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Existence and Exponential Decay of Solutions to a Class of Nonlinear Viscoelastic Equations

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Abstract

In this paper we consider a class of nonlinear viscoelastic equations, in which no dissipation mechanism is present, except for the convolution term accounting for the past memory of the variable. We prove the global existence of weak solutions and energy decay. The existence is proved by means of Faedo-Galerkin approximation method and asymptotic behavior is obtained by making use of the perturbed energy technique.

1 INTRODUCTION

Let Ω be a bounded open domain in $\mathbb{R}^n$ with smooth boundary Γ, ρ is a positive real number. In this paper, we consider the following nonlinear viscoelastic equation

$$\begin{aligned} &|u_t|^\rho u_{tt} - \Delta u + \int_0^t \mu(t-s)\Delta u(s)ds + g(u) = 0, \forall (x,t) \in \Omega \times \mathbb{R}^+; \\ &u(x,t) = 0, \qquad \forall (x,t) \in \Gamma \times \mathbb{R}^+; \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \qquad \forall x \in \Omega. \end{aligned} \tag{1.1}$$

Problems related to the equations are interesting not only from the point of view of PDE general theory, but also due to its applications in Mechanics. Problem (1.1) results from the mathematical description of small amplitude vibrations of an elastic string [7]. When $\rho = 0$, the material density is equal to 1, the equation describes the extensional vibrations of thin rods. Among the numerous works in this direction, we can cite Cavalcant et al [2, 3]. Cavalcanti et al [2] studied the equation

$$u_{tt} - \Delta u + \int_0^t \mu(t-s)\Delta u(s)ds + a(x)u_t + |u|^\gamma u = 0, \ \text{ in } \ \Omega \times \mathbb{R}^+.$$

Under the condition that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometry restrictions, the authors obtained an exponential rate of decay. In the same direction, Cavalcanti et al [3] have also studied, in a bounded domain, the equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t \mu(t-s)\Delta u(s)ds - \gamma \Delta u_t = 0.$$

They obtained the uniform decay rats of the energy assuming a strong damping $\triangle u_t$ acting in the domain. However, when the damping mechanism is given by the memory term, it is possible to prove that the exponential decay holds.

The purpose of this paper is to obtain the existence of global solutions and an exponential decay rate of the solutions for the equation (1.1) motivated by [1]. Our paper is organized as follows: In Section 2, we give some notations, assumptions and our main result. In Section 3 we obtain a global existence for global solutions and in Section 4 we derive the uniform decay of the energy.

2 PRELIMINARIES AND MAIN RESULT

Throughout this paper, we will consider the standard spaces $L^q(\Omega)$ with norm $\|u\|_q$, and we denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm on $L^2(\Omega)$. The symbols ∇ and $\triangle$ will stand for the gradient and the Laplacian respectively. The prime and the subscript t will denote time differentiation.

We will also consider the Sobolev space $H_0^1(\Omega)$, with the inner product $(\nabla u, \nabla v)$ and the norm $\|u\|_{H_0^1(\Omega)} = (\nabla u, \nabla v)$, and we will use the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $2 \le q \le 2n/(n-2)$ if $n \ge 3$ or $q \ge 2$ if $n = 1, 2$. C_q denote the embedding constant; i.e.

$$\|u\|_q \le C_q \|\nabla u\|. \tag{2.1}$$

First, we give the precise assumptions on the memory term μ and the nonlinear term g.

(h1) $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1-function such that $\mu(0) > 0$ and $1 - \int_0^\infty \mu(t-s)ds = l > 0$;

(h2) There exists positive constants ξ, ξ_1 such that $\xi\mu(t) \le -\mu'(t) \le \xi_1 \mu(t)$, for $t \ge 0$;

(g1) Let $g : \mathbb{R} \to \mathbb{R}$ be a non-decreasing C^1-function such that $g(0) = 0$;

(g2) For some $\gamma \ge 0$, there exists a constant $c > 0$, such that $|g'(s)| \le c(1 + |s|^\gamma)$.

Let $G(s) = \int_0^s g(t)dt$, then we infer from (g1), (g2) that, there exists a constant $c_1 > 0$ such that

$$|g(s)| \le c_1(1 + |s|^{\gamma+1}); \tag{2.2}$$

$$0 \le G(s) \le g(s)s. \tag{2.3}$$

Next, we introduce the energy

$$E(t) = \frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{1}{2}\big(1 - \int_0^t \mu(t-s)ds\big)\|\nabla u(t)\|^2 + \int_\Omega G(u)dx + \frac{1}{2}(\mu \circ \nabla u)(t),$$

where

$$(\mu \circ u)(t) = \int_0^t \mu(t-s)\|u(t) - u(s)\|^2 ds.$$

Finally, we state our main result.

Theorem 2.1. *Assume that μ and g satisfy the hypotheses (h1),(h2); (g1),(g2)respectively, and γ satisfies*

$$\begin{aligned} &0 \le \gamma, \rho && \textit{when } n = 1, 2; \\ &0 \le \gamma, \rho \le \tfrac{4}{n-2}, && \textit{when } n \ge 3 \end{aligned} \tag{2.4}$$

then for given $u_0 \in H_0^1(\Omega), u_1 \in L^{\rho+2}(\Omega)$, there exists a global solution to equations (1.1) such that

$$u \in L^\infty([0,\infty); H_0^1(\Omega)); \qquad u_t \in L^\infty([0,\infty); L^{\rho+2}(\Omega)). \tag{2.5}$$

Furthermore, there exist positive constants t_0, k and K, such that the solution given by (2.4) satisfies $E(t) \le Ke^{-kt}, \quad \forall t \ge t_0$.

3 EXISTENCE OF SOLUTIONS

In this section we are going to obtain the existence of global weak solutions to the problem (1.1) using Faedo-Galerkin's approximation. For this end we represent by $\{\omega_\nu | \nu = 1,2,...\}$ a orthonormal basis of $H_0^1(\Omega)$, by V_m the subspace of $H_0^1(\Omega)$ generated by the first m vectors $\omega_1, \omega_2, \cdots, \omega_m$, and we define

$$u_m(t) = \sum_{i=1}^m \xi_{im}(t)\omega_i,$$

where u_m is the solution to the following Cauchy problem:

$$\begin{aligned}
&(|u_m'(t)|^\rho u_m''(t), \omega) + (\nabla u_m(t), \nabla \omega) - \int_0^t \mu(t-s)(\nabla u_m(s), \nabla\omega)ds \\
&+(g(u_m), \omega) = 0; \qquad \forall \omega \in V_m \\
&u_m(0) = u_{0m} \to u_0(\text{in} \quad H_0^1(\Omega)); \qquad u_m'(0) = u_{1m} \to u_1(\text{in} \quad L^2(\Omega)).
\end{aligned} \tag{3.1}$$

The approximate system (3.1) is a normal one of differential equations which has a solution in $[0,T)$ for some $T > 0$. The extension to the whole interval $[0,\infty)$ is a consequence of the following estimate which we are going to prove below.

A simple computation gives us

$$\begin{aligned}
&\int_0^t \mu(t-s)(\nabla u_m'(t), \nabla u_m(s))ds \\
&= \frac{1}{2}(\mu' \circ \nabla u_m)(t) - \frac{1}{2}\frac{d}{dt}(\mu \circ \nabla u_m)(t) + \frac{1}{2}\frac{d}{dt}\|\nabla u_m(t)\|^2 \int_0^t \mu(s)ds;
\end{aligned} \tag{3.2}$$

$$\frac{d}{dt}\Big(\int_0^t \mu(s)ds\|\nabla u_m(t)\|^2\Big) = \mu(t)\|\nabla u_m(t)\|^2 + \int_0^t \mu(t-s)\frac{d}{dt}\|\nabla u_m(t)\|^2 ds. \tag{3.3}$$

Considering $\omega = u_m'(t)$ in (3.1), and combining with (3.2) and (3.3), we find

$$\frac{d}{dt}E_m(t) = \frac{1}{2}(\mu' \circ \nabla u_m)(t) - \frac{1}{2}\mu(t)\|\nabla u_m(t)\|^2 \le 0. \tag{3.4}$$

Where

$$E_m(t) = \frac{1}{\rho+2}\|u_m'(t)\|_{\rho+2}^{\rho+2} + \frac{1}{2}\big(1 - \int_0^t \mu(t-s)ds\big)\|\nabla u_m(t)\|^2 + \int_\Omega G(u_m)dx + \frac{1}{2}(\mu \circ \nabla u_m)(t).$$

Using (2.2), (2.3) and the two inequalities $\|u_m'(0)\|_{\rho+2} \le \|u_1\|_{\rho+2}$, $\| \nabla u_m(0)\| \le \| \nabla u_0\|$, we obtain

$$\int_\Omega G(u_m(0))dx \le c_1 + c_2\|u_0\|_{\gamma+2}^{\gamma+2}, \quad c_1, c_2 > 0.$$

Therefore, from (3.4) for $\forall t \in [0,T)$ we obtain

$$E_m(t) \le E_m(0) \le L_1, \tag{3.5}$$

where L_1 is a positive constant independent of m. So we obtain

$$\begin{aligned} u_m & \quad \text{bounded in} \quad L^\infty([0,T]; H_0^1(\Omega)); \\ u'_m & \quad \text{bounded in} \quad L^\infty([0,T]; L^{\rho+2}(\Omega)). \end{aligned}$$

From the energy estimate and (2.1), we deduce

$$|u'_\mu|^\rho u'_\mu \ \text{ bounded in } \ L^2([0,T]; H^{-1}(\Omega)). \tag{3.6}$$

By the compactness of weak topology, there exists a subsequence u_μ of u_m and a function u such that

$$u_\mu \to u \quad \text{weak star in} \quad L^\infty([0,T]; H_0^1(\Omega)); \tag{3.7}$$
$$u'_\mu \to u' \quad \text{weak star in} \quad L^\infty([0,T]; L^{\rho+2}(\Omega)); \tag{3.8}$$
$$|u'_\mu| u'_\mu \to |u'| u' \quad \text{weak star in} \quad L^2([0,T]; H^{-1}(\Omega)). \tag{3.9}$$

On the other hand, from Aubin-Lions theorem, we deduce that there exists a subsequence of u_μ, still represented by the same notation, such that

$$u_\mu \to u \ \text{ a.e. in } \ \Omega \times [0,T). \tag{3.10}$$

Applying (g1), (g2) and (2.3), we deduce that $g(u)$ is bounded. Consequently, by (3.10) we obtain

$$g(u_\mu) \to g(u) \text{ a.e. in } \ \Omega \times [0,T). \tag{3.11}$$

Multiplying (3.1) by $\theta \in D(0,T)$(here D(0,T) means the space of functions in C^∞ with compact support in (0,T)) and integrating the obtained result over (0,T), we infer

$$\begin{aligned} &-\frac{1}{\rho+1}\int_0^T (|u'_m(t)|^\rho u'_m(t), \omega)\theta'(t)dt + \int_0^T (\nabla u_m(t), \nabla\omega)\theta(t)dt \\ &-\int_0^T\int_0^t \mu(t-s)(\nabla u_m(s), \nabla\omega)\theta(t)dsdt + \int_0^T (g(u_m)dt, \omega)\theta(t)dt = 0. \end{aligned} \tag{3.12}$$

Convergences (3.7)-(3.9) and (3.11) are sufficient to pass to the limit in (3.12), in order to obtain

$$|u_t|^\rho u_{tt} - \Delta u + \int_0^t \mu(t-s)\Delta u(s)ds + g(u) = 0 \ \text{ in } \ L^2_{loc}((0,\infty); H^{-1}(\Omega)).$$

4 EXPONENTIAL DECAY

In this section we prove the exponential decay for regular solutions u_m of problem (3.1), and by using its limit we also can extend the same result to the weak solutions. For convenience we denote u_m by u.

We define the modified energy as

$$F(t) = E(t) + \varepsilon_1\phi(t) + \varepsilon_2\chi(t), \tag{4.1}$$

where ε_1 and ε_2 are positive constants to be specified later and

$$\begin{aligned} \phi(t) &= \frac{1}{\rho+1}\int_\Omega |u_t(t)|^\rho u_t(t)u(t)dx, \\ \chi(t) &= -\int_\Omega |u_t(t)|^\rho u_t(t)\big(\int_0^t \mu(t-s)(u(t)-u(s))ds\big)dx. \end{aligned}$$

From (2.1) and (3.5), we deduce

$$\|u(t)\|_{\rho+2}^{\rho+2} \le C_\rho^{\rho+2}\|\nabla u(t)\|^{\rho+2} \quad \le \quad C_\rho^{\rho+2}\big(\frac{2}{l}\big)^{\frac{\rho+2}{2}} E(0)^{\frac{\rho}{2}} E(t).$$

Therefore, using Young's inequality and $\int_0^t \mu(s)ds \le \int_0^\infty \mu(s)ds = 1-l$, we obtain

$$\begin{aligned}
|\phi(t)| &\le \frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{1}{(\rho+1)(\rho+2)}\|u(t)\|_{\rho+2}^{\rho+2} \le C_1 E(t);\\
|\chi(t)| &\le \int_0^t \mu(t-s)\|\|u_t(t)\|_{\rho+2}^{\rho+1}\|u(t)-u(s)\|_{\rho+2} ds\\
&\le \frac{1-l}{2}\|u_t(t)\|_{\rho+2}^{2\rho+2} + \frac{1}{2}\int_0^t \mu(t-s)\|u(t)-u(s)\|_{\rho+2}^2 ds\\
&\le \frac{1-l}{2}((\rho+2)E(0))^{\frac{\rho}{\rho+2}}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{C_\rho^2}{2}\int_0^t \mu(t-s)\|\nabla u(t)-\nabla u(s)\|_{\rho+2}^2 ds\\
&\le C_2 E(t),
\end{aligned}$$

It is straightforward to see that for ε_1 and ε_2 small enough, the inequality

$$\alpha_1 F(t) \le E(t) \le \alpha_2 F(t) \tag{4.2}$$

holds for two positive constants $\alpha_1 > 0$ and $\alpha_2 > 0$. Consequently, the uniformdecay of $E(t)$ is a consequence of the decay of $F(t)$.

In the following we estimate $F'(t)$. Using equation (3.1), it is easy to see that

$$\phi'(t) \;=\; \frac{1}{\rho+1}\|u_t(t)\|_{\rho+2}^{\rho+2} - \|\nabla u(t)\|^2 + \int_\Omega \nabla u(t)\int_0^t \mu(t-s)\nabla u(s)dsdx - (g(u),u). \tag{4.3}$$

We now estimate the third term in the right-hand side of (4.3). Using Cauchy-Schwarz and Young's inequality, for any $\eta > 0$,

$$\begin{aligned}
&\int_\Omega \nabla u(t)\int_0^t \mu(t-s)\nabla u(s)dsdx\\
\le\;& \frac{1}{2}\int_\Omega |\nabla u(t)|^2dx + \frac{1}{2}\int_\Omega \Big(\int_0^t \mu(t-s)(|\nabla u(s)-\nabla u(t)| + |\nabla u(t)|)ds\Big)^2 dx\\
=\;& \frac{1}{2}\int_\Omega |\nabla u(t)|^2dx + \int_\Omega \Big(\int_0^t \mu(t-s)|\nabla u(s)-\nabla u(t)|ds\Big)^2 dx + \int_\Omega \Big(\int_0^t \mu(t-s)|\nabla u(t)|ds\Big)^2 dx\\
&+2\int_\Omega \Big(\int_0^t \mu(t-s)|\nabla u(s)-\nabla u(t)|ds\Big)\Big(\int_0^t \mu(t-s)|\nabla u(t)|ds\Big)dx\\
\le\;& \frac{1}{2}\int_\Omega |\nabla u(t)|^2dx + (1+\eta)\int_\Omega \Big(\int_0^t \mu(t-s)|\nabla u(t)|ds\Big)^2 dx\\
&+(1+\frac{1}{\eta})\int_\Omega \Big(\int_0^t \mu(t-s)|\nabla u(s)-\nabla u(t)|ds\Big)^2 dx\\
\le\;& \frac{1}{2}(1+(1+\eta)(1-l)^2)\|\nabla u(t)\|^2\\
&+\frac{1}{2}(1+\frac{1}{\eta})(1-l)\int_\Omega\int_0^t \mu(t-s)|\nabla u(s)-\nabla u(t)|^2dsdx.
\end{aligned} \tag{4.4}$$

Using (2.3), we estimate the fourth term of the right-hand side of (4.3) as follows:

$$-(g(u),u) \le -\int_\Omega G(u)dx. \tag{4.5}$$

Combining (4.4), (4.5) with (4.3) and choosing $\eta = l/(1-l)$, we have

$$\phi'(t) \leq \frac{1}{\rho+1}\|u_t(t)\|_{\rho+2}^{\rho+2} - \frac{l}{2}\|\nabla u(t)\|^2 + \frac{1-l}{2l}(\mu \circ \nabla u)(t) - \int_\Omega G(u)dx. \tag{4.6}$$

Now we estimate

$$\begin{aligned}\chi'(t) &= (\rho+1)\{\int_\Omega \nabla u(t)\int_0^t \mu(t-s)(\nabla u(t)-\nabla u(s))dsdx \\ &\quad - \int_\Omega \big(\int_0^t \mu(t-s)\nabla u(s)ds\big)\big(\int_0^t \mu(t-s)(\nabla u(t)-\nabla u(s))ds\big)dx \\ &\quad + \int_\Omega g(u)\int_0^t \mu(t-s)(u(t)-u(s))dsdx\} \\ &\quad - \int_\Omega |u_t(t)|^\rho u_t(t)\int_0^t \mu'(t-s)(u(t)-u(s))dsdx - \int_0^t \mu(s)ds\int_\Omega |u_t(t)|^{\rho+2}dx. \end{aligned}\tag{4.7}$$

Similar to (4.3), we estimate the right-hand side terms of the inequality above. Using (2.2), for $\delta > 0$, we have: For the first term,

$$\begin{aligned}&(\rho+1)\int_\Omega \nabla u(t)\int_0^t \mu(t-s)(\nabla u(t)-\nabla u(s))dsdx \\ \leq\ &(\rho+1)\{\delta\int_\Omega |\nabla u(t)|^2dx + \frac{1-l}{4\delta}(\mu\circ\nabla u)(t)\}.\end{aligned}\tag{4.8}$$

For the second term,

$$\begin{aligned}&-(\rho+1)\int_\Omega \big(\int_0^t \mu(t-s)\nabla u(s)ds\big)\big(\int_0^t \mu(t-s)(\nabla u(t)-\nabla u(s))ds\big)dx \\ \leq\ &(\rho+1)\{\delta\int_0^t \mu(t-s)\|\nabla u(s)\|^2ds + \frac{1}{4\delta}\int_0^t \mu(t-s)\|\nabla u(t)-\nabla u(s)\|^2ds\} \\ \leq\ &(\rho+1)\{(2\delta+\frac{1}{4\delta})(1-l)(\mu\circ\nabla u)(t) + 2\delta(1-l)^2\int_\Omega |\nabla u(t)|^2dx\}.\end{aligned}\tag{4.9}$$

Similarly, for the third term,

$$\begin{aligned}&(\rho+1)\int_\Omega g(u)\int_0^t \mu(t-s)(u(t)-u(s))dsdx \\ \leq\ &(\rho+1)\{\delta(1-l)\big(\frac{2E(0)}{l}\big)^\gamma C_\gamma^{2\gamma+2}\|\nabla u(t)\|^2 + \frac{C_\gamma^2}{4\delta}(\mu\circ\nabla u)(t)\},\end{aligned}\tag{4.10}$$

and for the fourth term,

$$\begin{aligned}&-\int_\Omega |u_t(t)|^\rho u_t(t)\int_0^t \mu'(t-s)(u(t)-u(s))dsdx \\ \leq\ &\xi_1\{\delta(1-l)((\rho+2)E(0))^{\frac{\rho}{\rho+2}}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{C_\rho^2}{4\delta}(\mu\circ\nabla u)(t)\}.\end{aligned}\tag{4.11}$$

Combining (4.8)-(4.11) yields

$$\begin{aligned}\chi'(t) \leq\ &\delta(\rho+1)\{1+2(1-l)^2+(1-l)\big(\frac{2E(0)}{l}\big)^\gamma C_\gamma^{2\gamma+2}\}\|\nabla u(t)\|^2 \\ &+\{\big((\rho+1)\frac{(1-l)}{4\delta} + (\rho+1)(2\delta+\frac{1}{4\delta})(1-l) + \frac{C_\gamma^2}{4\delta}(\rho+1) + \frac{\xi_1 C_\rho^2}{4\delta}\}(\mu\circ\nabla u)(t) \\ &+\{\xi_1\delta(1-l)((\rho+2)E(0))^{\frac{\rho}{\rho+2}} - \int_0^t \mu(s)ds\}\|u_t(t)\|_{\rho+2}^{\rho+2}.\end{aligned}\tag{4.12}$$

Since $\mu(0) > 0$, then there exists $t_0 > 0$ such that

$$\int_0^t \mu(s)ds \geq \int_0^{t_0} \mu(s)ds = \mu_0, \quad \forall t \geq t_0. \tag{4.13}$$

Using (4.1), (4.6), (4.12) and (4.13), we obtain

$$\begin{aligned} F'(t) \leq & -\{\varepsilon_2(\mu_0 - \xi_1\delta(1-l)((\rho+2)E(0))^{\frac{\rho}{\rho+2}}) - \frac{\varepsilon_1}{\rho+1}\}\|u_t(t)\|_{\rho+2}^{\rho+2} \\ & -\{\frac{1}{2}\mu(t) + \frac{\varepsilon_1 l}{2} - \varepsilon_2\delta(\rho+1)\big(1+2(1-l)^2+(1-l)(\frac{2E(0)}{l})^\gamma C_\gamma^{2\gamma+2}\big)\}\|\nabla u(t)\|^2 \\ & -\varepsilon_1\int_\Omega G(u)dx \\ & -\{\frac{\xi}{2} - \frac{\varepsilon_1(1-l)}{2l} - \varepsilon_2\big((\rho+1)\frac{(1-l)}{4\delta} + (\rho+1)(2\delta + \frac{1}{4\delta}(1-l) \\ & + \frac{C_\gamma^2}{4\delta}(\rho+1) + \frac{\xi_1 C_\rho^2}{4\delta}\big)\}(\mu\circ\nabla u)(t). \end{aligned} \tag{4.14}$$

Now, we choose δ so small that

$$\mu_0 - \delta\xi_1(1-l)((\rho+2)E(0))^{\frac{\rho}{\rho+2}} > \frac{\mu_0}{2(\rho+1)}; \quad \delta(\rho+1)\{(1+2(1-l)^2+(1-l)(\frac{2E(0)}{l})^\gamma C_\gamma^{2\gamma+2}\} < \frac{l\mu_0}{16}.$$

When δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{\mu_0}{4}\varepsilon_2 = \varepsilon_1 \tag{4.15}$$

will make

$$k_1 = \varepsilon_2(\mu_0 - \xi_1\delta(1-l)((\rho+2)E(0))^{\frac{\rho}{\rho+2}}) - \frac{\varepsilon_1}{\rho+1} > 0;$$

$$k_2 = \frac{\varepsilon_1 l}{2} - \varepsilon_2\delta(\rho+1)\big(1+2(1-l)^2+(1-l)(\frac{2E(0)}{l})^\gamma\big) > 0.$$

We then pick ε_1 and ε_2 so small that (4.2) and (4.15) remain valid and

$$\frac{\xi}{2} - \frac{\varepsilon_1(1-l)}{2l} - \varepsilon_2\big((\rho+1)\frac{(1-l)}{4\delta} + (\rho+1)(2\delta + \frac{1}{4\delta}(1-l) + \frac{C_\gamma^2}{4\delta}(\rho+1) + \frac{\xi_1 C_\rho^2}{4\delta}\big) > 0.$$

Let $\beta > 0$ be the minimum of the right-hand side terms' coefficients of equality (4.14). Then we arrive at $F'(t) \leq -\beta E(t)$ for all $t \geq t_0$. This inequality and (4.2) yield

$$F'(t) \leq -\beta\alpha_1 F(t), \forall t \geq t_0.$$

A simple integration leads to $F(t) \leq F(t_0)e^{\beta\alpha_1(t_0-t)}, \forall t \geq t_0$. This inequality and (4.3) lead to

$$E(t) \leq \alpha_2 F(t_0)e^{\beta\alpha_1(t_0-t)}, \forall t \geq t_0.$$

This concludes the proof of Theorem 2.1.

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