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A FETI–DP FORMULATION FOR THE THREE-DIMENSIONAL STOKES PROBLEM WITHOUT PRIMAL PRESSURE UNKNOWNS *

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Abstract. A scalable FETI–DP (Dual-Primal Finite Element Tearing and Interconnecting) algorithm for the three-dimensional Stokes problem is developed and analyzed. This is an extension of the previous work for the two-dimensional problem in [8]. Advantages of this approach are the coarse problem without primal pressure unknowns and the use of a relatively cheap lumped preconditioner. Especially in three dimensions, these advantages provide a more robust and faster FETI-DP algorithm. In three dimensions, the velocity unknowns at subdomain corners and the averages of velocity unknowns over common faces are selected as the primal unknowns in the FETI-DP formulation. Its condition number bound is analyzed to be $C(H/h)$, where C is a positive constant which is independent of any mesh parameters and H/h is the number of elements across each subdomain. Numerical results are included.

Key words. FETI-DP, Stokes problem, lumped preconditioner

AMS subject classifications. 65N30, 65N55, 76D07

1. Introduction. FETI–DP (Dual-Primal Finite Element Tearing and Interconnecting) algorithms are known to be the most scalable domain decomposition methods, which are iterative substructuring methods based on Lagrange multipliers. The solution of a linear system, which is obtained from discretization of partial differential equations, is decoupled by a subdomain partition. The continuity on decoupled interface unknowns is then enforced in both primal and dual sense. Among these decoupled unknowns primal unknowns are selected to enforce strong continuity and Lagrange multipliers are introduced to enforce weak continuity at the remaining part of unknowns on the interfaces, which are called dual unknowns. The primal unknowns contribute to the coarse problem of the FETI-DP algorithms. The unknowns other than the Lagrange multipliers are eliminated by solving independent local problems and the global coarse problem. The resulting system on the Lagrange multipliers is solved by an iterative method combined with a preconditioner. The FETI-DP algorithms have been successfully developed for the elliptic problems and elasticity problems [1, 2, 5, 6, 9, 10]. This family of algorithms was also extended to the Stokes problem [7, 11, 12, 13]. These results can be applied to a more general form of Stokes equations with nonconstant viscosity.

In all these works for the Stokes problem, the compatibility condition on the dual velocity unknowns is required in each subdomain. As a consequence of this requirement, the velocity averages on edges in addition to the velocity unknowns at the subdomain corners are selected

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as primal unknowns in two dimensions. In three dimensions, introduction of face averages and more complicated primal unknowns related to edges is unavoidable. By enforcing the compatibility condition on the dual velocity unknowns in each subdomain, additional primal unknowns of pressure components, that are constant in each subdomain, appear in these algorithms. This gives an indefinite coarse problem with both primal velocity and primal pressure unknowns.

In our previous work [8], we developed a new FETI-DP algorithm for the Stokes problem in two dimensions. In this algorithm, only velocity unknowns at the subdomain corners are selected as primal variables to reduce complication of the implementation. The primal pressure components are not used contrary to other approaches for the Stokes problem. In this formulation, we can eliminate all the pressure unknowns by solving local Stokes problems, since such selection of the primal velocity unknowns results in the dual velocity unknowns which guarantee the solvability of the local Stokes problems without eliminating spurious pressure components. The Dirichlet-type preconditioners are no longer relevant to the FETI-DP formulation and a lumped preconditioner is naturally employed. Its condition number bound $C(H/h)(1 + \log(H/h))$ was proved with the constant C depending on the inf-sup constant of a certain pair of velocity and pressure spaces. Furthermore it was shown that the inf-sup constant is independent of any mesh parameters for rectangular subdomain partitions. This method can be considered as an extension of the work in [14] to the Stokes problem.

In this paper, we extend the FETI-DP algorithm without primal pressure unknowns to the three-dimensional Stokes problem. We note that in all the previous approaches to make the local Stokes problem satisfy compatibility condition, quite complicated primal velocity unknowns are selected in three dimensions; see [13]. This makes them less practical in three dimensions. Our goal is to develop a scalable FETI-DP algorithm for the three-dimensional problem with relatively less complicated primal unknowns and with the computationally more efficient lumped preconditioner. By relaxing the compatibility condition on the dual velocity unknowns, we can select relatively small set of primal unknowns, which are the primal velocity unknowns at the subdomain corners. For the scalability of the method in three dimensions, additional primal unknowns, which are velocity averages over common faces, are introduced. Note that this set of primal unknowns is not enough to produce the dual velocity unknowns which satisfy the compatibility condition in the three-dimensional Stokes problem. We also list requirements for the selection of primal velocity unknowns which can be used in the FETI-DP formulation without primal pressure unknowns; see Remarks 2.1 and 4.4.

The resulting coarse problem of our method consists of only the primal velocity unknowns and becomes symmetric and positive definite. This allows to use a more practical Cholesky solver for the coarse problem in contrast to indefinite coarse problems appeared in [11, 12, 13, 16]. Combined with the lumped preconditioner, a scalable condition number bound $C(H/h)$ is obtained for the FETI-DP algorithm, of which bound is improved by one less log factor than that of the two-dimensional case [8]. Note that the lumped preconditioner provides a computationally more efficient FETI-DP algorithm for three-dimensional problems; see [1, 14]. This bound is the same as that of the FETI-DP method for the elliptic problem with a lumped preconditioner and with the same set of primal unknowns, which are unknowns at the subdomain corners and averages of solutions over common faces; see [14, Lemma 4].

This paper is organized as follows. In Section 2, the FETI-DP formulation without any primal pressure unknowns will be derived and in Section 3 some preliminary results will be provided. The analysis of a condition number bound will be carried out in Section 4. In the final section, numerical results will be presented. Throughout this paper, C stands for a generic positive constant that does not depend on any mesh parameters.

2. FETI–DP formulation.

2.1. A model problem and finite element spaces. We consider the three-dimensional Stokes problem,

$$\begin{aligned} -\triangle \boldsymbol{u} + \nabla p &= \boldsymbol{f} \ \text{ in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \ \text{ in } \Omega, \\ \boldsymbol{u} &= 0 \ \text{ on } \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω is a bounded polyhedral domain in $\mathbf{R}^3$ and $\boldsymbol{f} \in [L^2(\Omega)]^3$.

We introduce an inf-sup stable finite element space $(\widehat{X}, \overline{P})$ for a given triangulation in Ω. Let P be a pressure finite element space with functions which can be discontinuous across the element boundaries and $\widehat{X}$ be a velocity finite element space with functions which are continuous across the element boundaries. We then enforce the average zero condition in Ω on the pressure finite element functions and the zero boundary condition on the velocity finite element functions. We denote the resulting velocity space by $\widehat{X}$ and the resulting pressure space by $\overline{P}$. In a more detail,

$$\overline{P} = P \bigcap L_0^2(\Omega),$$

where $L_0^2(\Omega)$ consists of square integrable functions with their average zero in Ω.

From the finite element space $(\widehat{X}, \overline{P})$, we obtain a discrete problem of (2.1):

find $(\widehat{\boldsymbol{u}}, \overline{p}) \in (\widehat{X}, \overline{P})$ satisfying

$$\begin{aligned} &\int_\Omega \nabla \widehat{\boldsymbol{u}} \cdot \nabla \boldsymbol{v}\, dx - \int_\Omega \overline{p}\, \nabla \cdot \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx, \ \forall \boldsymbol{v} \in \widehat{X}, \\ &- \int_\Omega \nabla \cdot \widehat{\boldsymbol{u}}\, q\, dx = 0, \ \forall q \in \overline{P}. \end{aligned} \tag{2.2}$$

We will develop a FETI-DP algorithm to look for a fast solution of the discrete problem. In the FETI-DP algorithm, a symmetric and positive definite system on Lagrange multipliers will be solved iteratively by employing a quite cheap lumped preconditioner.

2.2. A FETI-DP formulation. We first decompose Ω into a non–overlapping subdomain partition $\{\Omega_i\}_{i=1}^N$ in such a way that each subdomain aligns to the finite elements equipped for Ω. The subdomain finite element spaces are then obtained by

$$X^{(i)} = \widehat{X}|_{\Omega_i},\ P^{(i)} = P|_{\Omega_i},$$

that are the restrictions of $\widehat{X}$ and P to the individual subdomains. Among the subdomain velocity unknowns, we select some unknowns at the subdomain boundary as primal unknowns and we denote each part of the subdomain velocity unknowns by $\boldsymbol{u}_I^{(i)}$, $\boldsymbol{u}_\Pi^{(i)}$, and $\boldsymbol{u}_\Delta^{(i)}$, where I, Π, and Δ denote unknowns located at the subdomain interior, the primal unknowns, and the remaining dual unknowns at the subdomain boundary, respectively. In the present work, the velocity unknowns at the subdomain corners and the averages of the velocity unknowns over common faces are selected as the primal unknowns.

We introduce the corresponding velocity spaces, $X_I^{(i)}$, $X_\Pi^{(i)}$, and $X_\Delta^{(i)}$, to the unknowns $\boldsymbol{u}_I^{(i)}$, $\boldsymbol{u}_\Pi^{(i)}$, and $\boldsymbol{u}_\Delta^{(i)}$, respectively. We also introduce a space $X_r^{(i)}$ with both the interior and the dual velocity unknowns,

$$X_r^{(i)} = X_I^{(i)} \times X_\Delta^{(i)},$$

and use the notation $\boldsymbol{u}_r^{(i)}$ for the velocity unknowns in the space $X_r^{(i)}$.

Throughout the paper, for given spaces $W^{(i)}$ equipped for Ω_i we denote by W the product space of $W^{(i)}$ and by $\widetilde{W}$ the subspace of W, where the strong continuity at the primal unknowns is enforced. The subspace of W, where continuity at all interface unknowns is enforced, will be denoted by $\widehat{W}$. The unknowns at these spaces W, $\widetilde{W}$, and $\widehat{W}$ are then decoupled, partially coupled, and fully coupled across the subdomain interface, respectively. We also allow the same notational convention for the velocity unknowns; $\boldsymbol{u}_r$ denotes $(\boldsymbol{u}_r^{(1)}, \dots, \boldsymbol{u}_r^{(N)})$ and $\widetilde{\boldsymbol{u}}$ denotes velocity unknowns in the space $\widetilde{X}$. We will use the same notation $\boldsymbol{u}$ to denote velocity unknowns and the corresponding finite element function.

We now obtain an equivalent mixed form of the Stokes problem (2.2) in the finite element space $(\widetilde{X}, \overline{P})$ by enforcing the pointwise continuity on the remaining part of the interface unknowns using Lagrange multipliers $\boldsymbol{\lambda} \in M$:

find $((\boldsymbol{u}_I, \boldsymbol{u}_\Delta, \widehat{\boldsymbol{u}}_\Pi), \overline{p}, \boldsymbol{\lambda}) \in \widetilde{X} \times \overline{P} \times M$ such that

$$\begin{pmatrix} K_{II} & K_{I\Delta} & K_{I\Pi} & \overline{B}_I^T & 0 \\ K_{I\Delta}^T & K_{\Delta\Delta} & K_{\Delta\Pi} & \overline{B}_\Delta^T & J_\Delta^T \\ K_{I\Pi}^T & K_{\Delta\Pi}^T & K_{\Pi\Pi} & \overline{B}_\Pi^T & 0 \\ \overline{B}_I & \overline{B}_\Delta & \overline{B}_\Pi & 0 & 0 \\ 0 & J_\Delta & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_\Delta \\ \widehat{\boldsymbol{u}}_\Pi \\ \overline{p} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ \boldsymbol{f}_\Pi \\ 0 \\ 0 \end{pmatrix}, \tag{2.3}$$

where $\overline{B}_I$, $\overline{B}_\Delta$, and $\overline{B}_\Pi$ are from

$$-\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\boldsymbol{u}}\, q\, dx, \quad \forall q \in \overline{P},$$

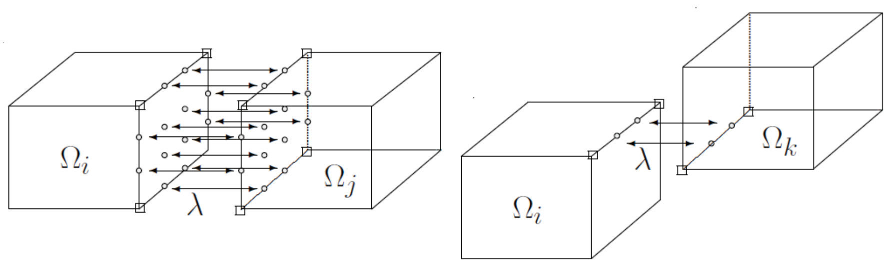

FIG. 1. *Face-based (left) and edge-based (right) Lagrange multipliers: the white dots are the nodes where Lagrange multipliers are used to enforce the continuity and the rectangles are the subdomain corners where strong continuity has been enforced by using primal velocity unknowns.*

J_Δ is a boolean matrix that computes jump of the dual unknowns across the subdomain interface Γ_{ij},

$$J_\Delta \boldsymbol{u}_\Delta|_{\Gamma_{ij}} = \boldsymbol{u}_\Delta^{(i)} - \boldsymbol{u}_\Delta^{(j)},$$

and the other terms are from

$$\sum_i \int_{\Omega_i} \nabla \widetilde{\boldsymbol{u}} \cdot \nabla \widetilde{\boldsymbol{v}} \, dx.$$

The common interface Γ_{ij} can be an edge or a face of subdomains Ω_i and Ω_j. We introduce fully redundant Lagrange multipliers in our FETI-DP formulation and denote by M the space of Lagrange multipliers. For $\boldsymbol{\lambda} \in M$, $\boldsymbol{\lambda}|_{F_{ij}}$ denotes the Lagrange multipliers which are related to the continuity constraints $\boldsymbol{u}_\Delta^{(i)} - \boldsymbol{u}_\Delta^{(j)} = 0$ on the common face $\overline{F}_{ij}$. Similarly, $\boldsymbol{\lambda}|_{E_{ik}}$ denotes the Lagrange multipliers related to the continuity constraints $\boldsymbol{u}_\Delta^{(i)} - \boldsymbol{u}_\Delta^{(k)} = 0$ on the common edge E_{ik}, which is the only common part of the two subdomains Ω_i and Ω_k. We call $\boldsymbol{\lambda}|_{F_{ij}}$ face-based Lagrange multipliers and $\boldsymbol{\lambda}|_{E_{ik}}$ edge-based Lagrange multipliers, respectively. In Figure 1, examples of face- and edge-based Lagrange multipliers are presented.

Let $\mathcal{N}(x)$ be the set of subdomain indices containing the node x and $|\mathcal{N}(x)|$ denote the number of elements in the set $\mathcal{N}(x)$. We may assume that $|\mathcal{N}(x)|$ is bounded by some constant number which does not depend on any mesh parameters. By employing the fully redundant Lagrange multipliers, for the velocity unknowns $\boldsymbol{w}_\Delta(x)$ at the node x, we have the identity

$$\boldsymbol{w}_\Delta(x)|_{\partial\Omega_i} = E_\Delta \boldsymbol{w}_\Delta(x)|_{\partial\Omega_i} + \frac{1}{|\mathcal{N}(x)|} J_\Delta^T J_\Delta \boldsymbol{w}_\Delta(x)|_{\partial\Omega_i}, \tag{2.4}$$

where each terms are defined by

$$E_\Delta \boldsymbol{w}_\Delta(x)|_{\partial\Omega_i} = \frac{1}{|\mathcal{N}(x)|} \sum_{\partial\Omega_k \ni x} \boldsymbol{w}_\Delta^{(k)}(x)$$

and

$$J_\triangle^T J_\triangle \boldsymbol{w}_\triangle(x)|_{\partial\Omega_i} = \sum_{\partial\Omega_k \ni x} (\boldsymbol{w}_\triangle^{(i)}(x) - \boldsymbol{w}_\triangle^{(k)}(x)).$$

We call $E_\triangle \boldsymbol{w}_\triangle$ and $J_\triangle^T J_\triangle \boldsymbol{w}_\triangle$ average and jump operators, respectively. This identity (2.4) will be useful in our analysis of a condition number bound.

We recall the pressure finite element space,

$$\overline{P} = P \bigcap L_0^2(\Omega),$$

where $P = \prod_{i=1}^N P^{(i)}$. These local pressure spaces $P^{(i)}$ do not satisfy the zero average condition. In order to eliminate all the pressure unknowns by solving independent local Stokes problems, we will use the pressure space P instead of $\overline{P}$ in our FETI-DP formulation. By adding a constant pressure component, we extend the pressure space $\overline{P}$ to the space P. The added constant component will give us an additional condition on $\widetilde{\boldsymbol{u}}$,

$$\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\boldsymbol{u}}\, q\, dx = 0, \quad q = c, \tag{2.5}$$

which is equivalent to

$$\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\boldsymbol{u}}\, c\, dx = c \sum_{ij} \int_{F_{ij}} (\boldsymbol{u}_\triangle^{(i)} - \boldsymbol{u}_\triangle^{(j)}) \cdot \boldsymbol{n}_{ij}\, ds = 0. \tag{2.6}$$

Here F_{ij} denotes the common face of two subdomains Ω_i and Ω_j. The above equation can be obtained as a linear combination of the continuity constraints on $\boldsymbol{u}_\triangle$,

$$J_\triangle \boldsymbol{u}_\triangle = 0. \tag{2.7}$$

Since $J_\triangle \boldsymbol{u}_\triangle = 0$ has been already enforced in (2.3), by adding (2.5) to the algebraic system (2.3), we obtain an extended algebraic system which is equivalent to (2.3).

REMARK 2.1. *The identity* (2.6) *holds for our choice of primal unknowns, which are velocity unknowns at subdomain corners and velocity averages over common faces. In the selection of primal unknowns, it is required that they satisfy the identity* (2.6). *In other words, any primal unknowns which are derived from a certain continuity condition across common faces fulfill the requirement. For example, the primal unknowns related to the constraints,*

$$\int_{F_{ij}} \boldsymbol{u}^{(i)} \cdot \boldsymbol{n}_{ij}\, ds = \int_{F_{ij}} \boldsymbol{u}^{(j)} \cdot \boldsymbol{n}_{ij}\, ds,$$

can be used, however, the primal unknowns associated to the averages across a common edge E,

$$\int_E \boldsymbol{u}^{(i)}\, ds = \int_E \boldsymbol{u}^{(j)}\, ds,$$

are not appropriate in our FETI-DP formulation.

We write the extended algebraic system with the pressure space P as follows: find $((\boldsymbol{u}_I, \boldsymbol{u}_\Delta, \widehat{\boldsymbol{u}}_\Pi), p, \boldsymbol{\lambda}) \in (\widetilde{X}, P, M)$ such that

$$\begin{pmatrix} K_{II} & K_{I\Delta} & K_{I\Pi} & B_I^T & 0 \\ K_{I\Delta}^T & K_{\Delta\Delta} & K_{\Delta\Pi} & B_\Delta^T & J_\Delta^T \\ K_{I\Pi}^T & K_{\Delta\Pi}^T & K_{\Pi\Pi} & B_\Pi^T & 0 \\ B_I & B_\Delta & B_\Pi & 0 & 0 \\ 0 & J_\Delta & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_\Delta \\ \widehat{\boldsymbol{u}}_\Pi \\ p \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ \boldsymbol{f}_\Pi \\ 0 \\ 0 \end{pmatrix}. \tag{2.8}$$

Here B_I, B_Δ, and B_Π are from

$$-\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\boldsymbol{u}}\, q\, dx, \quad \forall q \in P,$$

and the other terms are the same as those in (2.3).

In the new algebraic form, the unknowns $(\boldsymbol{u}_I, \boldsymbol{u}_\Delta, p)$ can be eliminated by solving independent local problems. The advantage of the extended algebraic system is that no pressure unknowns are left and only the primal velocity unknowns remain after solving the local problems. The primal velocity unknowns can be eliminated by solving the global coarse problem, which is smaller and more practical than those appeared in other domain decomposition algorithms for the Stokes problem [11, 12, 13, 16]. As a result a linear system on $\boldsymbol{\lambda}$ will be obtained. The introduction of fully redundant Lagrange multipliers and the extension of the pressure space make the resulting system singular. We will later provide details for a subspace of M where the system is symmetric and positive definite. The FETI-DP iteration will be performed on the subspace.

Let

$$S = \begin{pmatrix} K_{II} & K_{I\Delta} & B_I^T \\ K_{I\Delta}^T & K_{\Delta\Delta} & B_\Delta^T \\ B_I & B_\Delta & 0 \end{pmatrix}. \tag{2.9}$$

We recall that $X_r = X_I \times X_\Delta$. We can show that (X_r, P) satisfies the following condition: for any nonzero $p \in P$, there exists $\boldsymbol{v}_r \in X_r$ satisfying

$$\sum_{i=1}^{N} \int_{\Omega_i} \nabla \cdot \boldsymbol{v}_r p\, dx \neq 0, \tag{2.10}$$

so that S is invertible. This assertion can be proved in a similar way to [8, Lemma 3.1]. We then eliminate $(\boldsymbol{u}_I, \boldsymbol{u}_\Delta, p)$ from (2.8),

$$\begin{pmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_\Delta \\ p \end{pmatrix} = S^{-1} \left(\begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix} - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix} \widehat{\boldsymbol{u}}_\Pi - \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix} \boldsymbol{\lambda} \right). \tag{2.11}$$

This is solving the local Stokes problem with a Dirichlet boundary condition given at the primal unknowns $\widehat{\boldsymbol{u}}_\Pi$ and a Neumann boundary condition given at the other part of unknowns on the subdomain boundary.

Substituting $(\boldsymbol{u}_I, \boldsymbol{u}_\Delta, p)$ into (2.8) and then solving for $\widehat{\boldsymbol{u}}_\Pi$,

$$S_{\Pi\Pi}\widehat{\boldsymbol{u}}_\Pi = \boldsymbol{f}_\Pi - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T S^{-1} \left(\begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix} \boldsymbol{\lambda} \right), \tag{2.12}$$

we obtain the resulting algebraic system on $\boldsymbol{\lambda}$,

$$F_{DP}\boldsymbol{\lambda} = d, \tag{2.13}$$

where
(2.14)

$$\begin{aligned} F_{DP} &= \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}^T S^{-1} \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}^T S^{-1} \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix} S_{\Pi\Pi}^{-1} \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T S^{-1} \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}, \\ d &= \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}^T S^{-1} \left(\begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix} - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix} S_{\Pi\Pi}^{-1} \left(\boldsymbol{f}_\Pi - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T S^{-1} \begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix} \right) \right), \end{aligned}$$

and

$$S_{\Pi\Pi} = K_{\Pi\Pi} - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T S^{-1} \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}.$$

Since $X_r \subset \widetilde{X}$, the assertion (2.10) also holds for $(\widetilde{X}, P)$. This fact gives that $S_{\Pi\Pi}$ is invertible, in fact, symmetric and positive definite.

We introduce a lumped preconditioner $\widehat{M}^{-1}$ which is given by

$$\widehat{M}^{-1} = \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}^T S \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}.$$

We recall the matrix S in (2.9) and obtain the resulting form of the preconditioner

$$\widehat{M}^{-1} = J_\Delta K_{\Delta\Delta} J_\Delta^T. \tag{2.15}$$

This preconditioner was introduced for FETI-type algorithms of the elliptic problems to reduce the cost for solving a Dirichlet problem which appears in the optimal preconditioner [3].

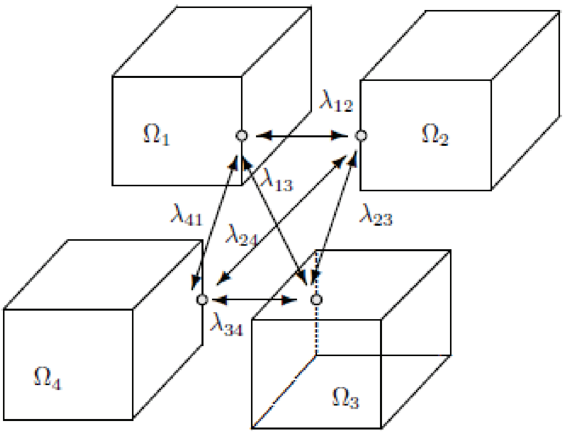

FIG. 2. *Six Lagrange multipliers are used for an unknown (white dot) on an edge which are the common part of four subdomains*

Later, FETI-DP algorithms with the lumped preconditioner was proved to give a good convergence for the elliptic problems [14].

The resulting system (2.13) for $\boldsymbol{\lambda} \in M$ is symmetric and positive semidefinite. We will now find a subspace of M where F_{DP} is positive definite. Let Null(J_Δ^T) be the space of vectors $\boldsymbol{\mu} \in M$ such that $J_\Delta^T \boldsymbol{\mu} = 0$. The introduction of the fully redundant Lagrange multipliers gives that for $\boldsymbol{\lambda} \in$ Null(J_Δ^T), $F_{DP}\boldsymbol{\lambda} = 0$, see Figure 2. The extension of the pressure space causes one more null space component which is given by

$$\boldsymbol{\mu}_0|_{F_{ij}} = \zeta_{ij}\boldsymbol{n}_{ij}, \ \forall F_{ij} \ \text{ and } \ \boldsymbol{\mu}_0|_{E_{lk}} = 0, \ \forall E_{lk}. \tag{2.16}$$

Here $\boldsymbol{\mu}_0|_{F_{ij}}$ and $\boldsymbol{\mu}_0|_{E_{lk}}$ are face-based and edge-based Lagrange multipliers, respectively, $\boldsymbol{n}_{ij}$ is the unit normal to the face F_{ij}, and at each nodal point $x_l \in \overline{F}_{ij}$, $\zeta_{ij}(x_l)$ is given by

$$\zeta_{ij}(x_l) = \int_{F_{ij}} \phi_l(x(s), y(s), z(s))\, ds, \tag{2.17}$$

where ϕ_l is the velocity basis function related to the node x_l. This can be shown by observing that $(\boldsymbol{u}_I, \boldsymbol{u}_\Delta, \widehat{\boldsymbol{u}}_\Pi) = 0$, $p = c$, and $\boldsymbol{\lambda} = c\boldsymbol{\mu}_0$ are solutions of (2.8) for the zero force terms $(\boldsymbol{f}_I, \boldsymbol{f}_\Delta, \boldsymbol{f}_\Pi) = 0$ with c as an arbitrary constant.

Let Range(J_Δ) be the range space of J_Δ. We then have

$$M = \text{Null}(J_\Delta^T) \bigoplus \text{Range}(J_\Delta).$$

We now introduce a subspace of M, which is orthogonal to the null space components of F_{DP},

$$M_c = \left\{ \boldsymbol{\mu} \in \text{Range}(J_\Delta) \,:\, \boldsymbol{\mu}^t \boldsymbol{\mu}_0 = 0 \right\}. \tag{2.18}$$

Moreover, M_c is in fact the range space of F_{DP} and $d \in M_c$; see the formula for F_{DP} and d in (2.14) and the result in [8, Lemma 3.2]. We build the orthogonal projection to the space M_c by finding a basis of Null(J_Δ^T). By applying the projection throughout the conjugate gradient iteration, we perform the iteration within the subspace M_c.

In a more detail, let $\{\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \cdots, \boldsymbol{\mu}_m\}$ be a basis of Null(J_Δ^T). We consider the following vector $\widetilde{\boldsymbol{\mu}}_0$ of the form,

$$\widetilde{\boldsymbol{\mu}}_0 = \alpha_0 \boldsymbol{\mu}_0 + \alpha_1 \boldsymbol{\mu}_1 + \alpha_2 \boldsymbol{\mu}_2 + \cdots + \alpha_m \boldsymbol{\mu}_m \tag{2.19}$$

and we then determine the values of $\alpha_1, \alpha_2, \cdots, \alpha_m$, and α_0 so that

$$\boldsymbol{\mu}_i^T \widetilde{\boldsymbol{\mu}}_0 = 0 \text{ for } i = 1, 2, \cdots, m \text{ and } \widetilde{\boldsymbol{\mu}}_0^T \widetilde{\boldsymbol{\mu}}_0 = 1.$$

We now introduce a projection

$$P_{M_c} = I - \widetilde{\boldsymbol{\mu}}_0 \widetilde{\boldsymbol{\mu}}_0^T.$$

For any $\boldsymbol{\lambda} \in$ Range(J_Δ), we can see that $P_{M_c}\boldsymbol{\lambda}$ belongs to M_c. We consider

$$\boldsymbol{\mu}_i^T P_{M_c} \boldsymbol{\lambda} = \boldsymbol{\mu_i}^T (\boldsymbol{\lambda} - \widetilde{\boldsymbol{\mu}}_0 \widetilde{\boldsymbol{\mu}}_0^T \boldsymbol{\lambda}) \quad \text{for } i = 1, \cdots, m.$$

By using Range$(J_\Delta)\perp$Null(J_Δ^T) and $\boldsymbol{\mu}_i^T \widetilde{\boldsymbol{\mu}}_0 = 0$ for $i = 1, \cdots, m$, we obtain that

$$\boldsymbol{\mu}_i^T P_{M_c} \boldsymbol{\lambda} = 0, \quad i = 1, \cdots, m, \text{ for } \boldsymbol{\lambda} \in \text{Range}(J_\Delta). \tag{2.20}$$

We now consider

$$\widetilde{\boldsymbol{\mu}}_0^T P_{M_c} \boldsymbol{\lambda} = \widetilde{\boldsymbol{\mu}}_0^T (\boldsymbol{\lambda} - \widetilde{\boldsymbol{\mu}}_0 \widetilde{\boldsymbol{\mu}}_0^T \boldsymbol{\lambda}).$$

Since $\widetilde{\boldsymbol{\mu}}_0^T \widetilde{\boldsymbol{\mu}}_0 = 1$, we have

$$\widetilde{\boldsymbol{\mu}}_0^T P_{M_c} \boldsymbol{\lambda} = 0. \tag{2.21}$$

From (2.19)-(2.21), we obtain for any $\boldsymbol{\lambda} \in$ Range(J_Δ)

$$\boldsymbol{\mu}_i^T P_{M_c} \boldsymbol{\lambda} = 0, \quad \text{for } i = 0, 1, \cdots, m.$$

This proves that $P_{M_c}\boldsymbol{\lambda}$ belongs to M_c. We note that all the iterates of the FETI-DP algorithm with the lumped preconditioner belong to Range(J_Δ). Therefore we will use the projection P_{M_c}, which can be built easily as in the above, during the FETI-DP iteration.

REMARK 2.2. *Compared to the two-dimensional problem, more complicated interface conditions appear in the three-dimensional case, as it is common to most FETI-DP formulation. Edge- and face-based fully redundant Lagrange multipliers and the additional primal velocity unknowns other than velocity unknowns at the corners are required.*

3. Preliminary results. In this section, we provide some preliminary results to analyze a condition number bound of the new FETI-DP algorithm equipped with the lumped preconditioner for the Stokes problem. For those results which can be obtained in a similar way to the two-dimensional case, we will refer our previous work [8].

For the proof of the lower bound analysis, we need an inf-sup stability of a certain pair of velocity and pressure finite element spaces. We introduce a pair of velocity and pressure finite element spaces,

$$(\widehat{E}_{I,\Pi}, \overline{P}),$$

where $\widehat{E}_{I,\Pi} = X_I + \widehat{E}_\Pi$. Here $\widehat{E}_\Pi$ is an enriched primal velocity space that is constructed as follows. We denote by F the common face where two subdomains intersect and by E the common edge where more than two subdomains intersect. For a given subdomain partition, $\mathcal{V}$, $\mathcal{F}$, and $\mathcal{E}$ denote the set of subdomain corners, the set of common faces, and the set of common edges, respectively. For a common edge E, we denote by $\mathcal{F}(E)$ the set of faces that share the edge E in common. We consider a function $\boldsymbol{v} \in \widehat{X}$ such that

$$\begin{aligned} \boldsymbol{v}(V) &= \boldsymbol{a}_V, \\ \int_F I^h(\theta_F \boldsymbol{v})(x)\, dx(s) &= \boldsymbol{a}_F, \\ \int_F I^h(\theta_E \boldsymbol{v})(x) \cdot \mathbf{n}_F\, dx(s) &= a_F^E, \quad \forall F \in \mathcal{F}(E), \end{aligned}$$

with the given values of $\boldsymbol{a}_V$, $\boldsymbol{a}_F$, and a_F^E, which are provided for all $V \in \mathcal{V}$, $F \in \mathcal{F}$, and $E \in \mathcal{E}$. Here θ_F and θ_E are face and edge cut-off functions, which are one at the interior nodes of the face F, and the edge E, respectively, and zero at the other part of the unknowns. In addition, $I^h(v)$ is the nodal interpolant of v to the velocity finite element space $\widehat{X}$. We note that for an edge E, the values of a_F^E are provided for each face F containing E, i.e., for all face $F \in \mathcal{F}(E)$.

The enriched primal velocity space then consists of such functions $\boldsymbol{v}$ which minimize discrete H^1-seminorm, i.e.,

$$\widehat{E}_\Pi = \left\{ \boldsymbol{v} \in \widehat{X} \;:\; \boldsymbol{v} \text{ minimizes } |\boldsymbol{v}|_{1,\Omega} \text{ with the given values } \boldsymbol{a}_V,\ \boldsymbol{a}_F,\ \text{and } a_F^E \right\}.$$

We introduce the pressure space $\overline{P}_\Pi$ with functions defined in Ω that are constant in each subdomain and have their average zero in Ω. The space $\overline{P}$ is then decomposed into

$$\overline{P} = P_I \bigoplus \overline{P}_\Pi,$$

where

$$P_I = \prod_{i=1}^{N} (P^{(i)} \bigcap L_0^2(\Omega_i)).$$

We will prove that the pair $(\widehat{E}_{I,\Pi}, \overline{P})$ is inf-sup stable. Let β_I and β_Π be inf-sup constants of (X_I, P_I) and $(\widehat{E}_\Pi, \overline{P}_\Pi)$. The inf-sup constant β of $(\widehat{E}_{I,\Pi}, \overline{P})$ is then bounded below by

$$\beta^2 \geq C \min\left\{ \frac{\beta_I^2}{\beta_I^2+1}\beta_\Pi^2, \beta_I^2 \right\},$$

see [8, Lemma 3.5]. Since the pair (X_I, P_I) is inf-sup stable, we only need to prove the inf-sup stability of $(\widehat{E}_\Pi, \overline{P}_\Pi)$.

For a given function $\boldsymbol{v} \in \widehat{X}$, we define $I_{\widehat{E}_\Pi}(\boldsymbol{v})$ by an interpolant to the space $\widehat{E}_\Pi$ with those values $\boldsymbol{a}_V$, $\boldsymbol{a}_F$, and a_F^E obtained from $\boldsymbol{v}$. We introduce the notations

$$|\boldsymbol{u}|_1^2 = \sum_{i=1}^N |\boldsymbol{u}|^2_{H^1(\Omega_i)}, \quad \|p\|_0^2 = \sum_{i=1}^N \|p\|^2_{L^2(\Omega_i)},$$

where $|\cdot|_{H^1(\Omega_i)}$ and $\|\cdot\|_{L^2(\Omega_i)}$ are the H^1-seminorm and the L^2-norm in Ω_i, respectively.

LEMMA 3.1. *The pair $(\widehat{E}_\Pi, \overline{P}_\Pi)$ satisfies that for any $q \in \overline{P}_\Pi$, there exists $\boldsymbol{v}_{\widehat{E}} \in \widehat{E}_\Pi$ such that*

$$\frac{\int_\Omega \nabla\cdot \boldsymbol{v}_{\widehat{E}} q\,dx}{|\boldsymbol{v}_{\widehat{E}}|_1 \|q\|_0} \geq \widehat{\beta},$$

where $\widehat{\beta}$ is the inf-sup constant of the pair $(\widehat{X}, \overline{P}_\Pi)$.

Proof. We note that $(\widehat{X}, \overline{P}_\Pi)$ is inf-sup stable with the constant $\widehat{\beta}$, in other words, for any $q \in \overline{P}_\Pi$ there exists $\boldsymbol{v} \in \widehat{X}$ such that

$$\frac{\int_\Omega \nabla\cdot \boldsymbol{v} q\,dx}{|\boldsymbol{v}|_1 \|q\|_0} \geq \widehat{\beta}. \tag{3.1}$$

For $q \in \overline{P}_\Pi$, which is constant in each subdomain, we let q_i denote those constant values in each subdomain Ω_i. For the given $\boldsymbol{v} \in \widehat{X}$, we find $I_{\widehat{E}_\Pi}(\boldsymbol{v}) \in \widehat{E}_\Pi$ and let $\boldsymbol{v}_{\widehat{E}} = I_{\widehat{E}_\Pi}(\boldsymbol{v})$. We then have

$$\begin{aligned}
\int_\Omega \nabla\cdot\boldsymbol{v} q\,dx &= \sum_i q_i \int_{\Omega_i} \nabla\cdot \boldsymbol{v}\,dx \\
&= \sum_i q_i \int_{\partial\Omega_i} \boldsymbol{v}\cdot\boldsymbol{n}_i\,ds \\
&= \sum_i q_i \int_{\partial\Omega_i} I^h((\theta_F+\theta_E+\theta_V)\boldsymbol{v})\cdot\boldsymbol{n}_i\,ds \\
&= \sum_i q_i \int_{\partial\Omega_i} I^h((\theta_F+\theta_E+\theta_V)\boldsymbol{v}_{\widehat{E}})\cdot\boldsymbol{n}_i\,ds \\
&= \int_\Omega \nabla\cdot\boldsymbol{v}_{\widehat{E}} q\,dx.
\end{aligned} \tag{3.2}$$

Here $\boldsymbol{n}_i$ is the unit normal to $\partial\Omega_i$. In the fourth equality, the following properties for $\boldsymbol{v}_{\widehat{E}}$ are used

$$\begin{aligned}
&\boldsymbol{v}_{\widehat{E}}(V) = \boldsymbol{v}(V),\ \forall V \in \mathcal{V},\\
&\int_F I^h(\theta_F \boldsymbol{v}_{\widehat{E}}) \cdot \boldsymbol{n}_F = \int_F I^h(\theta_F \boldsymbol{v}) \cdot \boldsymbol{n}_F,\ \forall F \in \mathcal{F},\\
&\int_F I^h(\theta_E \boldsymbol{v}_{\widehat{E}}) \cdot \boldsymbol{n}_F = \int_F I^h(\theta_E \boldsymbol{v}) \cdot \boldsymbol{n}_F,\ \forall F \in \mathcal{F}(E),\ \forall E \in \mathcal{E},
\end{aligned}$$

where $\boldsymbol{n}_F$ is the unit normal to the face F. By the definition, $\boldsymbol{v}_{\widehat{E}}$ satisfies that

$$|\boldsymbol{v}_{\widehat{E}}|_1 \le |\boldsymbol{v}|_1.$$

From the above bound, the identity in (3.2), and the bound in (3.1), we obtain the desired result for $\boldsymbol{v}_{\widehat{E}}$,

$$\frac{\int_\Omega \nabla \cdot \boldsymbol{v}_{\widehat{E}} q\, dx}{|\boldsymbol{v}_{\widehat{E}}|_1 \|q\|_0} \ge \frac{\int_\Omega \nabla \cdot \boldsymbol{v} q\, dx}{|\boldsymbol{v}|_1 \|q\|_0} \ge \widehat{\beta}.$$

□

REMARK 3.2. *The inf-sup constant of the pair $(\widehat{E}_\Pi, \overline{P}_\Pi)$ is bounded below by the value $\widehat{\beta}$, which is independent of any mesh parameters. In our previous work [8] for the two-dimensional Stokes problem, a similar result was proved for only rectangular subdomain partition. In this new approach, we do not need such an assumption on the subdomain partition.*

REMARK 3.3. *We note that the quite complicated enriched primal velocity space is introduced only for the proof of the lower bound analysis. The interpolant $I_{\widehat{E}_\Pi}(\boldsymbol{v})$ then preserves the flux across the subdomain interface, i.e.,*

$$\sum_i \int_{\Omega_i} \nabla \cdot (I_{\widehat{E}_\Pi}(\boldsymbol{v}))\, p\, dx = \sum_i \int_{\Omega_i} \nabla \cdot \boldsymbol{v}\, p\, dx, \quad \forall p \in \overline{P}_\Pi,$$

which plays a major role in the proof of the lower bound. In the actual implementation of the FETI-DP algorithm, the velocity unknowns at the subdomain corners and the velocity averages across the common faces are selected as the primal unknowns. In [13], all those unknowns, $\boldsymbol{a}_V$, $\boldsymbol{a}_F$, and a_F^E, are selected as primal unknowns, which produce quite a large coarse problem combined with primal pressure unknowns.

4. Condition number analysis. In this section, we will provide a condition number bound of the FETI-DP operator with the lumped preconditioner by proving the following inequalities:

$$C_1 \beta^2 \langle \widehat{M} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle \le \langle F_{DP} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle \le C_2 \left(\frac{H}{h} \right) \langle \widehat{M} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle, \quad \forall \boldsymbol{\lambda} \in M_c,$$

where β is the inf-sup constant of the pair $(\widehat{E}_{I,\Pi}, \overline{P})$. These inequalities yield the following condition number bound,

$$\kappa(\widehat{M}^{-1} F_{DP}) \le C \frac{1}{\beta^2} \left(\frac{H}{h} \right).$$

4.1. Lower bound analysis. We provide an analysis for the lower bound of the proposed FETI-DP algorithm. The following lemma is proved in [4, Lemma 2.3]:

LEMMA 4.1. *Consider the discrete saddle point problem*

$$\begin{pmatrix} \nu A & B^T \\ B & -1/\alpha C \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ g \end{pmatrix},$$

where A and C are positive definite and, if $\alpha = \infty$, B has full row rank. Let $\beta \geq 0$ be the best inf-sup constant of the pair (A, B) such that

$$p^T B A^{-1} B^T p \geq \beta^2 p^T C p, \quad \forall p.$$

Then,

$$\|\boldsymbol{u}\|_A \leq 1/\nu \|\boldsymbol{f}\|_{A^{-1}} + \frac{1}{\sqrt{\beta^2 + \nu/\alpha}} \|g\|_{C^{-1}},$$
$$\|p\|_C \leq \frac{1}{\sqrt{\beta^2 + \nu/\alpha}} \|\boldsymbol{f}\|_{A^{-1}} + \frac{\nu}{\beta^2 + \nu/\alpha} \|g\|_{C^{-1}}.$$

We introduce a matrix K, which gives the broken H^1-seminorm for $\boldsymbol{u} = (\boldsymbol{u}_I, \boldsymbol{u}_\Delta, \widehat{\boldsymbol{u}}_\Pi) \in \widetilde{X}$, i.e.,

$$\sum_{i=1}^{N} |\boldsymbol{u}|^2_{H^1(\Omega_i)} = \boldsymbol{u}^T K \boldsymbol{u},$$

where K is obtained from the block matrices in (2.3),

$$K = \begin{pmatrix} K_{II} & K_{I\Delta} & K_{I\Pi} \\ K_{I\Delta}^T & K_{\Delta\Delta} & K_{\Delta\Pi} \\ K_{I\Pi}^T & K_{\Delta\Pi}^T & K_{\Pi\Pi} \end{pmatrix}.$$

With the help of the pair $(\widehat{E}_{I,\Pi}, \overline{P})$, we obtain the following lemma, which is proved similarly to [8, Lemma 4.2]. We include its proof for the completeness.

LEMMA 4.2. *For any $\boldsymbol{\mu} \in M_c$, there exists $\boldsymbol{u} \in \widetilde{X}$ such that*

1. $J_\Delta \boldsymbol{u}_\Delta = \boldsymbol{\mu}$,
2. $\sum_i \int_{\Omega_i} \nabla \cdot \boldsymbol{u}\, q\, dx = 0, \quad \forall q \in P$,
3. $\langle K\boldsymbol{u}, \boldsymbol{u}\rangle \leq C \frac{1}{\beta^2} \langle K_{\Delta\Delta} J_\Delta^T \boldsymbol{\mu}, J_\Delta^T \boldsymbol{\mu}\rangle$, *where β is the inf-sup constant of the pair $(\widehat{E}_{I,\Pi}, \overline{P})$.*

Proof. For any $\boldsymbol{\mu} \in M_c$, there exists $\boldsymbol{v}_\Delta \in X_\Delta$ such that

$$J_\Delta \boldsymbol{v}_\Delta = \boldsymbol{\mu}.$$

We then find $\widehat{\boldsymbol{v}}_\Delta \in \widehat{X}_\Delta$, the space of fully coupled dual velocity unknowns, which gives that

$$E_\Delta(\boldsymbol{v}_\Delta + \widehat{\boldsymbol{v}}_\Delta) = 0.$$

Let $\boldsymbol{w}_\triangle = \boldsymbol{v}_\triangle + \widehat{\boldsymbol{v}}_\triangle$ and then we obtain

$$J_\triangle \boldsymbol{w}_\triangle = \boldsymbol{\mu} \text{ and } E_\triangle \boldsymbol{w}_\triangle = 0, \tag{4.1}$$

since $J_\triangle \widehat{\boldsymbol{v}}_\triangle = 0$.

For the $\boldsymbol{w}_\triangle$, we find $\boldsymbol{w}_I \in X_I$, $\widehat{\boldsymbol{w}}_E \in \widehat{E}_\Pi$, and $p \in \overline{P}$ such that

$$\begin{aligned}
\sum_i \int_{\Omega_i} \nabla(\boldsymbol{w}_I + \boldsymbol{w}_\triangle + \widehat{\boldsymbol{w}}_E) \cdot \nabla \boldsymbol{v}_I \, dx - \sum_i \int_{\Omega_i} \nabla \cdot \boldsymbol{v}_I p \, dx = 0, \quad \forall \boldsymbol{v}_I \in X_I,\\
\sum_i \int_{\Omega_i} \nabla(\boldsymbol{w}_I + \boldsymbol{w}_\triangle + \widehat{\boldsymbol{w}}_E) \cdot \nabla \widehat{\boldsymbol{v}}_E \, dx - \sum_i \int_{\Omega_i} \nabla \cdot \widehat{\boldsymbol{v}}_E p \, dx = 0, \quad \forall \widehat{\boldsymbol{v}}_E \in \widehat{E}_\Pi,\\
- \sum_i \int_{\Omega_i} \nabla \cdot (\boldsymbol{w}_I + \boldsymbol{w}_\triangle + \widehat{\boldsymbol{w}}_E) q \, dx = 0, \quad \forall q \in \overline{P}.
\end{aligned} \tag{4.2}$$

We let

$$\boldsymbol{u} = \boldsymbol{w}_I + \boldsymbol{w}_\triangle + \widehat{\boldsymbol{w}}_E$$

and will show that $\boldsymbol{u}$ satisfies the three requirements. We represent $\widehat{\boldsymbol{w}}_E$ with a vector of unknowns in the space $\widetilde{X}$,

$$\widehat{\boldsymbol{w}}_E = (\boldsymbol{z}_I, \boldsymbol{z}_\triangle, \widehat{\boldsymbol{w}}_\Pi),$$

and obtain $\boldsymbol{u}$ as in the form,

$$\boldsymbol{u} = (\boldsymbol{w}_I + \boldsymbol{z}_I, \boldsymbol{w}_\triangle + \boldsymbol{z}_\triangle, \widehat{\boldsymbol{w}}_\Pi),$$

so that we have

$$\boldsymbol{u}_\triangle = \boldsymbol{w}_\triangle + \boldsymbol{z}_\triangle.$$

Since $\widehat{\boldsymbol{w}}_E \in \widehat{X}$, $J_\triangle \boldsymbol{z}_\triangle = 0$. Combined with (4.1), this gives the first requirement

$$J_\triangle \boldsymbol{u}_\triangle = J_\triangle \boldsymbol{w}_\triangle = \boldsymbol{\mu} \tag{4.3}$$

and

$$\begin{aligned}
\sum_{i=1}^N \int_{\Omega_i} \nabla \cdot \boldsymbol{u} \, dx &= \sum_{ij} \int_{F_{ij}} (\boldsymbol{u}_\triangle^{(i)} - \boldsymbol{u}_\triangle^{(j)}) \cdot \boldsymbol{n}_{ij} \, ds\\
&= \sum_{ij} \int_{F_{ij}} (\boldsymbol{w}_\triangle^{(i)} - \boldsymbol{w}_\triangle^{(j)}) \cdot \boldsymbol{n}_{ij} \, ds\\
&= \sum_{ij} (\zeta_{ij} \boldsymbol{\mu}|_{F_{ij}}) \cdot \boldsymbol{n}_{ij} = \sum_{ij} \boldsymbol{\mu}|_{F_{ij}} \cdot (\zeta_{ij} \boldsymbol{n}_{ij}),
\end{aligned}$$

where $\boldsymbol{\mu}|_{F_{ij}}$ is the part of Lagrange multipliers $\boldsymbol{\mu}$ corresponding to the face $\overline{F}_{ij}$ and ζ_{ij} is defined in (2.17). Here we used that $J_\triangle \boldsymbol{w}_\triangle = \boldsymbol{\mu}$, i.e., $(\boldsymbol{w}_\triangle^{(i)} - \boldsymbol{w}_\triangle^{(j)})|_{F_{ij}} = \boldsymbol{\mu}|_{F_{ij}}$. Since

$\boldsymbol{\mu} \in M_c$ and $\boldsymbol{\mu}_0|_{F_{ij}} = \zeta_{ij}\boldsymbol{n}_{ij}$, the above equation is zero so that the second requirement is proved for $\boldsymbol{u}$, combined with the third equation in (4.2); see also (2.16) and (2.18).

We write the weak form in (4.2) into the algebraic equations,

$$\begin{pmatrix} K_{II} & K_{IE} & B_I^T \\ K_{EI} & K_{EE} & B_E^T \\ B_I & B_E & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{w}_I \\ \widehat{\boldsymbol{w}}_E \\ p \end{pmatrix} = \begin{pmatrix} -K_{I\Delta}\boldsymbol{w}_\Delta \\ -K_{E\Delta}\boldsymbol{w}_\Delta \\ -B_\Delta \boldsymbol{w}_\Delta \end{pmatrix}. \tag{4.4}$$

Let

$$A = \begin{pmatrix} K_{II} & K_{IE} \\ K_{EI} & K_{EE} \end{pmatrix}, \quad B = \begin{pmatrix} B_I & B_E \end{pmatrix}.$$

We introduce the mass matrix C which gives the L^2-norm of functions in the space $\overline{P}$, i.e.,

$$\langle Cq, q\rangle = \|q\|^2_{L^2(\Omega)}, \quad \text{for } q \in \overline{P}.$$

Since $(\widehat{E}_{I,\Pi}, \overline{P})$ is inf-sup stable, the pair (A, B) satisfies the condition in Lemma 4.1 with the constant β and the matrix B has full row rank. We apply Lemma 4.1 to the mixed problem (4.4) with $\nu = 1$ and $\alpha = \infty$ to obtain

$$\left\| \begin{pmatrix} \boldsymbol{w}_I \\ \widehat{\boldsymbol{w}}_E \end{pmatrix} \right\|_A^2 \le 2 \left\| \begin{pmatrix} K_{I\Delta}\boldsymbol{w}_\Delta \\ K_{E\Delta}\boldsymbol{w}_\Delta \end{pmatrix} \right\|_{A^{-1}}^2 + \frac{2}{\beta^2} \|B_\Delta \boldsymbol{w}_\Delta\|^2_{C^{-1}}. \tag{4.5}$$

Here $\|\boldsymbol{v}\|_A^2 = \langle A\boldsymbol{v}, \boldsymbol{v}\rangle$.

Similarly to the proof in [8, Lemma 4.2], the two terms in (4.5) are bounded by $\langle K_{\Delta\Delta}\boldsymbol{w}_\Delta, \boldsymbol{w}_\Delta\rangle$. We then obtain for $\boldsymbol{u} = \boldsymbol{w}_I + \boldsymbol{w}_\Delta + \widehat{\boldsymbol{w}}_E$,

$$\langle K\boldsymbol{u}, \boldsymbol{u}\rangle \le 2\langle K_{\Delta\Delta}\boldsymbol{w}_\Delta, \boldsymbol{w}_\Delta\rangle + 2 \left\| \begin{pmatrix} \boldsymbol{w}_I \\ \widehat{\boldsymbol{w}}_E \end{pmatrix} \right\|_A^2 \le C\frac{1}{\beta^2}\langle K_{\Delta\Delta}\boldsymbol{w}_\Delta, \boldsymbol{w}_\Delta\rangle.$$

From the above bound combined with (2.4) and (4.1), we obtain

$$\langle K\boldsymbol{u}, \boldsymbol{u}\rangle \le \frac{C}{\beta^2}\langle K_{\Delta\Delta}J_\Delta^T\boldsymbol{\mu}, J_\Delta^T\boldsymbol{\mu}\rangle.$$

□

We introduce

$$\widetilde{X}(\text{div}) = \left\{ \boldsymbol{v} \in \widetilde{X} \; : \; \int_{\Omega_i} \nabla \cdot \boldsymbol{v} q \, dx = 0, \quad \forall q \in P \right\}. \tag{4.6}$$

We then have the identity,

$$\langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle = \max_{\boldsymbol{v} \in \widetilde{X}(\text{div})} \frac{\langle J_\Delta \boldsymbol{v}_\Delta, \boldsymbol{\lambda}\rangle^2}{\langle K\boldsymbol{v}, \boldsymbol{v}\rangle}. \tag{4.7}$$

The following lower bound can be obtained from Lemma 4.2 and (4.7), see [8, Theorem 4.3]:

THEOREM 4.3. *For any* $\boldsymbol{\lambda} \in M_c$, *we have*

$$C_1 \beta^2 \langle \widehat{M} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle \leq \langle F_{DP} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle,$$

where β *is the inf-sup constant of the pair* $(\widehat{E}_{I,\Pi}, \overline{P})$ *and* C_1 *is a positive constant that does not depend on any mesh parameters.*

REMARK 4.4. *We note that the condition on the dual velocity unknowns,*

$$J_\Delta \boldsymbol{u}_\Delta \in M_c,$$

i.e.,

$$\sum_{F_{ij}} (\boldsymbol{u}_\Delta^{(i)} - \boldsymbol{u}_\Delta^{(j)})|_{F_{ij}} \cdot \boldsymbol{\mu}_0|_{F_{ij}} = \sum_{F_{ij}} \int_{F_{ij}} (\boldsymbol{u}_\Delta^{(i)} - \boldsymbol{u}_\Delta^{(j)}) \cdot \boldsymbol{n}_{F_{ij}} \, ds = 0,$$

is required for the proof of the lower bound in Theorem 4.3. Here $\boldsymbol{\mu}_0$ *is in* (2.16) *and* $\boldsymbol{n}_{F_{ij}}$ *is the unit normal to the common face* F_{ij}*. The primal velocity unknowns have been chosen so that the dual velocity unknowns satisfy such a requirement.*

4.2. Upper bound analysis. The following result is obtained from a Poincaré inequality, see [14, Lemma 4]:

LEMMA 4.5. *Let* Ω_i *be a three-dimensional subdomain. For any function* $v \in H^1(\Omega_i)$,

$$\|v - c_F\|^2_{L^2(F)} \leq CH |v|^2_{H^1(\Omega_i)},$$

where F *is a face of the subdomain* Ω_i *and* c_F *is given by*

$$c_F = \frac{\int_F I^h(\theta_F v) dx(s)}{\int_F dx(s)}.$$

We note that for $\boldsymbol{u} = (\boldsymbol{u}^{(1)}, \cdots, \boldsymbol{u}^{(N)}) \in \widetilde{X}$, $\boldsymbol{u}$ satisfies the primal constraints so that

$$\int_{F_{ij}} I^h(\theta_{F_{ij}} \boldsymbol{u}^{(i)}) \, dx(s) = \int_{F_{ij}} I^h \left(\theta_{F_{ij}} \boldsymbol{u}^{(j)} \right) \, dx(s),$$

where F_{ij} is the common face of Ω_i and Ω_j.

LEMMA 4.6. *There exists a constant* C *such that*

$$\langle K_{\Delta\Delta} J_\Delta^T J_\Delta \boldsymbol{u}_\Delta, J_\Delta^T J_\Delta \boldsymbol{u}_\Delta \rangle \leq C \frac{H}{h} \langle K \boldsymbol{u}, \boldsymbol{u} \rangle, \quad \textit{for any } \boldsymbol{u} \in \widetilde{X}.$$

Proof. Let $\boldsymbol{w}_\Delta = J_\Delta^T J_\Delta \boldsymbol{u}_\Delta$. We note that

$$\langle K_{\Delta\Delta} J_\Delta^T J_\Delta \boldsymbol{u}_\Delta, J_\Delta^T J_\Delta \boldsymbol{u}_\Delta \rangle = \sum_{i=1}^N |\boldsymbol{w}_\Delta^{(i)}|^2_{H^1(\Omega_i)}.$$

From the inverse inequality and

$$\|\boldsymbol{w}_\Delta^{(i)}\|^2_{L^2(\Omega_i)} \leq Ch \|\boldsymbol{w}_\Delta^{(i)}\|^2_{L^2(\partial\Omega_i)},$$

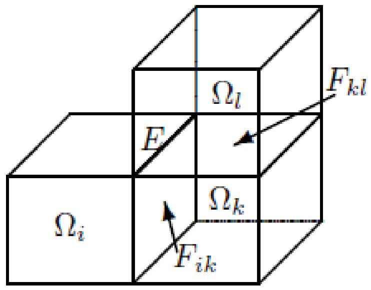

FIG. 3. *Subdomains Ω_i and Ω_l connected through faces F_{ik} and F_{kl} containing the edge E*

we obtain

$$\langle K_{\Delta\Delta} J_\Delta^T J_\Delta \boldsymbol{u}_\Delta, J_\Delta^T J_\Delta \boldsymbol{u}_\Delta \rangle \leq C h^{-1} \sum_{i=1}^{N} \|\boldsymbol{w}_\Delta^{(i)}\|^2_{L^2(\partial\Omega_i)}. \tag{4.8}$$

We note that for $x \in \partial\Omega_i$

$$\boldsymbol{w}_\Delta^{(i)}(x) = \begin{cases} \left(\boldsymbol{u}_\Delta^{(i)}(x) - \boldsymbol{u}_\Delta^{(j)}(x)\right), & \text{when } x \in F_{ij}, \\ \sum_{l \in \mathcal{N}(x)} \left(\boldsymbol{u}_\Delta^{(i)}(x) - \boldsymbol{u}_\Delta^{(l)}(x)\right), & \text{when } x \in E. \end{cases} \tag{4.9}$$

Here F_{ij} is an open face of Ω_i, which is the common part of two subdomains Ω_i and Ω_j, E is an open edge of Ω_i, and $\mathcal{N}(x)$ is the set of subdomain indices sharing the node x. We decompose $\boldsymbol{w}_\Delta^{(i)}$ into

$$\boldsymbol{w}_\Delta^{(i)} = \sum_{F_{ij} \subset \partial\Omega_i} I^h(\theta_{F_{ij}} \boldsymbol{w}_\Delta^{(i)}) + \sum_{E \subset \partial\Omega_i} I^h(\theta_E \boldsymbol{w}_\Delta^{(i)}) \tag{4.10}$$

and then compute each part using the formula in (4.9).

For the first term of the above equation, by Lemma 4.5 we obtain

$$\begin{aligned} \|I^h(\theta_{F_{ij}} \boldsymbol{w}_\Delta^{(i)})\|^2_{L^2(\partial\Omega_i)} &= \|I^h(\theta_{F_{ij}} \boldsymbol{w}_\Delta^{(i)})\|^2_{L^2(F_{ij})} \\ &\leq C \|\boldsymbol{u}_\Delta^{(i)} - \boldsymbol{u}_\Delta^{(j)}\|^2_{L^2(F_{ij})} \\ &= C \|\boldsymbol{u}^{(i)} - \boldsymbol{u}^{(j)}\|^2_{L^2(F_{ij})} \\ &\leq C \left(\|\boldsymbol{u}^{(i)} - \boldsymbol{c}_{F_{ij}}\|^2_{L^2(F_{ij})} + \|\boldsymbol{u}^{(j)} - \boldsymbol{c}_{F_{ij}}\|^2_{L^2(F_{ij})} \right) \\ &\leq CH \left(|\boldsymbol{u}^{(i)}|^2_{H^1(\Omega_i)} + |\boldsymbol{u}^{(j)}|^2_{H^1(\Omega_j)} \right), \end{aligned} \tag{4.11}$$

where

$$\boldsymbol{c}_{F_{ij}} = \frac{\int_{F_{ij}} I^h(\theta_{F_{ij}} \boldsymbol{u}^{(i)})\, dx(s)}{\int_{F_{ij}} dx(s)} = \frac{\int_{F_{ij}} I^h(\theta_{F_{ij}} \boldsymbol{u}^{(j)})\, dx(s)}{\int_{F_{ij}} dx(s)}.$$

The edge term can be bounded by

$$\|I^h(\theta_E \boldsymbol{w}_\Delta^{(i)})\|^2_{L^2(\partial\Omega_i)} \leq Ch \sum_{l \in \mathcal{N}(x)} \|\boldsymbol{u}_\Delta^{(i)}(x) - \boldsymbol{u}_\Delta^{(l)}(x)\|^2_{L^2(E)}. \tag{4.12}$$

For the term related to the edge E, we consider all the subdomain faces sharing the edge E. Among them we select a path from Ω_i to Ω_l, $\{\Omega_i, \Omega_{k_1}, \cdots, \Omega_{k_n}, \Omega_l\}$ which are connected through their common faces, see Figure 3. We note that $\boldsymbol{u}$ has then the same averages across the common faces F. We denote them by $\boldsymbol{c}_F$. For a simple presentation, we assume that the path consists of only three subdomains, i.e., $\{\Omega_i, \Omega_k, \Omega_l\}$. The following assertion can be applied to a more general case without any difficulty.

We then obtain

$$\begin{aligned}
&h\|\boldsymbol{u}_\Delta^{(i)} - \boldsymbol{u}_\Delta^{(l)}\|^2_{L^2(E)} \\
&\leq 2h\|\boldsymbol{u}_\Delta^{(i)} - \boldsymbol{u}_\Delta^{(k)}\|^2_{L^2(E)} + 2h\|\boldsymbol{u}_\Delta^{(k)} - \boldsymbol{u}_\Delta^{(l)}\|^2_{L^2(E)} \\
&\leq C\left(\|\boldsymbol{u}_\Delta^{(i)} - \boldsymbol{u}_\Delta^{(k)}\|^2_{L^2(F_{ik})} + \|\boldsymbol{u}_\Delta^{(k)} - \boldsymbol{u}_\Delta^{(l)}\|^2_{L^2(F_{kl})}\right) \\
&\leq CH\left(|\boldsymbol{u}^{(i)}|^2_{H^1(\Omega_i)} + |\boldsymbol{u}^{(k)}|^2_{H^1(\Omega_k)} + |\boldsymbol{u}^{(l)}|^2_{H^1(\Omega_l)}\right).
\end{aligned} \tag{4.13}$$

Here we have used the fact that both F_{ik} and F_{kl} contain the edge E and the inequality used in (4.11). Combining (4.8) with (4.10)–(4.13), and using

$$\sum_{i=1}^{N} |\boldsymbol{u}^{(i)}|^2_{1,\Omega_i} = \langle K\boldsymbol{u}, \boldsymbol{u}\rangle,$$

the desired bound has been proved. □

The identity in (4.7) combined with Lemma 4.6 gives the following upper bound, see [8, Theorem 4.6]:

THEOREM 4.7. *For any $\boldsymbol{\lambda} \in M_c$, we have*

$$\langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle \leq C_2 \frac{H}{h} \langle \widehat{M}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle,$$

where C_2 is a positive constant that does not depend on any mesh parameters.

5. Numerical results. We consider a model Stokes problem defined in the unit cubic domain $\Omega = [0,1]^3$ with the exact solution

$$\boldsymbol{u} = \boldsymbol{0} \text{ and } p(x,y,z) = xyz - \frac{1}{8}.$$

A prism finite element [11, 15] is used for discretization. For a given mesh size h, the domain is divided uniformly into smaller cubes with its side length h. Each small cube is then divided into eight prisms as in Figure 4. The velocity basis functions are piecewise linear in each prism element and the pressure basis functions are piecewise constant in each cube consisting of the eight prisms.

The domain is then uniformly partitioned into cubical subdomains, which align to the given triangulation in Ω. The notation $Nd = 3^3$ means that the domain Ω is divided by equally spaced three subintervals in each direction and H denotes the length of the subintervals. The notation H/h is used to denote the number of prism elements across each subdomain. The conjugate gradient iteration of the FETI-DP algorithm proceeds until the l^2-norm

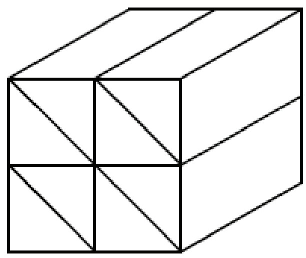

FIG. 4. *Eight prism elements for a cube*

	With preconditioner				Without preconditioner			
Nd	Iter	κ	$\lambda_{\min}$	$\lambda_{\max}$	Iter	κ	$\lambda_{\min}$	$\lambda_{\max}$
3^3	27	22.13	1.417	31.38	96	723.9	0.3132	226.7
4^3	28	23.10	1.452	33.43	100	758.6	0.4219	320.1
5^3	29	24.57	1.405	34.52	102	775.9	0.5321	412.9
6^3	30	25.39	1.384	35.15	103	786.5	0.6415	504.5
7^3	30	25.76	1.380	35.55	104	790.0	0.7535	595.3
8^3	30	25.98	1.378	35.81	104	793.0	0.8643	685.3

TABLE 1

Performance as increase of the number of subdomains Nd with a fixed local problem size ($H/h = 8$) in each subdomain. Iter: the number of iterations, κ: the condition number, λ_{min}: the minimum eigenvalue, and λ_{max}: the maximum eigenvalue.

of the relative residual is reduced by a factor of 10^6. Performance of the method will be tested with respect to the increase of the number of subdomains Nd and the increase of the size of the local problem H/h.

In Table 1, the FETI-DP algorithm is performed by increasing the number of subdomains. Here the size of local problems is fixed with $H/h = 8$. The number of iterations and condition numbers are presented for both the case combined with the lumped preconditioner and the case without employing the preconditioner. The scalability of the method, which does not depend on the number of subdomains, is observed for the both cases and the lumped preconditioner results in considerable reduction on the number of iterations.

In Table 2, we present the convergence behavior depending on the increase of the size of local problems in a given subdomain partition with $Nd = 3^3$. For the case without the preconditioner, both the minimum eigenvalues and the maximum eigenvalues show bad behaviors. The preconditioner dramatically reduces the number of iterations. The number of iterations and condition numbers increase slightly for the case with the lumped preconditioner as the size of local problems increases, which confirms the scalability result of our analytical bound.

	With preconditioner				Without preconditioner			
H/h	Iter	κ	$\lambda_{\min}$	$\lambda_{\max}$	Iter	κ	$\lambda_{\min}$	$\lambda_{\max}$
6	27	20.86	1.344	28.05	80	313.3	0.4524	141.7
8	27	22.13	1.417	31.38	96	723.9	0.3123	226.7
10	29	21.86	1.583	34.61	111	1.387E+3	0.2360	327.4
12	30	23.76	1.586	37.71	125	2.359E+3	0.1892	446.4
14	31	25.71	1.583	40.72	138	3.696E+3	0.1577	582.8
16	33	27.96	1.561	43.66	150	5.428E+3	0.1356	736.1

TABLE 2

Performance as increase of the local problem size H/h in a fixed subdomain partition with $Nd = 3^3$. Iter: the number of iterations, κ: the condition number, λ_{min}: the minimum eigenvalue, and λ_{max}: the maximum eigenvalue.

In our FETI-DP formulation, we are allowed to select primal unknowns which are based on common faces and produce the dual velocity unknowns $\boldsymbol{u}_\triangle$ satisfying $J_\triangle \boldsymbol{u}_\triangle \in M_c$; see Remarks 2.1 and 4.4. We selected the primal unknowns that are velocity unknowns at subdomain corners and the averages of each velocity component over common faces. Even though this set of primal constraints reduces a considerable amount of primal unknowns required in the previous approaches [12, 13], we observed that our coarse problem is a bottleneck of the computation as the number of subdomains gets larger. We test our algorithm for a smaller set of primal unknowns by selecting the primal unknowns which are averages of the normal velocity component over common faces, instead of averages of each velocity component. In a more detail, we select

$$\int_{F_{ij}} \boldsymbol{v}^{(i)} \cdot \boldsymbol{n}_{ij}\, ds = \int_{F_{ij}} \boldsymbol{v}^{(j)} \cdot \boldsymbol{n}_{ij}\, ds$$

as the primal unknowns, where F_{ij} is the common face of two subdomains Ω_i and Ω_j, and $\boldsymbol{n}_{ij}$ is the unit normal to the face F_{ij}.

In Table 3, the performance of the algorithm with the smaller set of primal unknowns is presented regarding to the number of subdomains. In contrast to the case with the larger set of primal unknowns, the number of iterations and the condition numbers increase as the number of subdomains increases. We note that this smaller set of primal unknowns is not enough to resolve all the rigid body motions of the Stokes problem in three dimensions; see [13, Section 7]. For this case, we still have stable behavior of the minimum eigenvalues, since such primal unknowns give $\boldsymbol{u}_\triangle$ satisfying the requirement of the lower bound analysis discussed in Remark 4.4. When Nd gets larger than 8^3 with the larger set of primal unknowns, we observed that computational cost is mostly done on solving the coarse problem of which size is larger than 5000. This fact makes the algorithm with the larger set of primal unknowns less efficient, even though the larger set of primal unknowns gives less number of iterations.

	Corners and face averages					Corners and face normals				
Nd	Iter	κ	$\lambda_{\min}$	$\lambda_{\max}$	coarse dofs.	Iter	κ	$\lambda_{\min}$	$\lambda_{\max}$	coarse dofs.
3^3	27	22.13	1.417	31.38	186	33	40.98	1.587	65.06	78
4^3	28	23.10	1.452	33.43	513	42	75.92	1.463	111.0	225
5^3	29	24.57	1.405	34.52	1,092	54	103.8	1.475	153.2	492
6^3	30	25.39	1.384	35.15	1,995	61	126.8	1.492	189.3	915
7^3	30	25.76	1.380	35.55	3,294	67	149.0	1.463	218.0	1,530
8^3	30	25.98	1.378	35.81	5,061	73	165.5	1.455	240.9	2,373
9^3	30	26.19	1.374	35.99	7,368	76	176.9	1.463	258.9	3,480
10^3	30	26.32	1.372	36.13	10,287	79	187.1	1.461	273.3	4,887

TABLE 3

Performance as increase of the number of subdomains Nd with a fixed local problem size ($H/h = 8$). Iter: the number of iterations, κ: the condition number, λ_{min}: the minimum eigenvalue, λ_{max}: the maximum eigenvalue, and coarse dofs.: the number of primal unknowns.

In Table 4, the scalability of the method with the smaller set of primal unknowns is tested for the size of the local problems H/h. Here the domain is divided uniformly into 3^3 subdomains. The minimum eigenvalues present stable behavior and the maximum eigenvalues increase depending on the size of local problems. Compared to the case with the larger set of primal unknowns, we observe a modest increase on the number iterations. In Figure 5, the maximum eigenvalues versus the size of local problems are plotted for the two cases. The case with the larger set of primal unknowns fits to the straight line $y = x$, which agrees well to our theoretical bound. In fact, the result shows that our bound is sharp. For the case with the smaller set of primal unknowns, the maximum eigenvalues fit between $y = x$ and $y = x^2$ and are much closer to $y = x$. We can see that the smaller choice of primal unknowns provides relatively good scalability, of which bound is closer to $C(H/h)$, depending on the size of local problems. We note that for the three-dimensional elliptic problems the case with primal unknowns only at subdomain corners gives the bound of maximum eigenvalues $C(H/h)^2$ for the FETI-DP algorithm with a lumped preconditioner; see [14, 17].

In our last numerical experiment, we select only the velocity unknowns at subdomain corners as the primal unknowns, which gives the smallest possible set of primal unknowns in three-dimensional problems. In Table 5, the performance of the method is presented. Stable behaviors of the number of iterations and condition numbers are observed regarding to the number of subdomains. However quite a large number of iterations is required than in the previous two choices of primal unknowns. As the size of local problems gets larger, we observe substantial increases in the number of iterations and the condition numbers. In a more detail, both the minimum eigenvalue and the maximum eigenvalue are affected by the

	Corners and face averages				Corners and face normals			
H/h	Iter	κ	$\lambda_{\min}$	$\lambda_{\max}$	Iter	κ	$\lambda_{\min}$	$\lambda_{\max}$
6	27	20.86	1.344	28.05	31	38.62	1.361	52.58
8	27	22.13	1.417	31.38	33	40.98	1.587	65.05
10	29	21.86	1.583	34.61	35	46.72	1.644	76.81
12	30	23.76	1.586	37.71	37	56.88	1.552	88.31
14	31	25.71	1.583	40.72	38	61.68	1.615	99.63
16	33	27.96	1.561	43.66	40	69.10	1.604	110.8

TABLE 4

Performance as increase of the local problem size H/h in a fixed subdomain partition with $Nd = 3^3$. Iter: the number of iterations, κ: the condition number, λ_{min}: the minimum eigenvalue, and λ_{max}: the maximum eigenvalue.

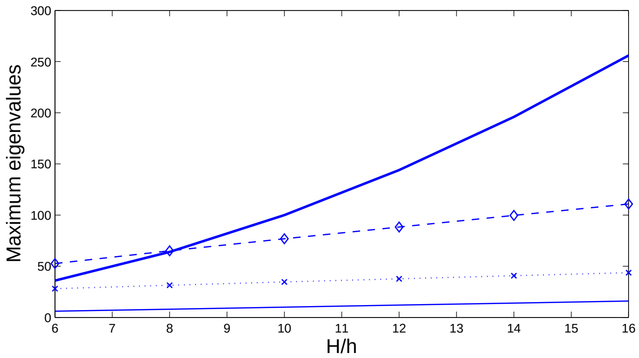

FIG. 5. *Plot of estimated maximum eigenvalues with respect to the size of local problems H/h; dotted line (with primal unknowns at subdomain corners and the face averages), dashed line (with primal unknowns at subdomain corners and the face normals), solid line ($y = x$), and thick solid line ($y = x^2$).*

local problem size. We note that such selection of primal unknowns dose not satisfy the requirement, $J_\triangle \boldsymbol{u}_\triangle \in M_c$, for the lower bound analysis. The minimum eigenvalue decreases as more prism elements are introduced for the local problems. The behavior of the maximum eigenvalues is presented in Figure 6, which follows the quadratic function of the local problem size.

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Corners only									
Nd $(\frac{H}{h}=8)$	Iter	κ	$\lambda_{\min}$	$\lambda_{\max}$	H/h $(Nd=3^3)$	Iter	κ	$\lambda_{\min}$	$\lambda_{\max}$
3^3	61	247.3	0.982	242.9	6	49	118.6	1.1252	144.1
4^3	84	273.2	1.060	289.7	8	61	247.3	0.982	242.9
5^3	93	283.2	1.103	312.6	10	73	432.0	0.836	361.4
7^3	97	290.4	1.434	332.0	12	82	697.6	0.722	503.9
9^3	100	296.5	1.147	340.1	14	93	1.060E+3	0.633	671.6
10^3	101	303.9	1.126	342.4	16	102	1.525E+3	0.563	859.3

TABLE 5

Performance of the method with the primal velocity unknowns only at subdomain corners. H/h: the number of elements across each subdomain, Nd: the number of subdomains, Iter: the number of iterations, κ: the condition number, λ_{min}: the minimum eigenvalue, and λ_{max}: the maximum eigenvalue.

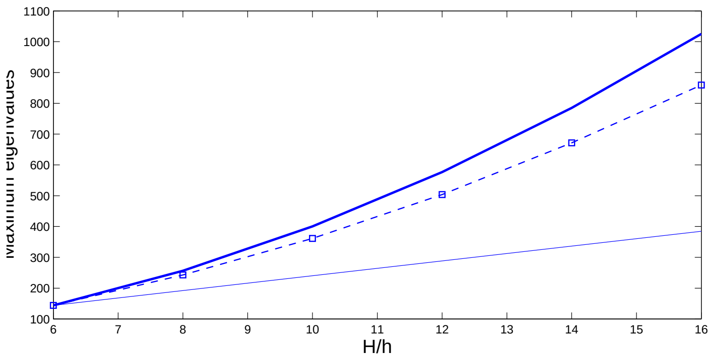

FIG. 6. *Plot of estimated maximum eigenvalues with respect to the size of local problems H/h; dashed line (with primal unknowns at subdomain corners), solid line ($y = Cx$), and thick solid line ($y = Cx^2$).*

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