moreover, a theorem of Alekseevsky asserts that there are no other compact homogeneous positive quaternionic Kähler manifolds (e.g., see [1]).

Quite recently, in the paper [10] R. Kobayashi and K. Onda announced a proof of the LeBrun-Salamon conjecture saying that every irreducible positive quaternioinic Kähler manifold is isometric to one of the Wolf spaces. Their proof crucially uses the Ricci flow technique developed by G. Perelman, S. Bando and W. X. Shi (see [12], [2] and [15]).

Every positive quaternionic Kähler manifold admits a twistor space which is a complex contact Fano manifold. It is also true that a contact Fano manifold is a twistor space of a positive quaternionic Kähler manifold if and only if it admits a Kähler-Einstein metric. However, it is generally believed that every contact Fano manifold admits a Kähler-Einstein manifold. In view of the recent result of Kobayashi-Onda and Wolf, it seems to be reasonable to conjecture that every contact Fano manifold with the second Betti number $b_2 = 1$ is indeed a symmetric space.

In this paper, we take a quite different approach to classify positive quaternionic Kähler manifolds, which is likely to contain much weaker, but more geometric and so meaningful, results than those of Kobayashi and Onda. In other words, we exclusively consider positive quaternionic Kähler manifolds with an isometry group of certain rank. For instance, it is known in [14] that every 16-dimensional positive quaternionic Kähler manifold admits an isometry group of dimension 8. Hence such a manifold admits an isometric torus action of rank at least 2.

In the papers [4] and [5], Fang showed that for m even integer every positive quaternionic Kähler $4m$-manifold with sym-rank$(M) \geq \frac{m}{2}+3$ is isometric to $\mathbb{H}P^m$ or $Gr_2(\mathbb{C}^{m+2})$. (Here the symmetry rank sym-rank (M, g) (or simply sym-rank (M)) of a Riemannian manifold with a Riemannian metric g is defined as the rank of the isometry group Isom (M, g). Equivalently, it can be defined as the largest number r such that a r-dimensional torus acts effectively and isometrically on M). Moreover, the first named author showed in [8] that every positive quaternionic Kähler $4m$-manifold with sym-rank$(M) \geq [\frac{m}{2}]+3$ is isometric to $\mathbb{H}P^m$ or $Gr_2(\mathbb{C}^{m+2})$, regardless of the parity of the integer m. One of the crucial arguments in the proof is an extended version of connectedness theorem in [4] in the presence of the group action. Recently, in the paper [9] we were able to improve the lower bound of the paper [8] further by one under the assumption of the fourth Betti number $b_4 = 1$ by applying more delicate arguments. On the other hands, the classification of low dimensional positive quaternionic Kähler manifolds has already been done by N. Hitchin, Poon and Salamon, Herrera and Herrera (see [7], [13] and [6] for more details).

The aim of this paper is to give a classification result of positive quaternionic Kähler manifolds satisfying both the fourth Betti number $b_4(M) = 1$ and sym-rank$(M) \geq \frac{m}{2} + 1$. Our first main result is

Theorem 1.1. *Let M be a positive quaternionic Kähler manifold of dimension $4m$ with an isometric T^k-action. If k is greater than or equal to 6, then the dimension of any fixed point component of the S^1-action induced from the T^k-action is greater than or equal to $4k$.*

We will prove Theorem 1.1 by contradiction, in Section 2. Namely, otherwise we would find two fixed point components under the induced T^{k-3}-action. Then let γ denote a minimizing geodesic joining between two fixed point components, and use the second variation of γ. Since every positive quaternionic Kähler manifold has a positive scalar curvature of M, we can derive a contradiction to Proposition 1.2 in [4]. (See Section 2 for more details.)

Our second main result is

Theorem 1.2. *Let M be a positive quaternionic Kähler manifold of dimension $4m$ with $b_4(M) = 1$. If the symmetry rank satisfies* sym-rank$(M) \geq \frac{m}{2} + 1$ *for $m \geq 10$, then M is isometric to $\mathbb{H}P^m$.*

This theorem is actually a corollary of Theorem 1.1. We will prove it by using a mathematical induction starting from the case $m = 10$. The restriction on the dimension in Theorem 1.2 might be improved or removed, but to do so we probably need new insights or techniques. So we hope to return this issue in a future paper. Along the proof of Theorem 1.2. the following rigidity theorem of LeBrun and Salamon will be crucially used.

Theorem 1.3 ([11])**.** *The second homotopy group $\pi_2(M)$ of a positive quaternionic Kähler manifold M is a finite group with 2-torsion or $\mathbb{Z}$, and M is isometric to $\mathbb{H}P^n$ or $Gr_2(\mathbb{C}^{n+2})$, according to $\pi_2(M) = 0$ or $\mathbb{Z}$.*

We also need the following connectedness theorem whose statement without the presence of a group action is due to Fang in [4]:

Theorem 1.4 ([8])**.** *Let M be a positive quaternionic Kähler manifold of dimension $4m$. If N is a quaternionic Kähler submanifold of dimension $4n$, then the inclusion $N \hookrightarrow M$ is $(2n - m + 1)$-connected. Furthermore, if there is a Lie group G acting isometrically on M and fixing N pointwise, then the inclusion map is $(2n - m + 1 + \delta(G))$-connected, where $\delta(G)$ is the dimension of the principal orbit of G.*

We organize this paper as follows. In Section 2, we give a proof of Theorem 1.1 which is one of the key ingredients to prove Theorem 1.2. In order to prove Theorem 1.2, we use the mathematical induction on the dimension of a positive quaternionic Kähler manifold with certain symmetry rank. To do so, we need to deal with a positive quaternionic Kähler manifold of dimension 40 with an isometric T^6-action, and Section 3 is devoted to the classification of such manifolds. Finally, in Section 4 we give a proof of our main Theorem 1.2 which will be based on the classification of a positive quaternionic Kähler manifold of dimension 40 with an isometric T^6-action established in Section 3.

2 Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1 which will play a crucial role in the proof of Theorem 1.2. To do so, as before let M be a positive quaternionic Kähler manifold of dimension $4m$. Assume that M admits an isometric T^k-action. Let V denote a fixed point component of the S^1-action induced from the T^k-action.

We begin with this section by setting up some notations for later use. When T^i acts on M, we denote by $\mathrm{Fix}(T^i, M)$ the fixed point set of T^i-action. One of the connected components of $\mathrm{Fix}(T^i, M)$ will be often denoted by $\mathrm{Fix}(T^i, M)_0$. We shall also denote by N_i one of the connected components of $\mathrm{Fix}(T^i, M)$, and d_i will always mean its dimension. With these notations in place, note that we can choose N_{i+1} in $\mathrm{Fix}(S^1, N_i) \subset \mathrm{Fix}(T^{i+1}, M)$. Thus one can consider the following chain at x:

$$x \in N_k \subset N_{k-1} \subset \cdots \subset N_1 \subset M.$$

Therefore, from now on we will always assume without further mentioning that there is the relation $N_i \subset N_{i-1} \subset M$ for $2 \leq i \leq k$.

As mentioned in Section 1, we prove Theorem 1.1 by contradiction. Hence suppose that V has the dimension less than or equal to $4k-4$. Note that, since V admits T^{k-1}-action by its construction, it follows from Theorem B in [4] that the dimension of V should be greater than or equal to $4k-8$. Then we claim the following lemma.

Lemma 2.1. *There exist two fixed-point components contained in V under the T^{k-3}-action induced from the T^{k-1}-action on V ($k \geq 6$).*

Proof. We divide the proof into two cases:

Case 1: Assume first that the dimension of V is equal to $4k-4$.

For the sake of simplicity, let $N_0 = V$. Then, for each $0 \leq i \leq k-2$, we can define a fixed-point component N_i inductively in such a way that $N_{i+1} = \mathrm{Fix}(S^1, N_i)_0$. So clearly we have the following inclusions:

$$V = N_0 \supset N_1 \supset \cdots \supset N_{k-1}.$$

In particular, note that N_{k-3} admits an isometric T^2-action. Thus we have $\dim(N_{k-3}) \geq 4$ by Theorem B in [4]. Moreover, we have $\dim(N_{k-3}) \leq \dim(N_0) - 4(k-3) = (4k-4) - 4(k-3) = 8$ in this case.

If $\dim(N_{k-3}) = 8$ then $\dim(N_{k-4}) = 12$ by the effectiveness of the isometric action. Then it follows from Theorem 1.2 in [5] that N_{k-4} should be isometric to $\mathbb{H}P^3$ or $Gr_2(\mathbb{C}^5)$.

Now note that $\dim(N_i) - \dim(N_{i+1}) = 4$ for $0 \leq i \leq k-5$. Let $d_i = \dim N_i$. Then it is easy to see that

$$2 + d_{i+1}/2 - d_i/4 = 2 + d_{i+1}/2 - (d_{i+1} + 4)/4 = 1 + d_{i+1}/4 \geq 3,$$

since $d_{i+1} \geq 12$. Thus we have

$$\pi_2(V) = \pi_2(N_1) = \ldots = \pi_2(N_{k-4}).$$

This implies that V is actually isometric to $\mathbb{H}P^{k-1}$ or $Gr_2(\mathbb{C}^{k+1})$.

Next assume that $\dim(N_{k-3}) = 4$. If $\dim(N_1) = 4k-8$, then by Theorem 1.2 in [5], V is isometric to $\mathbb{H}P^{k-1}$ or $Gr_2(\mathbb{C}^{k+1})$.

If $\dim(N_1) = 4k-12$, N_1 admits the maximal symmetry rank. Note that $\dim(N_i) = 4k-8-4i$ for $1 \leq i \leq k-3$. Hence $\dim(N_1) - \dim(N_2) = 4$. Again by Theorem 1.2 in [5] and $4k-12 \geq 12$ $(k \geq 6)$, N_1 is isometric to $\mathbb{H}P^{k-3}$ or $Gr_2(\mathbb{C}^{k-1})$. Since

$$1/2(4k-12) - 1/4(4k-4) + 2 \geq k-3 \geq 3$$

(i.e., $N_1 \to V$ is at least 3-connected), V is isometric to $\mathbb{H}P^{k-1}$ or $Gr_2(\mathbb{C}^{k+1})$.

Now we are ready to consider the following subcases in more detail:

Subcase 1.1: V is isometric to $\mathbb{H}P^{k-1}$.

Note that the Euler characteristic $\chi(V) = k$ of V is greater than or equal to 6, since $k \geq 6$. If $\dim(N_{k-3}) = 8$, N_{k-3} is isometric to $\mathbb{H}P^2$. Thus $\chi(N_{k-3}) = 3$.

Similarly, for the case of $\dim(N_{k-3}) = 4$, by Hitchin in [7], N_{k-3} is S^4 or $\mathbb{C}P^2$. Thus $\chi(N_{k-3}) \leq 3$. Notice that $\dim(\mathrm{Fix}(T^2, N_{k-3}))$ is less than or equal to 4, due to the conditions of $\dim(N_{k-3}) \leq 8$ and the existence of an isometric T^2-action on N_{k-3}.

For $i = 1, 2$, let W_i denote the fixed point components of dimension 4 under the T^{k-1}-action on V. Since the Euler characteristic $\chi(V)$ of V is greater than or equal to 6, we can consider the following three possibilities: either $\mathrm{Fix}(T^{k-1}, V)$ contains six isolated fixed points $x_1, \ldots, x_6$ or three isolated fixed points x_1, x_2, x_3 and W_1 or the union $W_1 \cup W_2$.

Assume first that $\mathrm{Fix}(T^{k-1}, V)$ contains six isolated fixed points $x_1, \ldots, x_6$. Let

$$T^{k-1} = S^1_1 \times S^1_2 \times \cdots \times S^1_{k-1},$$

where S^1_j $(1 \leq j \leq k-1)$ is the unit circle group. Then, by considering the isotropy representation at x_1 of the T^{k-1}-action, we can choose K_1 to be $\mathrm{Fix}(T^{k-1}/(S^1_1 \times S^1_2), V)_0$ containing the isolated fixed point x_1. Since $\chi(K_1) \leq 3$, we may assume without loss of generality that $\mathrm{Fix}(S^1_1 \times S^1_2, K_1) \backslash \{x_1, x_2, x_3\} = \emptyset$. Note that there exists no fixed point component of dimension 2.

Similarly, by considering the isotropy representation of the T^{k-1}-action at x_4, we can choose another fixed point component K_2 to be $\mathrm{Fix}(T^{k-1}/(S^1_1 \times$

$S_2^1), V)_0$ containing the isolated fixed point x_4. We then claim that $K_1 \cap K_2$ is empty. Indeed, otherwise then K_1 should coincide with K_2. But then K_1 would contain x_4, which is clearly a contradiction. Similar arguments apply to other two cases. Therefore we have completed the proof of Lemma 2.1 in this Subcase 1.1.

Subcase 1.2: V is isometric to $Gr_2(\mathbb{C}^{k+1})$.

Let $m = k - 1$. It is well-known that

$$\chi(Gr_2(\mathbb{C}^{m+2})) = (m+2)(m+1)/2. \tag{2.1}$$

If $\dim(N_{k-3}) = 8$, then $\chi(N_{k-3}) \leq 6$. On the other hands, if $\dim(N_{k-3}) = 4$, then by Hitchin ([7]), $N_{k-3} = \mathbb{C}P^2$ or S^4 and so $\chi(N_{k-3}) \leq 3$. Thus we always have $\chi(N_{k-3}) \leq 6$. Since $k \geq 6$, $\chi(Gr_2(\mathbb{C}^{k+1})) = (k+1)k/2 \geq 21$.

Without a loss of generality, we let $K_1 = N_{k-3} = \mathrm{Fix}(T^{k-1}/(S_1^1 \times S_2^1), V)_0$. Since $\chi(K_1) \leq 6$ and $\chi(V) \geq 21$, there exists an $x_4 \in \mathrm{Fix}(T^{k-1}, V \setminus K_1)$. Consider the fixed point component $K_2 = \mathrm{Fix}(T^{k-1}/(S_1^1 \times S_2^1), V)_0$ containing x_4. Then we have $K_1 \cap K_2 = \emptyset$.

Case 2: Assume next that the dimension of V is equal to $4k - 8$ (≥ 16).

Observe that the fixed point set for the T^{k-1}-action is isolated. We can assume that

$$T^{k-1} = S_0^1 \times ... \times S_{k-2}^1,$$

where S_l^1 $(0 \leq l \leq k-2)$ is the unit circle group.

At $x \in \mathrm{Fix}(T^{k-1}, V)$, we can assume that $T^{k-2} = T^{k-1}/S_0^1$ acts left quaternionically linearly on $T_x(V)$.

Let $N_l = \mathrm{Fix}(S_1^1 \times ... \times S_l^1, V)_0$. Then $\dim(N_l) = 4k - 8 - 4l$. Assume that $\mathrm{Fix}(T^{k-1}, V)$ consists of x_1, x_2, x_3, ... Let $K_1 = \mathrm{Fix}(S_1^1 \times ... \times S_{k-3}^1, V)_0 \ni x_1$. Since $\chi(K_1) \leq 3$, $\mathrm{Fix}(S_0^1 \times S_{k-2}^1, K_1) \setminus \{x_1, x_2, x_3\} = \emptyset$. Let $K_2 = \mathrm{Fix}(S_1^1 \times ... \times S_{k-3}^1, V)_0 \ni x_4$. Then $K_1 \cap K_2 = \emptyset$.

This completes the proof of Lemma 2.1. □

Now we are ready to prove Theorem 1.1. By Lemma 2.1, we can take two disjoint totally geodesic submanifolds K_1 and K_2 in $\mathrm{Fix}(T^{k-3}, V)$ whose dimension satisfies $4 \leq \dim K_i \leq 8$ for $i = 1, 2$. Let $L = K_1 \times K_2$, $f_i : K_i \to V$ the quaternionic immersions for each $i = 1, 2$, and $f = (f_1, f_2)$. Let s be the distance between K_1 and K_2 and let $\gamma : [0, s] \to V$ be a minimizing geodesic from K_1 to K_2.

Let $\mathcal{V}$ be the real linear vector space spanned by vectors $(W(0), W(1))$, where W is a parallel vector field along γ so that W is orthogonal to $\dot\gamma$, $I\dot\gamma$, $J\dot\gamma$, and $K\dot\gamma$. Then the quaternionic dimension of $\mathcal{V}$ is equal to $\dim_{\mathbb{H}}(V) - 1$. Notice that IW as well as JW and KW is orthogonal to all of $\dot\gamma$, $I\dot\gamma$, $J\dot\gamma$, and $K\dot\gamma$. This implies that $(IW(0), IW(1))$, $(JW(0), JW(1))$, and $(KW(0), KW(1))$ are actually in $\mathcal{V}$.

Lemma 2.2.

$$\dim_{\mathbb{H}}(\mathcal{V} \cap f_*(T_{(\gamma(0)=x, \gamma(s)=y)} L)) \geq 1.$$

Proof. For the sake of simplicity, let $G = T^{k-3}$.

We first consider the case that $\dim V = 4k-4$ and $\dim K_1 = \dim K_2 = 4$. Suppose on the contrary that $\dim_{\mathbb{H}}(\mathcal{V} \cap f_*(T_{(x,y)}L)) = 0$. Let v_1 be a unit vector in the tangent space of $(K_1)_{\gamma(0)}$, P a parallel transport from $\gamma(0)$ to $\gamma(s)$ along γ, and v the parallel vector field along γ such that $v(0) = v_1$. Then, by assumption, $v(s) = P(v_1)$ is in the normal space $((K_2)_{\gamma(s)})^{\perp}$ of the tangent space $(K_2)_{\gamma(s)}$.

Since G acts on $((K_2)_{\gamma(s)})^{\perp} \setminus \{0\}$ fixed-point freely, there must exist $g \neq 1 \in G$ such that $dg(P(v_1)) \neq P(v_1)$, so that $\langle dg(P(v_1)), P(v_1)\rangle <\| P(v_1) \|^2$. Since P is an isometry and $v(0)$ is a tangent vector of K_1 fixed by the group action of G, we have

$$\begin{aligned} 1 =\| v_1 \|^2_{\gamma(0)} &= \langle dg(v_1), v_1\rangle_{\gamma(0)} = \langle P(dg(v_1)), P(v_1)\rangle_{\gamma(s)} \\ &= \langle dg(P(v_1)), P(v_1)\rangle_{\gamma(s)} <\| P(v_1) \|^2 = 1, \end{aligned}$$

which is a contradiction. Hence $\mathcal{V}$ has at least quaternionic dimension 1.

Next we deal with the case that $\dim K_1 = 8$ and $\dim K_2 = 4$. Again suppose that $\dim_{\mathbb{H}}(\mathcal{V} \cap f_*(T_{(x,y)}L)) = 0$. Then notice that any vector in $(K_1)_{\gamma(0)}$ is transported to $((K_2)_{\gamma(s)})^{\perp}$ and that any vector in $(K_1)_{\gamma(0)}$ is transported to $\mathrm{Fix}(G, ((K_2)_{\gamma(s)})^{\perp})$. Thus $V_{\gamma(s)}$ contains

$$P((K_1)_{\gamma(0)}) \oplus ((K_2)_{\gamma(s)})^{\perp} \setminus \mathrm{Fix}(G, ((K_2)_{\gamma(s)})^{\perp}) \oplus (K_2)_{\gamma(s)}.$$

Notice that G acts on $((K_2)_{\gamma(s)})^{\perp} \setminus \mathrm{Fix}(G, ((K_2)_{\gamma(s)})^{\perp})$ nontrivially. Since

$$\begin{aligned} &\dim_{\mathbb{H}}(P((K_1)_{\gamma(0)}) + \dim_{\mathbb{H}}(((K_2)_{\gamma(s)})^{\perp} \setminus \mathrm{Fix}(G, ((K_2)_{\gamma(s)})^{\perp}))+ \\ &\dim_{\mathbb{H}}((K_2)_{\gamma(s)}) \geq \dim_{\mathbb{H}} K_1 + \dim G + \dim_{\mathbb{H}} K_2 = k, \end{aligned} \tag{2.2}$$

and $\dim_{\mathbb{H}} V = k-1$, this is a contradiction. That is, $\mathcal{V}$ has at least quaternionic dimension 1.
If $\dim V = 4k-4$ and $\dim K_1 = \dim K_2 = 8$, we will prove by similar process in the above case (cf. (2.2)).

$$\dim V_{\gamma(s)} \geq \dim_{\mathbb{H}} K_1 + \dim G + \dim_{\mathbb{H}} K_2 = 2 + (k-3) + 2 = k+1$$

It is a contradiction. That is, $\mathcal{V}$ has at least quaternionic dimension 1.
Finally, if $\dim V = 4k-8$, $\dim K_1 = 4$, and $\dim K_2 \geq 4$, then this time we have

$$\dim G + \dim_{\mathbb{H}} K_1 + \dim_{\mathbb{H}} K_2 \geq k-1.$$

It is again a contradiction since $\dim_{\mathbb{H}} V = k-2$. Thus $\mathcal{V}$ has at least quaternionic dimension 1.

This completes the proof of Lemma 2.2. □

By Lemma 2.2, there exists a parallel field X along γ. Since K_1 and K_2 are positive quaternionic Kähler manifolds by a theorem of Berger in [3], we have

$$\begin{aligned}
&L''_X(0) + L''_{IX}(0) + L''_{JX}(0) + L''_{KX}(0) = -\int_0^s \langle R(T,X)T, X\rangle \\
&+ \langle R(T,IX)T, IX\rangle + \langle R(T,JX)T, JX\rangle + \langle R(T,KX)T, KX\rangle dt \\
&= -\int_0^s \frac{\lambda}{n+2} \parallel T \parallel^2 \parallel X \parallel^2 dt < 0,
\end{aligned} \tag{2.3}$$

where $\lambda > 0$ is Einstein constant and $T = \gamma'(t)$. Since γ is minimizing, it is a contradiction. This proves Theorem 1.1.

3 Positive quaternionic Kähler manifolds of dimension 40

The aim of this section is to classify positive quaternionic Kähler manifolds of dimension 40 which admits an isometric T^6-action. We need to consider this case in detail in order to apply the mathematical induction.

Throughout this section, let M be a positive quaternionic Kähler manifold of dimension 40 with an isometric T^6-action. By Theorem 0.3 in [11], all odd Betti numbers are equal to zero. Thus $\chi(M) > 0$, so there exist $x \in \mathrm{Fix}(T^6, M)_0$.

Consider the following chain at x ($\in N_6$):

$$M \supset N_1 \supset \cdots \supset N_5 \supset N_6,$$

and let d_i denote the dimension of N_i for $1 \le i \le 6$, as before. By the representation $T^6 \to \mathrm{Sp}(n)_x\mathrm{Sp}(1)_x$, there exists subgroup T^5 such that T^5 acts left quaternionically linearly (cf. 7 page in [4]). Thus without loss of generality we can assume that N_1 admits T^5-action acting left quaternionically linearly at x.

Then we have the following lemma.

Lemma 3.1. *The following three cases of* 6*-tuples* $(d_1, d_2, d_3, d_4, d_5, d_6)$ *do not occur:*

$$(32, 24, 16, 8, 4, 0),\ (32, 24, 12, 8, 4, 0),\ (32, 16, 12, 8, 4, 0).$$

Proof. Suppose not. Then, the fixed point set should be isolated. In this proof, we will use the Euler characteristic of N_3.

Case 1: Assume that $(d_1, d_2, d_3, d_4, d_5, d_6) = (32, 24, 16, 8, 4, 0)$. Then we claim that $\chi(N_3) \ge 6$. If $b_2(N_3) = 1$, N_3 is isometric to $Gr_2(\mathbb{C}^6)$ and thus $\chi(N_3) \ge 6$.

Recall that by a theorem of Salamon in [14] we have the following formula:

$$3b_4 = b_6 + 2b_8 + 1, \tag{3.1}$$

where $b_2(N_3)$ =0. From (3.1), $b_4(N_3) \ge 1$. If $b_6(N_3) = b_8(N_3) = 0$, it is a contradiction from (3.1). Thus $b_6(N_3) + b_8(N_3) \ge 1$. Thus $\chi(N_3) = 2 + 2(b_4(N_3) + b_6(N_3) + b_8(N_3)) \ge$6. This prove the claim.

Set $T^6 = S^1_1 \times \ldots \times S^1_6$ and let $N_3 = \mathrm{Fix}(S^1_4 \times S^1_5 \times S^1_6, M)_0$. Then, by the assumption, $\mathrm{Fix}(S^1_1 \times S^1_2 \times S^1_3, N_3)$ is isolated. So we may assume that

$$\mathrm{Fix}(S^1_1 \times S^1_2 \times S^1_3, N_3) \supseteq \{x_1, ..., x_4\}.$$

Now, let C_1 be $\mathrm{Fix}(T^6/S^1_1, M)_0$ containing x_1. Since $\dim C_1 = 4$, we have $\chi(C_1) \le 3$ by [7]. So we may assume that $\mathrm{Fix}(S^1_1, C_1) \setminus \{x_1, x_2, x_3\} = \emptyset$. Let C_4 be $\mathrm{Fix}(S^1_2 \times S^1_3, N_3)_0$ containing x_4. Note that by the definition of a fixed point component, $C_1 \cap C_4 = \emptyset$. Note also that by construction $\mathrm{Fix}(S^1_2 \times S^1_3, N_3) \supset C_1 \cup C_4$. Recall that $\dim C_4 = 4$ by assumption.

Let γ be a minimizing geodesic from C_1 to C_4 such that $\gamma(0) = p \in C_1$ and $\gamma(l) = q \in C_4$ where l is the distance from C_1 to C_4. Note that $S^1_2 \times S^1_3$ acts on $(C_4)^{\perp}_q \setminus \{0\}$ fixed-point freely.

Let P denote the parallel transport from p to q along γ. Then we claim that for some $v \neq 0 \in (C_1)_p$, $P(v) \in (C_4)_q$. To see it, suppose on the contrary that for any $v \neq 0 \in (C_1)_p$ such that $\| v \| = 1$, $P(v) \in (C_4)^{\perp}_q$. Clearly there exists $g \in S^1_2 \times S^1_3$ such that $g \cdot P(v) \neq P(v)$. Then apply an argument as in the previous section. That is, we have

$$\begin{aligned} 1 = \| v \|^2 &= \langle g \cdot v, v \rangle = \langle P(g \cdot v), P(v) \rangle \\ &= \langle g \cdot P(v), P(v) \rangle < \| P(v) \|^2 = 1. \end{aligned}$$

It is a contradiction. This proves the claim. Let X be a parallel vector field along γ such that $X(0) = v \in (C_1)_p$ and $X(l) \in (C_4)_q$.

Finally, we have

$$L''_X(0) + L''_{IX}(0) + L''_{JX}(0) + L''_{KX}(0) < 0.$$

(cf. (2.3) in Section 2) Since γ is minimizing, it is a contradiction.

Case 2: Assume that $(d_1, d_2, d_3, d_4, d_5, d_6) = (32, 24, 12, 8, 4, 0)$ or $(32, 16, 12, 8, 4, 0)$. Since $d_3 - d_4 = 4$, by Theorem 1.2 in [5], N_3 is isometric to $\mathbb{H}P^3$ or $Gr_2(\mathbb{C}^5)$. Thus $\chi(N_3) \geq 4$. The rest of the proof is completely similar to Case 1. So we leave it to the reader. This completes the proof of Lemma 3.1. □

Recall that an isometric G-action on a quaternionic Kähler manifold M is called *of quaternionic type* if $\overline{\rho}_x : G \to \mathrm{Sp}(1)_x$ is trivial for any G-fixed point point x.

Theorem 3.2. *Let M be a positive quaternionic Kähler manifold of dimension 40 with an isometric T^6-action. If $b_4(M)$ is equal to 1, then M is isometric to $\mathbb{H}P^{10}$.*

Proof. $\chi(M) > 0$. Thus $\mathrm{Fix}(T^6, M) \ni x$. Since M admits an isometric T^6-action, we may assume that at least T^5 acts left quaternionically linearly on $T_x(M)$. Especially, we can assume that T^5 acts left quaternionically linearly on $T_x(N_1)$ where $N_1 = \mathrm{Fix}(S^1, M)_0$. If $N_i = \mathrm{Fix}(T^i, M)_0$, Consider a chain at x :

$$N_1 \supseteq N_2 \supseteq N_3 \ni x.$$

Notice that by Theorem 1.1 d_1 should be greater than or equal to 24. Now we consider all possible dimensions which the submanifold N_1 can have.

If $d_1 = 36$, by Theorem 1.2 in [4], M is isometric to $\mathbb{H}P^{10}$ or $Gr_2(\mathbb{C}^{12})$. Since $b_4(Gr_2(\mathbb{C}^{12})) > 1$, M is isometric to $\mathbb{H}P^{10}$. On the other hands, Consider the case $(d_1, d_2) = (32, 20)$. Since N_3 admits an isometric T^3 acting quaternionically linearly on $T_x(N_3)$, $d_3 \geq 12$. Thus for $(d_1, d_2, d_3, d_4) = (32, 20, 12, 8)$ (*resp.* $(28, 20, 16, d_4)$), it follows from Theorem 1.2 in [4] that N_3 (*resp.* N_2) is isometric to quaternionic Kähler space or complex Grassmannian space, since $d_3 - d_4 = 4$ (*resp.* $d_2 - d_3 = 4$). By Theorem 1.4, we have $\pi_2(N_3) = \pi_2(M)$ (*resp.* $\pi_2(N_2) = \pi_2(M)$). Thus M is isometric to $\mathbb{H}P^{10}$, since $b_4(M) = 1$.

The proof of other remaining cases is completely similar. So we leave it to the reader. This completes the proof of Theorem 3.2. □

Finally we close this section with a remark.

Remark 3.3. Let M be a positive quaternionic Kähler manifold manifold of dimension $4m + 4$ with an isometric T^m. If $4m \geq 24$, then $m \geq [(m+1)/2] + 3$. Thus it follows from [8] that M is isometric to $\mathbb{H}P^{m+1}$ or $Gr_2(\mathbb{C}^{m+3})$.

4 Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2. Essentially the proof is a direct corollary of Theorem 1.1. We use the mathematical induction on the dimension of a positive quaternionic Kähler manifold.

To do so, notice that the case of odd m is already dealt with by the main result of the paper [9]. Thus it suffices to consider the case that m is even. In the following proof, we continue to use the notations in Section 2. In particular, recall that N_1 is a fixed point component in M of the induced S^1-action. The proof goes in two steps as follows.

Step 1: Assume first that $m = 10$. This case has already been proved in Theorem 3.2.

Step 2: Next assume that the theorem holds for all dimensions less than or equal to $8s_0$ ($s_0 \geq 5$). For simplicity, let $l = 2s_0 + 2$. Let M be a positive quaternionic Kähler $4l$-manifold with $b_4(M) = 1$ and an isometric $T^{\frac{l}{2}+1}$-action. We then want to show that M is actually isometric to $\mathbb{H}P^l$.

Indeed, if $\dim N_1 = 4(l-1)$, N_1 has codimension 4 in M. Since $l \geq 12$, M is isometric to $\mathbb{H}P^l$ or $Gr_2(\mathbb{C}^{l+2})$ (cf. Theorem 1.2 in [5]). If $p = s_0 + 3$ and $4p \leq \dim(N_1) \leq 4(l-2)$, it follows from the induction hypothesis that N_1 is isometric to a quaternionic projective space, since $b_4(M) = b_4(N_1)$ and sym-rank$(N_1) \geq l/2$.

From $\pi_2(M) = \pi_2(N_1) = 0$ (cf. Theorem 1.4), by the rigidity theorem of LeBrun and Salamon in [11], M is isometric to $\mathbb{H}P^l$.

Now it remains to consider the case that $\dim N_1 \leq 4(s_0 + 2)$. To do so, first note that $\dim N_1 \geq 4(l/2 - 1) = 4s_0$ (cf. Theorem B in [4]). Hence we need to consider the following two subcases:

(1) $4s_0 \leq \dim N_1 \leq 4s_0 + 4$.

(2) $\dim N_1 = 4s_0 + 8$.

However, for the case (1), it is easy to see that the case that N_1 has dimension $4s_0$ or $4s_0+4$ with symmetry rank $s_0+1 \geq 5$ does not occur, since by Theorem 1.1 the dimension of N_1 should be greater than or equal to

$$4\left(\frac{l}{2}+1\right) = 2l + 4 = 4s_0 + 8.$$

On the other hand, if N_1 has dimension $4s_0+8$ with symmetry rank $s_0+1 \geq 6$ as in the case (2), we have $\pi_2(N_1) = 0$ or $\mathbb{Z}$ by Remark 3.3. Thus we have $\pi_2(M) = 0$ or $\mathbb{Z}$, thanks to Theorem 1.4 and so the relation $\pi_2(N_1) = \pi_2(M)$. By Theorem 1.3, M is isometric to $\mathbb{H}P^l$ or $Gr_2(\mathbb{C}^{l+2})$. Since $b_4(M) = 1$, M is isometric to $\mathbb{H}P^l$. This completes the proof of Theorem 1.2.

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