On the selection of primal unknowns for a FETI-DP formulation of the Stokes problem

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ON THE SELECTION OF PRIMAL UNKNOWNS FOR A FETI-DP FORMULATION OF THE STOKES PROBLEM *

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Abstract. Selection of primal unknowns is important in the convergence of FETI-DP (dual-primal finite element tearing and interconnecting) methods, which are known to be the most scalable dual iterative substructuring methods. A FETI-DP algorithm for the Stokes problem without primal pressure unknowns was developed and analyzed by the authors [5]. Only the velocity unknowns at the subdomain vertices are selected to be the primal unknowns and convergence of the algorithm with a lumped preconditioner is determined by the condition number bound $C(H/h)(1+\log(H/h))$, where H/h is the number of elements across subdomains. In this work, primal unknowns corresponding to the averages on edges are introduced and a better condition number bound $C(H/h)$ is proved for such a selection of primal unknowns. Numerical results are included.

Key words. FETI-DP, Stokes problem, lumped preconditioner

AMS subject classifications. 65N30, 65N55, 76D07

1. Introduction. A FETI-DP algorithm for the Stokes problem without primal pressure unknowns was developed by the authors in [5]. It belongs to a family of dual iterative substructuring methods, see [1, 2, 3, 4, 6, 7, 8, 10]. A pair of inf-sup stable velocity and pressure finite element spaces is given for a triangulation of the domain and unknowns in the finite element spaces are decoupled across subdomain interfaces by introducing a partition of the given domain into many smaller subdomains. Among the decoupled unknowns, some important unknowns are selected to be primal unknowns. A strong continuity will be enforced to the primal unknowns and at the remaining part of unknowns the continuity will be imposed weakly by using Lagrange multipliers. After elimination of the unknowns other than the Lagrange multipliers, a system on the dual unknowns, i.e., the Lagrange multipliers, is obtained and it is solved iteratively with a preconditioner that accelerates the convergence of the iteration.

In the previous approaches for the Stokes problem [4, 9, 10, 11, 14], the coarse space of domain decomposition algorithms consists of both the velocity and pressure basis elements. These algorithms require a certain inf-sup stability of the coarse space, which results in the use of a relatively large number of velocity basis elements. In FETI-DP algorithms, the primal unknowns are related to coarse basis elements. Differently to the previously developed

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algorithms, in the work [5] by the authors no primal pressure unknowns are selected and only the velocity unknowns at the subdomain vertices are selected as the primal unknowns in the FETI-DP formulation. Such a selection of primal unknowns gives a symmetric and positive-definite coarse problem with its size smaller than those appeared in the previous approaches. This leads to a more efficient FETI-DP algorithm, which allows the use of a more practical lumped preconditioner and a more practical solver for the coarse problem.

The FETI-DP algorithm in [5] can be considered as an extension of the work in [13] to the Stokes problem. In [13], FETI-DP algorithms with various selections of the primal unknowns are introduced and analyzed for elliptic problems combined with inexact local solvers. From these results, we observed that in the two-dimensional elliptic problems the selection of primal unknowns based on averages over common edges gives a better convergence of the algorithm compared to the choice based on the subdomain vertices.

Motivated by this observation, for the two-dimensional Stokes problem we consider a different set of primal unknowns that are averages of velocity unknowns over subdomain edges, which are common part of two subdomains. The primal unknowns from the velocity values at subdomain vertices, which have been selected in our previous work [5], result in a FETI-DP algorithm with its condition number bound $C(H/h)(1+\log(H/h))$, where H/h is the number of elements across subdomains. We will prove that selection of the primal unknowns, which are averages of velocity unknowns on edges, gives a better condition number bound $C(H/h)$ as in [13].

This paper is organized as follows. In section 2, the FETI-DP algorithm without primal pressure unknowns is introduced and in section 3, this algorithm is described for the choice of primal velocity unknowns based on averages over common edges and a bound of its condition number is analyzed. In section 4, numerical results are presented to confirm the obtained bound and to compare two different choices of primal unknowns. Throughout this paper, C denotes a generic positive constant which does not depend on any mesh parameters and the number of subdomains.

2. A FETI-DP algorithm without primal pressure unknowns. We recall the FETI-DP algorithm introduced in our previous work [5]. We consider the two-dimensional Stokes problem,

$$\begin{aligned} -\triangle \boldsymbol{u} + \nabla p &= \boldsymbol{f} \ \text{ in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \ \text{ in } \Omega, \\ \boldsymbol{u} &= 0 \ \text{ on } \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω is a bounded polygonal domain in $\mathbf{R}^2$ and $\boldsymbol{f} \in [L^2(\Omega)]^2$. A pair of velocity and pressure finite element spaces $(\widehat{X}, \overline{P}) \subset \big([H_0^1(\Omega)]^2, L_0^2(\Omega)\big)$ is equipped for a given triangulation in Ω. Functions in the velocity space $\widehat{X}$ are continuous across the triangles, square integrable up to their first weak derivatives, and have their values zero on $\partial\Omega$. The pressure space $\overline{P}$ is obtained from a pressure space P, which consists of functions that are discontinuous across element boundaries, i.e.,

$$\overline{P} = P \bigcap L_0^2(\Omega).$$

Here $L_0^2(\Omega)$ is the space of square integrable functions with their average zero in Ω. We assume that the pair $(\widehat{X}, \overline{P})$ is inf-sup stable and obtain a discrete problem for (2.1):

find $(\widehat{\boldsymbol{u}}, \overline{p}) \in (\widehat{X}, \overline{P})$ satisfying

$$\begin{aligned} &\int_\Omega \nabla \widehat{\boldsymbol{u}} \cdot \nabla \boldsymbol{v} \, dx - \int_\Omega \overline{p} \, \nabla \cdot \boldsymbol{v} \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx, \ \forall \boldsymbol{v} \in \widehat{X}, \\ &- \int_\Omega \nabla \cdot \widehat{\boldsymbol{u}} \, \overline{q} \, dx = 0, \ \forall \overline{q} \in \overline{P}. \end{aligned} \tag{2.2}$$

We now decompose Ω into a non-overlapping subdomain partition $\{\Omega_i\}_{i=1}^N$ in such a way that the subdomain boundaries align the given triangulation in Ω. We introduce local finite element spaces,

$$X^{(i)} = \widehat{X}|_{\Omega_i}, \ P^{(i)} = P|_{\Omega_i},$$

and the product spaces X and P,

$$X = \prod_{i=1}^N X^{(i)}, \quad P = \prod_{i=1}^N P^{(i)},$$

where functions can be discontinuous across subdomain boundaries. Among those unknowns in X, we select some unknowns on the subdomain interface as primal unknowns and enforce strong continuity at the primal unknowns to obtain $\widetilde{X}$, where functions can be discontinuous at the remaining part of the interface unknowns. We call the remaining part of unknowns dual unknowns. The notations $\boldsymbol{u}_I^{(i)}$, $\boldsymbol{u}_\Delta^{(i)}$, and $\boldsymbol{u}_\Pi^{(i)}$ are used to denote unknowns located at the interior part of $\Omega^{(i)}$, the dual unknowns on $\partial\Omega^{(i)}$, and the primal unknowns, respectively. The spaces $X_I^{(i)}$, $X_\Delta^{(i)}$, and $X_\Pi^{(i)}$ consist of the corresponding velocity unknowns $\boldsymbol{u}_I^{(i)}$, $\boldsymbol{u}_\Delta^{(i)}$, and $\boldsymbol{u}_\Pi^{(i)}$, respectively. Those product spaces are denoted by X_I, X_Δ, and X_Π.

By enforcing the continuity on the dual unknowns using Lagrange multipliers $\boldsymbol{\lambda} \in M$, we obtain an equivalent discrete problem to (2.2):

find $((\boldsymbol{u}_I, \boldsymbol{u}_\Delta, \widehat{\boldsymbol{u}}_\Pi), \overline{p}, \boldsymbol{\lambda}) \in \widetilde{X} \times \overline{P} \times M$ such that

$$\begin{pmatrix} K_{II} & K_{I\Delta} & K_{I\Pi} & \overline{B}_I^T & 0 \\ K_{I\Delta}^T & K_{\Delta\Delta} & K_{\Delta\Pi} & \overline{B}_\Delta^T & J_\Delta^T \\ K_{I\Pi}^T & K_{\Delta\Pi}^T & K_{\Pi\Pi} & \overline{B}_\Pi^T & 0 \\ \overline{B}_I & \overline{B}_\Delta & \overline{B}_\Pi & 0 & 0 \\ 0 & J_\Delta & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_\Delta \\ \widehat{\boldsymbol{u}}_\Pi \\ \overline{p} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ \boldsymbol{f}_\Pi \\ 0 \\ 0 \end{pmatrix}, \tag{2.3}$$

where $\overline{B}_I$, $\overline{B}_\Delta$, and $\overline{B}_\Pi$ are from

$$-\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\boldsymbol{u}} \, q \, dx, \quad \forall q \in \overline{P},$$

$J_\triangle$ is a boolean matrix that computes jump of the dual unknowns across the subdomain interface Γ_{ij},

$$J_\triangle \boldsymbol{u}_\triangle|_{\Gamma_{ij}} = \boldsymbol{u}_\triangle^{(i)} - \boldsymbol{u}_\triangle^{(j)},$$

and the other terms are from

$$\sum_i \int_{\Omega_i} \nabla\widetilde{\boldsymbol{u}} \cdot \nabla\widetilde{\boldsymbol{v}}\, dx.$$

We note that M is the space of vector unknowns of Lagrange multipliers.

In [5], to remove all the pressure unknowns by solving the independent local Stokes problems the pressure space $\overline{P}$ is replaced with P. The space P has one more pressure component than $\overline{P}$, which is constant in Ω. The added constant pressure component gives an additional condition on $\widetilde{\boldsymbol{u}}$,

$$\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\boldsymbol{u}}\, q\, dx = 0, \quad q = c, \tag{2.4}$$

which is equivalent to

$$\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\boldsymbol{u}}\, c\, dx = c \sum_{ij} \int_{\Gamma_{ij}} (\boldsymbol{u}_\triangle^{(i)} - \boldsymbol{u}_\triangle^{(j)}) \cdot \boldsymbol{n}_{ij}\, ds = 0.$$

Here, Γ_{ij} is the common edge of Ω_i and Ω_j. The additional condition is in fact a linear sum of $J_\triangle \boldsymbol{u}_\triangle = 0$. By using the pressure space P instead of $\overline{P}$, we still obtain an equivalent algebraic system to (2.3):

find $((\boldsymbol{u}_I, \boldsymbol{u}_\triangle, \widehat{\boldsymbol{u}}_\Pi), p, \boldsymbol{\lambda}) \in (\widetilde{X}, P, M)$ such that

$$\begin{pmatrix} K_{II} & K_{I\triangle} & K_{I\Pi} & B_I^T & 0 \\ K_{I\triangle}^T & K_{\triangle\triangle} & K_{\triangle\Pi} & B_\triangle^T & J_\triangle^T \\ K_{I\Pi}^T & K_{\triangle\Pi}^T & K_{\Pi\Pi} & B_\Pi^T & 0 \\ B_I & B_\triangle & B_\Pi & 0 & 0 \\ 0 & J_\triangle & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_\triangle \\ \widehat{\boldsymbol{u}}_\Pi \\ p \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\triangle \\ \boldsymbol{f}_\Pi \\ 0 \\ 0 \end{pmatrix}. \tag{2.5}$$

Here B_I, $B_\triangle$, and B_Π are from

$$-\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\boldsymbol{u}}\, q\, dx, \quad \forall q \in P,$$

and the other terms are the same as those in (2.3).

The unknowns $(\boldsymbol{u}_I, \boldsymbol{u}_\triangle, p)$ can be eliminated by solving independent local Stokes problems,

$$\begin{pmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_\triangle \\ p \end{pmatrix} = S^{-1} \left(\begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\triangle \\ 0 \end{pmatrix} - \begin{pmatrix} K_{I\Pi} \\ K_{\triangle\Pi} \\ B_\Pi \end{pmatrix} \widehat{\boldsymbol{u}}_\Pi - \begin{pmatrix} 0 \\ J_\triangle^T \\ 0 \end{pmatrix} \boldsymbol{\lambda} \right), \tag{2.6}$$

where S is given by

$$S = \begin{pmatrix} K_{II} & K_{I\Delta} & B_I^T \\ K_{I\Delta}^T & K_{\Delta\Delta} & B_\Delta^T \\ B_I & B_\Delta & 0 \end{pmatrix}. \tag{2.7}$$

Substituting $(\boldsymbol{u}_I, \boldsymbol{u}_\Delta, p)$ into (2.5) and then solving for $\widehat{\boldsymbol{u}}_\Pi$

$$S_{\Pi\Pi}\widehat{\boldsymbol{u}}_\Pi = \boldsymbol{f}_\Pi - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T S^{-1} \left(\begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix} \boldsymbol{\lambda} \right), \tag{2.8}$$

we obtain the resulting algebraic system on $\boldsymbol{\lambda}$,

$$F_{DP}\boldsymbol{\lambda} = d, \tag{2.9}$$

where

(2.10)
$$\begin{aligned} F_{DP} &= \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}^T S^{-1} \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}^T S^{-1} \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix} S_{\Pi\Pi}^{-1} \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T S^{-1} \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}, \\ d &= \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}^T S^{-1} \left(\begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix} - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix} S_{\Pi\Pi}^{-1} \left(\boldsymbol{f}_\Pi - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T S^{-1} \begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix} \right) \right), \end{aligned}$$

and

$$S_{\Pi\Pi} = K_{\Pi\Pi} - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T S^{-1} \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}.$$

The resulting system on $\boldsymbol{\lambda}$ is symmetric and positive semidefinite. In a more detail, when the velocity unknowns at subdomain vertices are selected as the primal unknowns, it has one null space component which is given by

$$\begin{pmatrix} \mu_0^{(1)} \\ \mu_0^{(2)} \end{pmatrix} \Big|_{\Gamma_{ij}} = \begin{pmatrix} \zeta_{ij}\boldsymbol{n}_{ij}^{(1)} \\ \zeta_{ij}\boldsymbol{n}_{ij}^{(2)} \end{pmatrix}, \ \forall \Gamma_{ij}. \tag{2.11}$$

Here, $\mu_0^{(1)}$ and $\mu_0^{(2)}$ are Lagrange multipliers related to each x and y- components of velocity unknowns, $\boldsymbol{n}_{ij}^{(k)}$ are each component of $\boldsymbol{n}_{ij}$, the unit normal vector to Γ_{ij}, and

$$\zeta_{ij}(x_l) = \int_{\Gamma_{ij}} \phi_l(x(s), y(s))\, ds, \tag{2.12}$$

where ϕ_l is the velocity basis element related to the node x_l at Γ_{ij}. For the details, we refer [5, Section 2.2].

We now introduce a subspace of M, which is orthogonal to the null space of F_{DP},

$$M_c = \left\{ \boldsymbol{\mu} \in M \; : \; \sum_{ij} \boldsymbol{\mu}_{ij} \cdot \zeta_{ij} \boldsymbol{n}_{ij} = 0 \right\},$$

where $\boldsymbol{\mu}_{ij} = \boldsymbol{\mu}|_{\Gamma_{ij}}$. Then F_{DP} is positive definite on M_c. The system in (2.9) is then solved by the conjugate gradient method with a lumped preconditioner of the form,

$$\widehat{M}^{-1} = J_\Delta K_{\Delta\Delta} J_\Delta^T. \tag{2.13}$$

In our previous work [5], we proved the following condition number bound for the FETI-DP algorithm equipped with the lumped preconditioner and with the velocity unknowns at subdomain vertices as primal unknowns,

$$\kappa(\widehat{M}^{-1} F_{dp})) \leq C(H/h)(1 + \log(H/h)),$$

which determines the convergence of the conjugate gradient iteration. The same bound has been proved to be optimal for the FETI-DP algorithm of the elliptic problems with a lumped preconditioner, see [13].

In the work by Li and Widlund [11], both the velocity unknowns at the subdomain vertices and the velocity averages on subdomain edges are used as the primal velocity unknowns. In addition, primal pressure unknowns are included in their FETI-DP formulation. They introduced a quite expensive Dirichlet preconditioner and obtained a condition number bound $C(1 + \log(H/h))^2$. Due to the introduction of the primal pressure unknowns, their approach needs both of them, i.e., velocity unknowns at subdomain vertices and averages of the velocity on edges, to provide the stability of the coarse problem matrix as well as to make them satisfy zero flux condition across subdomain interfaces. In their experimental work, a FETI-DP algorithm with primal velocity unknowns at subdomain vertices is tested. Its convergence depends on the number of subdomains and additional primal unknowns are required to achieve a scalable algorithm.

On the other hand, for two dimensional elliptic problems, it is well known that either primal unknowns at subdomain vertices or primal unknowns related to the averages on subdomain edges are enough to obtain a scalable condition number bound, which means that the condition number bound only depends on the local problem size, see [12].

No pressure primal unknowns in the FETI-DP algorithm of our work [5] resulted in a scalable method for the Stokes problems with only the primal velocity unknowns at subdomain vertices. Its condition number bound is the same as that of elliptic problems with an inexact lumped preconditioner, see [13]. We note that in the work [13] a better condition

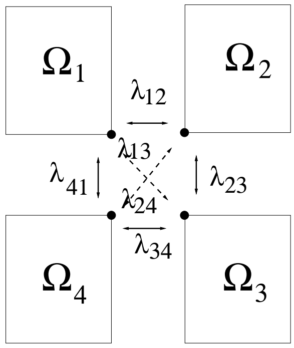

FIG. 1. *Example of fully redundant Lagrange multipliers: λ_{12}, λ_{23}, λ_{34}, and λ_{41} are Lagrange multipliers used to enforce continuity over common (closed) edges, and λ_{13} and λ_{24} are Lagrange multipliers used to enforce continuity among the subdomains sharing only the common vertex.*

number bound, CH/h, was obtained for elliptic problems with a choice of primal unknowns which are averages over common edges.

Motivated by this fact, we will consider a set of primal velocity unknowns, which are averages of the velocity unknowns on subdomain edges. In this case, the space $\widetilde{X}$ consists of the velocity unknowns that can be discontinuous at subdomain vertices and across subdomain edges except that their averages over common edges are the same. We will show that such a choice of primal unknowns gives an improved condition number bound,

$$\kappa(\widehat{M}^{-1}F_{DP}) \leq CH/h,$$

compared to the previous choice of the primal velocity unknowns at subdomain vertices.

3. Primal unknowns based on edge averages. In this section, we will provide an analysis of the condition number bound for the FETI-DP algorithm with a new set of primal unknowns. Most part of the analysis can be done similarly to our previous work [5].

We first describe the FETI-DP algorithm with such a selection of the primal unknowns. Differently to the primal unknowns at subdomain vertices, the velocity space $\widetilde{X}$ has its elements that can be discontinuous at subdomain vertices. We introduce fully redundant Lagrange multipliers to enforce the continuity across Γ_{ij}, see Figure 1,

$$\boldsymbol{u}_{\Delta}^{(i)} - \boldsymbol{u}_{\Delta}^{(j)} = 0,$$

and our analysis is based on the fully redundant Lagrange multipliers.

We note that the resulting FETI-DP equations have the null space which has more than one dimension. In a more detail, F_{DP} is positive definite on M_c which is define by

$$M_c = \left\{ \boldsymbol{\lambda} \in M \,:\, \boldsymbol{\lambda} \perp \text{Null}(J_{\Delta}^T) \text{ and } \boldsymbol{\lambda}^T \boldsymbol{\mu}_0 = 0 \right\}.$$

Here $\text{Null}(J_{\Delta}^T)$ is the space of null components of J_{Δ}^T and $\boldsymbol{\mu}_0$ is introduced in (2.11). The null components in $\text{Null}(J_{\Delta}^T)$ are caused by the use of the fully redundant Lagrange multipliers. All these components can be removed by the l^2-orthogonal projection on M_c. We then perform the conjugate gradient iteration on the subspace by projecting the residuals on M_c. The other part of the FETI-DP algorithm is the same as in the previous section.

We recall the enriched primal velocity space introduced in [5],

$$\widehat{E}_\Pi = \left\{ \boldsymbol{v} \in \widehat{X} \,:\, \boldsymbol{v} \text{ minimizes the discrete } H^1\text{-seminorm for given values } \boldsymbol{a}_V,\ \boldsymbol{a}_E \right\}.$$

Here $\boldsymbol{a}_V$ and $\boldsymbol{a}_E$ denote the given values of $\boldsymbol{v}$ at the subdomain vertices V and the given average values of $\boldsymbol{v}$ on subdomain edges E, respectively. The pair of velocity and pressure spaces, $(X_I + \widehat{E}_\Pi, \overline{P})$, is then inf-sup stable with a constant β, which does not depend on any mesh parameters, see [5, Lemma 3.5].

Let

$$\widehat{E}_{I,\Pi} = X_I + \widehat{E}_\Pi.$$

We will provide a condition number bound by proving the following inequalities:

$$C_1\beta^2 \langle \widehat{M}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle \le \langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle \le C_2 \frac{H}{h} \langle \widehat{M}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle, \quad \forall \boldsymbol{\lambda} \in M_c,$$

where β is the inf-sup constant of the pair $(\widehat{E}_{I,\Pi}, \overline{P})$. These inequalities then lead to the desired condition number bound

$$\kappa(\widehat{M}^{-1}F_{DP}) \le C \frac{1}{\beta^2} \frac{H}{h}.$$

3.1. Lower bound analysis. Let $\mathcal{N}(x)$ be the set of subdomain indices sharing the point x. We introduce an average operator,

$$E_\Delta \boldsymbol{w}_\Delta(x)|_{\partial\Omega_i} = \frac{1}{|\mathcal{N}(x)|} \sum_{j \in \mathcal{N}(x)} \boldsymbol{w}_\Delta^{(j)}(x),$$

where $|\mathcal{N}(x)|$ is the number of elements in $\mathcal{N}(x)$. We then have the identity,

$$\boldsymbol{w}_\Delta(x)|_{\partial\Omega_i} = E_\Delta \boldsymbol{w}_\Delta(x)|_{\partial\Omega_i} + \frac{1}{|\mathcal{N}(x)|} J_\Delta^T J_\Delta \boldsymbol{w}_\Delta(x)\Big|_{\partial\Omega_i}, \tag{3.1}$$

since

$$J_\Delta^T J_\Delta \boldsymbol{w}_\Delta(x)|_{\partial\Omega_i} = \sum_{j \in \mathcal{N}(x)} (\boldsymbol{w}_\Delta^{(i)}(x) - \boldsymbol{w}_\Delta^{(j)}(x)).$$

We introduce the matrix K which gives the discrete H^1-seminorm on $\boldsymbol{u} \in \widetilde{X}$, i.e.,

$$\langle K\boldsymbol{u}, \boldsymbol{u}\rangle = \sum_{i=1}^N |\boldsymbol{u}|^2_{H^1(\Omega_i)}.$$

LEMMA 3.1. *For any $\boldsymbol{\mu} \in M_c$, there exists $\boldsymbol{u} \in \widetilde{X}$ such that*
1. $J_\Delta \boldsymbol{u}_\Delta = \boldsymbol{\mu}$,

2. $\sum_i \int_{\Omega_i} \nabla \cdot \boldsymbol{u}\, q\, dx = 0, \quad \forall q \in P,$

3. $\langle K\boldsymbol{u}, \boldsymbol{u}\rangle \leq C\frac{1}{\beta^2}\langle K_{\triangle\triangle} J_\triangle^T J_\triangle \boldsymbol{u}_\triangle, J_\triangle^T J_\triangle \boldsymbol{u}_\triangle\rangle$*, where* β *is the inf-sup constant of the pair* $(\widehat{E}_{I,\Pi}, \overline{P})$ *and* $\boldsymbol{u}_\triangle$ *is the part of dual unknowns of* $\boldsymbol{u}$*.*

Proof. Most part of the proof is identical to [5, Lemma 4.2]. For a given $\boldsymbol{\mu} \in M_c$, we select $\boldsymbol{w}_\triangle$ to satisfy

$$J_\triangle \boldsymbol{w}_\triangle = \boldsymbol{\mu} \text{ and } E_\triangle \boldsymbol{w}_\triangle = 0. \tag{3.2}$$

Similarly to the proofs in [5], from such a $\boldsymbol{w}_\triangle$ we can find $\boldsymbol{u} \in \widetilde{X}$ which satisfies the first two conditions and

$$\langle K\boldsymbol{u}, \boldsymbol{u}\rangle \leq C\frac{1}{\beta^2}\langle K_{\triangle\triangle}\boldsymbol{w}_\triangle, \boldsymbol{w}_\triangle\rangle.$$

From the above bound combined with (3.1), (3.2), and $J_\triangle \boldsymbol{u}_\triangle = J_\triangle \boldsymbol{w}_\triangle$, the third requirement on $\boldsymbol{u}$ then follows. □

REMARK 3.2. *In the above Lemma,* $\boldsymbol{w}_\triangle$ *in* (3.2) *can be constructed as follows. Since* $M_c \subset$ *Range*$(J_\triangle)$*, for a given* $\boldsymbol{\mu} \in M_c$ *there exists* $\boldsymbol{v}_\triangle \in \widetilde{X}$ *such that*

$$J_\triangle \boldsymbol{v}_\triangle = \boldsymbol{\mu}.$$

For the $\boldsymbol{v}_\triangle$*, we can find* $\boldsymbol{z}_\triangle \in \widehat{X}$ *which gives that*

$$E_\triangle(\boldsymbol{v}_\triangle + \boldsymbol{z}_\triangle) = 0.$$

Since $\boldsymbol{z}_\triangle \in \widehat{X}$ *is continuous across the subdomain interface,* $J_\triangle \boldsymbol{z}_\triangle = 0$*. We then obtain* $\boldsymbol{w}_\triangle = \boldsymbol{v}_\triangle + \boldsymbol{z}_\triangle$*.*

We introduce

$$\widetilde{X}(\mathrm{div}) = \left\{ \boldsymbol{v} \in \widetilde{X} \,:\, \int_{\Omega_i} \nabla \cdot \boldsymbol{v} q\, dx = 0 \quad \forall q \in P \right\}. \tag{3.3}$$

We then have the identity,

$$\langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle = \max_{\boldsymbol{v} \in \widetilde{X}(\mathrm{div})} \frac{\langle J_\triangle \boldsymbol{v}_\triangle, \boldsymbol{\lambda}\rangle^2}{\langle K\boldsymbol{v}, \boldsymbol{v}\rangle}, \tag{3.4}$$

where $\boldsymbol{v}_\triangle$ is the part of dual unknowns of $\boldsymbol{v}$.

By using Lemma 3.1 and (3.4) we obtain the lower bound, see [5, Theorem 4.3]:

THEOREM 3.3. *For any* $\boldsymbol{\lambda} \in M_c$*, we have*

$$C_1 \beta^2 \langle \widehat{M}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle \leq \langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle,$$

where β *is the inf-sup constant of the pair* $(\widehat{E}_{I,\Pi}, \overline{P})$ *and* C_1 *is a positive constant that does not depend on any mesh parameters.*

3.2. Upper bound analysis. For a given edge E, let θ_E denote the cut-off function which is one inside E and is zero, otherwise. Similarly, we define the cut-off function θ_V related to a vertex V. We need the following result for the upper bound analysis, see [13, Lemma 4] for the proof:

LEMMA 3.4. *Let Ω_i be a two dimensional subdomain. For any $u^{(i)} \in X^{(i)}$,*

$$\|u^{(i)} - c_E\|^2_{L^2(E)} \leq CH|u^{(i)}|^2_{H^1(\Omega_i)},$$

where E is an edge of the subdomain Ω_i and c_E is given by

$$c_E = \frac{\int_E I^h(\theta_E u^{(i)})\, dx(s)}{\int_E \, dx(s)}.$$

LEMMA 3.5. *There exists a constant C such that for any $\boldsymbol{u} \in \widetilde{X}$,*

$$\langle K_{\Delta\Delta} J_\Delta^T J_\Delta \boldsymbol{u}_\Delta, J_\Delta^T J_\Delta \boldsymbol{u}_\Delta \rangle \leq C\frac{H}{h}\langle K\boldsymbol{u}, \boldsymbol{u}\rangle,$$

where $\boldsymbol{u}_\Delta$ is the part of dual unknowns of $\boldsymbol{u}$.

Proof. Let $\boldsymbol{w}_\Delta = J_\Delta^T J_\Delta \boldsymbol{u}_\Delta$. We note that

$$\langle K_{\Delta\Delta} J_\Delta^T J_\Delta \boldsymbol{u}_\Delta, J_\Delta^T J_\Delta \boldsymbol{u}_\Delta \rangle = \sum_{i=1}^N |\boldsymbol{w}_\Delta^{(i)}|^2_{H^1(\Omega_i)}.$$

From the inverse inequality and

$$\|\boldsymbol{w}_\Delta^{(i)}\|^2_{L^2(\Omega_i)} \leq Ch\|\boldsymbol{w}_\Delta^{(i)}\|^2_{L^2(\partial\Omega_i)},$$

we obtain

$$\langle K_{\Delta\Delta} J_\Delta^T J_\Delta \boldsymbol{u}_\Delta, J_\Delta^T J_\Delta \boldsymbol{u}_\Delta \rangle \leq Ch^{-1}\sum_{i=1}^N \|\boldsymbol{w}_\Delta^{(i)}\|^2_{L^2(\partial\Omega_i)}. \tag{3.5}$$

We note that for $x \in \partial\Omega_i$

$$\boldsymbol{w}_\Delta^{(i)}(x) = \begin{cases} \boldsymbol{u}_\Delta^{(i)}(x) - \boldsymbol{u}_\Delta^{(j)}(x), & \text{when } x \in E_{ij}, \\ \sum_{m\in\mathcal{N}(x)} \left(\boldsymbol{u}_\Delta^{(i)}(x) - \boldsymbol{u}_\Delta^{(m)}(x)\right), & \text{when } x \in \mathcal{V}(\partial\Omega_i), \end{cases} \tag{3.6}$$

where E_{ij} is an open edge of Ω_i, which is the common part of two subdomains Ω_i and Ω_j, and $\mathcal{V}(\partial\Omega_i)$ is the set of vertices of Ω_i. We decompose $\boldsymbol{w}_\Delta^{(i)}$ into

$$\boldsymbol{w}_\Delta^{(i)} = \sum_{E_{ij}\subset\partial\Omega_i} I^h(\theta_{E_{ij}}\boldsymbol{w}_\Delta^{(i)}) + \sum_{V\in\mathcal{V}(\partial\Omega_i)} I^h(\theta_V \boldsymbol{w}_\Delta^{(i)}) \tag{3.7}$$

and then compute each part using the formula (3.6).

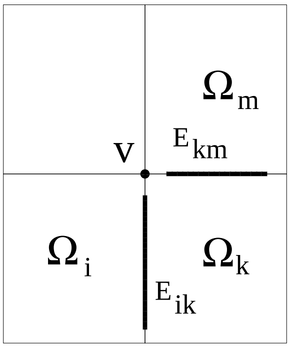

FIG. 2. *Example of a path of subdomains sharing the vertex V: E_{ik} and E_{km} are edges connecting the subdomains in the path.*

For the first term of the above equation, by Lemma 3.4 we obtain

$$\begin{aligned}
\|I^h(\theta_{E_{ij}}\boldsymbol{w}_\Delta^{(i)})\|^2_{L^2(\partial\Omega_i)} &= \|I^h(\theta_{E_{ij}}\boldsymbol{w}_\Delta^{(i)})\|^2_{L^2(E_{ij})} \\
&\le C\|\boldsymbol{u}_\Delta^{(i)} - \boldsymbol{u}_\Delta^{(j)}\|^2_{L^2(E_{ij})} \\
&= C\|\boldsymbol{u}^{(i)} - \boldsymbol{u}^{(j)}\|^2_{L^2(E_{ij})} \\
&\le C\left(\|\boldsymbol{u}^{(i)} - c_{E_{ij}}\|^2_{L^2(E_{ij})} + \|\boldsymbol{u}^{(j)} - c_{E_{ij}}\|^2_{L^2(E_{ij})}\right) \\
&\le CH\left(|\boldsymbol{u}^{(i)}|^2_{H^1(\Omega_i)} + |\boldsymbol{u}^{(j)}|^2_{H^1(\Omega_j)}\right),
\end{aligned} \tag{3.8}$$

where

$$c_{E_{ij}} = \frac{\int_{E_{ij}} I^h(\theta_{E_{ij}}\boldsymbol{u}^{(i)})\,dx(s)}{\int_{E_{ij}} dx(s)} = \frac{\int_{E_{ij}} I^h(\theta_{E_{ij}}\boldsymbol{u}^{(j)})\,dx(s)}{\int_{E_{ij}} dx(s)}.$$

We note that $\boldsymbol{u} = (\boldsymbol{u}^{(1)}, \cdots, \boldsymbol{u}^{(N)}) \in \widetilde{X}$, which has common edge averages across each E_{ij}.

For the term given at a vertex V, we consider subdomains in $\mathcal{N}(V)$, which share the vertex V. Among them, we select a path from Ω_i to Ω_m, $\{\Omega_i, \Omega_{k_1}, \cdots, \Omega_{k_n}, \Omega_m\}$, which are connected through their common edges. We may assume that the path consists of $\{\Omega_i, \Omega_k, \Omega_m\}$, see Figure 2. The following can also be applied to a more general case.

At the vertex $V \in \mathcal{V}(\partial\Omega_i)$, we then have that

$$\begin{aligned}
&\|I^h(\theta_V\boldsymbol{w}_\Delta^{(i)})\|^2_{L^2(\partial\Omega_i)} \\
\le\, & Ch\sum_{m\in\mathcal{N}(V)} |\boldsymbol{u}_\Delta^{(i)}(V) - \boldsymbol{u}_\Delta^{(m)}(V)|^2 \\
\le\, & C\sum_{m\in\mathcal{N}(V)} \left(h|\boldsymbol{u}_\Delta^{(i)}(V) - \boldsymbol{u}_\Delta^{(k)}(V)|^2 + h|\boldsymbol{u}_\Delta^{(k)}(V) - \boldsymbol{u}_\Delta^{(m)}(V)|^2\right) \\
\le\, & C\sum_{m\in\mathcal{N}(V)} \left(\|\boldsymbol{u}_\Delta^{(i)} - \boldsymbol{u}_\Delta^{(k)}\|^2_{L^2(E_{ik})} + \|\boldsymbol{u}_\Delta^{(k)} - \boldsymbol{u}_\Delta^{(m)}\|^2_{L^2(E_{km})}\right) \\
\le\, & CH\sum_{m\in\mathcal{N}(V)} \left(|\boldsymbol{u}^{(i)}|^2_{H^1(\Omega_i)} + |\boldsymbol{u}^{(k)}|^2_{H^1(\Omega_k)} + |\boldsymbol{u}^{(m)}|^2_{H^1(\Omega_m)}\right).
\end{aligned} \tag{3.9}$$

Here we have used the fact that both $\overline{E}_{ik}$ and $\overline{E}_{km}$ contain the vertex V and the inequality used in (3.8). Note that $\overline{E}$ denotes the closure of an open edge E. Combining (3.5) with (3.7), (3.8), and (3.9), and using

$$\sum_{i=1}^{N} |\boldsymbol{u}^{(i)}|^2_{1,\Omega_i} = \langle K\boldsymbol{u}, \boldsymbol{u}\rangle,$$

the desired bound has been proved. □

From the identity in (3.4) and Lemma 3.5, an upper bound then follows, see [5, Lemma 4.5]:

THEOREM 3.6. *For any $\boldsymbol{\lambda} \in M_c$, we have*

$$\langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle \leq C_2 \frac{H}{h} \langle \widehat{M}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle,$$

where C_2 is a positive constant that does not depend on any mesh parameters.

4. Numerical results. In this section, we provide numerical results of the FETI-DP algorithms corresponding to the choice of primal unknowns. In the first choice, the velocity unknowns at subdomain vertices are selected and in the second the unknowns related to the velocity averages on common edges are selected. In the second case, we use the fully redundant Lagrange multipliers to enforce the continuity at the remaining dual velocity unknowns across the subdomain interface.

We will present the number of iterations and approximated condition numbers of the two FETI-DP algorithms to confirm the bound of the condition number carried out in the previous section. The conjugate gradient iteration is performed up to the relative residual norm reduced by a factor of 10^6 and the condition numbers are approximated by the extreme eigenvalues of the tridiagonal Lanczos matrix generated by the iteration. Our test problem is defined in the unit rectangular domain $[0,1]^2$ with the exact solution,

$$\boldsymbol{u}(x,y) = \begin{pmatrix} \sin^3(\pi x)\sin^2(\pi y)\cos(\pi y) \\ -\sin^2(\pi x)\sin^3(\pi y)\cos(\pi x) \end{pmatrix} \text{ and } \; p(x,y) = x^2 - y^2.$$

In Table 1, the results are presented as increasing the number of subdomains with a fixed local problem size in each subdomain. The domain is partitioned into uniform rectangular subdomains. For example, $N = 4^2$ means that the domain Ω is divided into four subdomains in each x- and y-directional edges of Ω. The first choice of primal unknowns and the second choice of the primal unknowns are denoted by vertex-based and edge-based, respectively. For the both cases, the number of iterations and condition numbers do not depend on the number of subdomains. In other words, the results shows a good scalability with respect to the number of subdomains. For the second choice, we observe slightly less iterations and smaller condition numbers.

In Table 2, the number of iterations and condition numbers of the two choices are presented as increasing the size of local problems. Here the domain Ω is divided into uniform

	vertex-based				edge-based			
N	Iter	κ	$\boldsymbol{\lambda}_{min}$	$\boldsymbol{\lambda}_{max}$	Iter	κ	$\boldsymbol{\lambda}_{min}$	$\boldsymbol{\lambda}_{max}$
2^2	9	4.31e+00	2.60	1.12e+01	11	8.90	2.63	2.34e+01
4^2	16	1.17e+01	2.55	2.98e+01	18	1.05e+01	2.60	2.72e+01
8^2	21	1.36e+01	2.50	3.42e+01	20	1.16e+01	2.53	2.93e+01
12^2	21	1.40e+01	2.50	3.50e+01	19	1.18e+01	2.51	2.98e+01
16^2	22	1.41e+01	2.49	3.52e+01	19	1.19e+01	2.50	2.97e+01

TABLE 1

Scalability as increase of the number of subdomains, N, with a fixed local problem size ($H/h = 8$): the number of iterations Iter, the condition numbers κ, the minimum eigenvalues $\boldsymbol{\lambda}_{min}$, and the maximum eigenvalues $\boldsymbol{\lambda}_{max}$ of the vertex-based primal unknowns (the velocity values at the subdomain vertices are selected as the primal unknowns) and the edge-based one (the averages of velocity over common edges are selected as the primal unknowns)

rectangular subdomains with $N = 4^2$. The results from the second choice present weaker increase of iterations and condition numbers as the increase of the local problem size, H/h, compared to those from the first choice. As in the analysis, the minimum eigenvalues are almost identical for the both cases and do not depend on the size of local problems. Only the maximum eigenvalues increase with respect to the size of local problems and the second choice gives less increase in the maximum eigenvalues than the first choice. In Figure 3, the constant C in the bound of condition numbers for the second choice,

$$\kappa(M^{-1}F_{DP}) \leq C\frac{H}{h},$$

is estimated as H/h increases. We observe that the estimated values of C tend to converge to some number as H/h increases. The numerical results confirm that the bound of condition numbers are sharp.

The second choice of primal unknowns based on subdomain edges shows better performance than the first choice based on the subdomain vertices. The only complication in the second choice is introduction of additional null space components of the FETI-DP operator caused by using fully redundant Lagrange multipliers. This additional null space components can be eliminated by projecting the residual at each iteration. It is more practical to use the second choice of primal unknowns especially when the size of local problems are relatively large.

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	vertex-based				edge-based			
H/h	Iter	κ	$\boldsymbol{\lambda}_{min}$	$\boldsymbol{\lambda}_{max}$	Iter	κ	$\boldsymbol{\lambda}_{min}$	$\boldsymbol{\lambda}_{max}$
4	12	5.09e+00	2.64	1.34e+01	15	7.17	3.23	2.32e+01
8	16	1.17e+01	2.55	2.98e+01	18	1.05e+01	2.59	2.72e+01
16	24	2.78e+01	2.54	7.08e+01	21	1.31e+01	2.58	3.38e+01
20	26	3.64e+01	2.57	9.36e+01	21	1.43e+01	2.59	3.72e+01
26	29	5.04e+01	2.58	1.30e+02	22	1.62e+01	2.61	4.23e+01
32	32	6.51e+01	2.59	1.68e+02	24	1.84e+01	2.60	4.79e+01

TABLE 2

Performance as increase of the size of local problems, H/h, in a fixed subdomain partition ($N = 4^2$): the number of iterations Iter, the condition numbers κ, the minimum eigenvalues $\boldsymbol{\lambda}_{min}$, and the maximum eigenvalues $\boldsymbol{\lambda}_{max}$ of the vertex-based primal unknowns (the velocity values at the subdomain vertices are selected as the primal unknowns) and the edge-based one (the averages of velocity over common edges are selected as the primal unknowns)

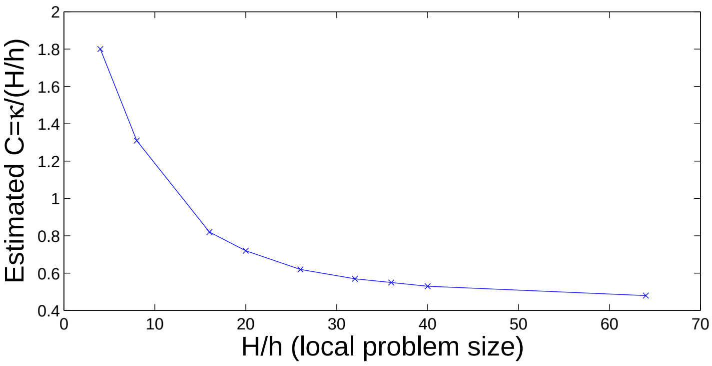

FIG. 3. *Plot of estimated constant $C = \kappa/(H/h)$ with respect to H/h for the edge-based primal unknowns*

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