The number of orientable small covers over cubes

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Abstract. In the present paper, we count orientable small covers over cubes. We also prove that the ratio of orientable small covers to all things approaches to 0 as n increases.

1. Introduction

A *small cover*, defined by Davis and Januszkiewicz in [2], is an n-dimensional closed smooth manifold M with a smooth action of standard real torus $\mathbb{Z}_2^n$ action such that the action is locally isomorphic to a standard action of $\mathbb{Z}_2^n$ on $\mathbb{R}^n$ and the orbit space $M/\mathbb{Z}_2^n$ is a simple convex polytope. For instance, $\mathbb{R}P^n$ with a natural action of $\mathbb{Z}_2^n$ is a small cover over an n-simplex. In general, real toric manifolds, the set of real points of a toric manifold, provide examples of small covers. Hence we may think of small covers as a topological generalization of real toric manifolds in algebraic geometry.

Small covers over hypercubes are known to be *real Bott manifolds*, which is obtained as iterated $\mathbb{R}P^1$ bundles starting with a point, where each fibration is the projectivization of a Whitney sum of two real line bundles, see [3] for details. The author found the 1-1 correspondence between the set of real Bott manifolds and the set of acyclic digraphs in the previous work [1], and hence he calculate the number of them.

In the present paper, we investigate the number of orientable small covers over cubes. Orientable small covers are well studied by Nakayama and Nishimura [4]. They find an orientability condition for a small cover. Using this, we establish the formula of the number of orientable small covers over cubes and show that the ratio O_n/R_n approaches 0 as n increases, where O_n is the number of orientable small covers and R_n is the number of small covers over an n-cube.

2. Orientable small covers over cubes

Let P be an n-dimensional simple polytope with m facets. Two small covers M_1 and M_2 over P are said to be *Davis-Januszkiewicz equivalent* (or simply, *D-J equivalent*) if there is a weak $\mathbb{Z}_2^n$-equivariant homeomorphism $f : M_1 \to M_2$ which

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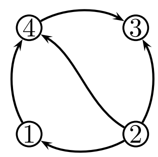

$$\overset{\phi}{\longmapsto} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

FIGURE 1. A bijection ϕ

commutes the diagram:

$$\begin{tikzcd} M_1 \arrow[rr, "f"] \arrow[dr] & & M_2 \arrow[dl] \\ & P & \end{tikzcd}.$$

It is well-known that all small covers over P can be distinguished by the map λ from the set of facets of P to $\mathbb{Z}_2^n$, called the *characteristic function*, which satisfies the *non-singularity condition*; $\{\lambda(F_{i_1}), \dots, \lambda(F_{i_n})\}$ is a basis of $\mathbb{Z}_2^n$ whenever the intersection $F_{i_1} \cap \cdots \cap F_{i_n}$ is non-empty, where $\{F_1, \dots, F_m\}$ is the set of facets of P.

We may assign an $n \times m$-matrix Λ to the characteristic function λ by ordering the facets and choosing a basis for $\mathbb{Z}_2^n$, i.e.,

$$\Lambda = (\lambda(F_1) \cdots \lambda(F_n)) .$$

If P is an n-cube, with the assumption that the facets F_j and F_{n+j} do not intersect for $1 \leq j \leq n$, then D-J equivalence classes of small covers are classified by $n \times 2n$ matrices of the following form

$$\Lambda \sim (E_n | \Lambda_*),$$

where E_n is the identity matrix and Λ_* is the square matrix of size n all of whose principal minors are 1. We refer the reader to [1] or [3] for details.

Let $M(n)$ be the set of $\mathbb{Z}_2$-matrices of size n all of whose principal minors are 1 and let $\mathcal{G}_n$ be the set of acyclic simple digraphs with labelled n nodes. In [1], we have a bijection $\phi : \mathcal{G}_n \to M(n)$ by

$$\phi : G \mapsto A(G) + E_n,$$

where $A(G)$ is the vertex adjacency matrix of G and E_n is the identity matrix of size n (see Figure 1).

On the other hand, we have the nice orientability condition for small covers due to Nakayama and Nishimura in [4].

Theorem 2.1 ([4])**.** *Let M be a small cover over P with Λ. Then M is orientable if and only if the sum of entries of the i-th column of Λ is odd for all $i = 1, \dots, n$.*

Corollary 2.2. *Let M be a small cover over I^n with $\Lambda = (E_n | \Lambda_*)$. Then M is orientable if and only if $\phi^{-1}(\Lambda_*)$ is the graph all of whose vertices have even indegrees.*

Proof. Let G be a graph and A its vertex adjacency matrix. Then the sum of entries of the i-th column of A means the indegree of i-th vertex of G (see Appendix). Note that the sum of entries of each column of $\Lambda_* - E_n$ is even, and hence the corollary follows Theorem 2.1 immediately. $\square$

3. The number of orientable small covers

Let R_n be the number of acyclic digraphs with labelled n nodes. The following is the recursive formula for R_n due to R.W.Robinson in [5].

$$R_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{k(n-k)} R_{n-k}.$$

Let $\mathcal{O}_n \subset \mathcal{G}_n$ be the set of graphs all of whose vertices have even indegrees and let O_n be the cardinality of $\mathcal{O}_n$ (we use the alphabet 'O' instead of 'E' although they have 'even' indegree vertices, because the 'O' is the abbreviation of the word '*Orientable*'). Then O_n is equal to the number of orientable small covers over I^n up to D-J equivalence.

Theorem 3.1. *Let R_k be the number of labelled acyclic digraphs with k vertices. Then,*

$$O_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{(k-1)(n-k)} R_{n-k}.$$

Proof. We count matrices in $M(n)$ all of whose the sum of entries of each column are odd. Let us denote the sum of entries of the i-th column of an $n \times n$-matrix A by $c_i(A)$. Since an acyclic digraph has always a 0-indegree vertex, there is some i such that $c_i(A) = 1$ for all $A \in M(n)$. Assume $c_{i_1} = \cdots = c_{i_k} = 1$, where $k \geq 1$. Since all principal minors of A are 1, the diagonal entries of A are all 1. Thus, by a conjugation, we may assume that A is of the following form:

$$\begin{pmatrix} E_k & S \\ 0 & T \end{pmatrix}, \tag{1}$$

where E_k is the identity matrix of size k, T is an $(n-k) \times (n-k)$-matrix and S is a $k \times (n-k)$-matrix. Note that $A \in M(n)$ if and only if $T \in M(k)$. Thus we may control only one row of S for making all $c_i(A)$'s are odd. This implies the number of A's of the form (1) whose $c_i(A)$'s are odd for all i is $2^{(k-1)(n-k)} R_{n-k}$. By applying the Principle of Inclusion-Exclusion, we get the formula for O_n. $\square$

Here are a few values of O_n.

n	0	1	2	3	4	5	6	7	$\cdots$
O_n	1	1	1	4	43	1156	74581	11226874	$\cdots$

Let us consider the *chromatic generating functions* of R_n and O_n, i.e., we set

$$R(x) = \sum_{n=0}^{\infty} R_n \frac{x^n}{n! 2^{\binom{n}{2}}}, \text{ and } O(x) = \sum_{n=0}^{\infty} O_n \frac{x^n}{n! 2^{\binom{n}{2}}}.$$

Corollary 3.2. *Let $F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n! 2^{\binom{n}{2}}}$. Then*

$$O(x) = \frac{1 - F(-x)}{F'(-x)}.$$

Proof. Let us consider chromatic generating functions $A(x), B(x)$ and $C(x)$ with respect to the sequence A_n, B_n and C_n, respectively. Note that if $C(x) = A(x)B(x)$, then $C_n = \sum_{k=0}^{n} A_k B_{n-k} \binom{n}{k} 2^{k(n-k)}$. Thus, we have $F(-x)R(x) = 1$ (see [6]) and

$$R\left(\frac{x}{2}\right) F(-x) + O(x) = \sum_{n=0}^{\infty} \frac{R_n}{2^n} \frac{x^n}{n!\binom{n}{2}} = R\left(\frac{x}{2}\right).$$

Hence we have $O(x) = F\left(-\frac{x}{2}\right)^{-1}(1 - F(-x))$. On the other hand, $F'(x) = F\left(\frac{x}{2}\right)$, which proves the corollary. □

Let $G(x) = \frac{F(\frac{x}{2})}{1-F(x)}$. We obtain estimates for O_n by analyzing the behavior of the function $G(x)$. Since $F(x)$ has an isolated zero $\alpha \approx -1.488$, $G(x)$ has an isolated zero 2α. Hence

$$G(x) \sim G'(2\alpha)(x - 2\alpha).$$

Hence we have

$$O(x) = \frac{1}{G(-x)} \sim \frac{1}{G'(2\alpha)(-x - 2\alpha)} = -\frac{(1 - F(2\alpha)}{\alpha F(\frac{\alpha}{2})} \sum_{n=0}^{\infty} \left(-\frac{x}{2\alpha}\right)^n,$$

and hence $O_n \sim K 2^{\binom{n}{2}} n! \left(-\frac{1}{2\alpha}\right)^n$, where $K = -\frac{(1-F(2\alpha)}{\alpha F\left(\frac{\alpha}{2}\right)} \approx 2.197$.

Corollary 3.3. *The ratio $\frac{O_{n+1}}{R_{n+1}} / \frac{O_n}{R_n}$ approaches to $\frac{1}{2}$ as n increases. In particular, the ratio $\frac{O_n}{R_n}$ approaches 0 as n increases.*

Proof. Since $R(x)F(-x) = 1$ and $F(x)$ has an isolated zero α, we have $R_n \sim C 2^{\binom{n}{2}} n! \left(-\frac{1}{\alpha}\right)^n$, where $C = -\frac{1}{F\left(\frac{\alpha}{2}\right)}$. Therefore $\frac{O_n}{R_n} \sim \frac{K}{C 2^n}$, and this proves the corollary. □

Appendix. Graph theory terminology

We review the terminology and notation in graph theory, following [7]. A *directed graph* or *digraph* G is a triple (V, E, φ), where $V = \{v_1, \ldots, v_n\}$ is a set of *vertices*, E is a set of directed *edges*, and φ is a map from E to $V \times V$. If $\varphi(e) = (u, v)$, then e is called an edge *from u to v*, with *initial vertex u* and *final vertex v*. This is denoted $u = \operatorname{int} e$ and $v = \operatorname{fin} e$. If $u = v$ then e is called a *loop*. If φ is injective and has no loops, then G is called *simple*. In this case, we denote e by (u, v) for simplify and G can be represented by (V, E). We shall assume that every graph is simple through the paper. A *walk* of length k from vertex u to v is a sequence $v_0, v_1, \ldots, v_k$ such that $v_0 = u$ and $v_k = v$, where $(v_i, v_{i+1}) \in E$ for all $i = 0, \ldots, k-1$. If all the v_i's are distinct except for $v_o = v_k$, then the walk is called a *cycle*. G is *acyclic* if there is no cycle of any length in G. The *outdegree* of a vertex v, denoted $\operatorname{outdeg}(v)$, is the number of edges of G with initial vertex v. Similarly the *indegree* of v, denoted $\operatorname{indeg}(v)$, is the number of edges of G with final vertex v.

On the other hand, all digraphs can be represented by matrices. Define an $n \times n$-matrix $A = (A_{ij})$ by

$$A_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E; \\ 0, & \text{otherwise.} \end{cases}$$

The matrix A is called the *(vertex) adjacency matrix* of G. Remark that the sum of entries of i-th column of A is equal to $\operatorname{indeg}(v_i)$ and the sum of entries of j-th row of A is equal to $\operatorname{outdeg}(v_j)$.

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