# On the fundamental groups of positively curved 5-manifolds with maximal local symmetry rank

by

Jin Hong Kim and Hee Kwon Lee

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**Abstract**

Let $M$ be a closed oriented Riemannian manifold of dimension 5 with positive sectional curvature. If $M$ admits a $\pi_1$-invariant isometric $T^k$ $(k = 2, 3)$, it has been shown by Fang and Rong that $M$ is homeomorphic to a spherical space form. In this short paper, we show that if $M$ admits a $\pi_1$-invariant isometric $T^3$-action, then $\pi_1(M)$ is actually cyclic. Furthermore, we show that if $\pi_1(M)$ is not isomorphic to $\mathbf{Z}_3$ as well, then $M$ is diffeomorphic to a lens space.

## 1 Introduction and Main Results

It is now a theorem by Brendle and Schoen ([2, 3]) that if a closed oriented Riemannian manifold $M$ admits a pointwise (or weakly) quarter pinched metric, then its universal covering is diffeomorphic to a sphere $S^n(1)$ with constant sectional curvature 1. Thus $M$ is diffeomorphic to a spherical space form $S^n(1)/\Gamma$, where $\Gamma$, called a *spherical $n$-space group*, is a subgroup of $O(n+1)$. Here the positive curvature condition plays a crucial role, and the so-called *spherical space form problem* for such positively curved Riemannian manifolds is completely solved.

In a similar vein, the aim of this short paper is to study the fundamental group of a positively curved oriented manifold which admits a $\pi_1$-invariant isometric $T^k$-action and give some sharpened results for positively curved oriented manifolds of dimension 5 with a $\pi_1$-invariant isometric $T^3$-action. A well-known theorem of Synge says that the fundamental group is trivial or finite, depending on the parity of the dimension. So we restrict our attention to the case where the dimension is odd. In particular, we investigate the case of dimension 5 in detail.

To describe our results, we need to set up some basic definitions. A $\pi_1$-*invariant torus* $T^k$-*action* on a manifold $M$ of dimension $n$ is defined to be an effective $T^k$-action on its universal covering space $\tilde{M}$ that extends to the action by the semi-direct product $T^k \times_\rho \pi_1(M)$, where $\rho : \pi_1(M) \to \mathrm{Aut}(T^k)$ is a homomorphism. The *local symmetry rank* of $M$ is then defined to be the largest $k$ for all possible $\pi_1$-invariant $T^k$-actions, and the *maximal local symmetry rank* is equal to $\left[\frac{n+1}{2}\right]$ by a theorem of Grove and Searle in [7]. It is easy to see that the $T^k$-action on $\tilde{M}$ is the lifting of a $T^k$-action on $M$ if and only if $\rho$ is trivial, i.e., the $T^k$-action on $\tilde{M}$ commutes with the action of $\pi_1(M)$ as deck transformations. It is also known as in [4] and [5] that, when $\pi_1(M)$ is finite, a $\pi_1$-invariant $T^k$-action is equivalent to a pure $F$-structure introduced by Cheeger and Gromov.

In their paper [6], among other things, Fang and Rong showed that if a positively curved closed 5-manifold $M$ admits a $\pi_1$-invariant isometric $T^k$-action for some $k > 1$, then $M$ is homeomorphic to a spherical space form $S^5(1)/\pi_1(M)$ where $\pi_1(M)$ is a finite subgroup of $O(6)$. As mentioned earlier, the symmetry rank $k$ should be less than or equal to 3 by a result of Grove and Searle in [7]. Moreover, Rong showed in [10] that if a closed positively curved manifold of dimension $n$ admits a $\pi_1$-invariant isometric $T^k$-action, then $\pi_1(M)$ has a cyclic subgroup with index less than or equal to a constant $w(n)$ (see also Theorem 0.5 in [11]). In view of these previous works, our main results sharpen their results for some particular cases, as follows.

**Theorem 1.1.** *Let $M$ be a closed oriented Riemannian manifold of dimension 5 with positive sectional curvature. If $M$ admits a $\pi_1$-invariant isometric $T^3$-action, then $\pi_1(M)$ is always cyclic.*

In Section 2, we give a proof of Theorem 1.1. In fact, in Section 2 we will show much stronger result that $M$ should be diffeomorphic to a lens space, provided that $\pi_1(M)$ is not isomorphic to $\mathbf{Z}_3$ (see Theorem 2.2 for more details).

For the case of positively curved 5-manifolds with a $\pi_1$-invariant $T^2$-action, Fang and Rong showed in Lemma 6.3 of [6] that if the $T^2$-action on the universal cover $\tilde{M}$ is not pseudo-free as well, then $\pi_1(M)$ is cyclic. More generally, they showed in [6] that if the fixed point set of $T^2$ on $\tilde{M}$ is empty, $\pi_1(M)$ is either cyclic, or every subgroup of order $3p$ ($p$ is prime) of $\pi_1(M)$ is cyclic, or has a normal cyclic subgroup of order 3. It would be interesting to find some other topological conditions under which non-cyclic cases in the above classification can be removed.

## 2 Proof of Theorem 1.1

The aim of this section is to give a proof of Theorem 1.1. To do so, we begin with the following lemma whose proof can be found in Lemmas 5.2 and 5.3 in [6].

**Lemma 2.1.** *Let $M$ be a closed oriented Riemannian manifold of dimension* 5 *with positive sectional curvature. If $M$ admits a $\pi_1$-invariant isometric $T^3$-action, then the following properties hold:*

(1) *$T^3$ has a circle subgroup $T^1$ which acts freely on $\tilde{M}$ which commutes with the action of $\pi_1(M)$.*

(2) *Let $T^1$ be as in* (1). *If $\rho : \pi_1(M) \to \mathrm{Aut}(T^3) = \mathrm{GL}(3, \mathbf{Z})$ is non-trivial, then the induced $\pi_1(M)$-action on the quotient space $\tilde{M}^* = M/T^1$ is pseudo-free, and $\pi_1(M)/\ker\rho \cong \mathbf{Z}_3$.*

Note that by a well-known theorem of Freedman $\tilde{M}^*$ is homeomorphic to the complex projective space $\mathbf{CP}^2$. Furthermore, since the remaining rank-2 subgroup $T^2$ of $T^3$ also acts effectively on $\tilde{M}^*$, it follows (e.g., see [9] or [8]) that $\tilde{M}^*$ is in fact diffeomorphic to $\mathbf{CP}^2$.

With Lemma 2.1 in place, we can now prove the following theorem.

**Theorem 2.2.** *Let $M$ be a closed oriented Riemannian manifold of dimension* 5 *with positive sectional curvature. If $M$ admits a $\pi_1$-invariant isometric $T^3$-action and $\pi_1(M)$ is not isomorphic to $\mathbf{Z}_3$ as well, then $\rho : \pi_1(M) \to \mathrm{Aut}(T^3)$ is trivial. In particular, $M$ is diffeomorphic to a lens space and so $\pi_1(M)$ is cyclic.*

*Proof.* Assume first that $\rho : \pi_1(M) \to \mathrm{Aut}(T^3)$ is trivial. Then the $T^3$-action on the universal cover $\tilde{M}$ is just the lifting of the $T^3$-action on $M$. But then it follows from a well-known theorem of Grove and Searle in [7] that $M$ is diffeomorphic to a lens space. In particular, $\pi_1(M)$ is cyclic.

Next, we assume that $\rho : \pi_1(M) \to \mathrm{Aut}(T^3)$ is non-trivial. By adapting basic ideas from the paper [6], especially the proofs of Lemmas 5.2 and 5.3 in [6], we will then derive a contradiction. To do so, note first that by assumption $\pi_1(M)$ is not isomorphic to $\mathbf{Z}_3$. Thus by Lemma 2.1 (2), there exists a non-trivial element $\gamma$ in $\pi_1(M)$ so that $\rho(\gamma) = \mathrm{Id}$.

Note that $\tilde{M}/T^3$ is homeomorphic to a simplex $\Delta^2$ as a stratified set such that the three vertices are the projection of three isolated circle orbits in $\tilde{M}$, the three edges are the projection of three components of $T^2$-orbits, and one 2-dimensional facet is the projection of one component of $T^3$-orbits.

One can then show that $\gamma \in \pi_1(M)$ acts trivially on the orbit space $\Delta^2$ if and only if $\rho(\gamma) = \mathrm{Id} \in \mathrm{Aut}(T^3) = \mathrm{GL}(3, \mathbf{Z})$. Indeed, if $\gamma \in \pi_1(M)$ acts trivially on $\Delta^2$, then the holonomy representation $\rho(\gamma)$ preserves all the isotropy groups of the $T^3$-action on $\tilde{M}$. In particular, when $I_1$, $I_2$, and $I_3$ denote three 2-dimensional isotropy groups, $\rho(\gamma)$ preserves the three circle isotropy groups $I_1 \cap I_2$, $I_2 \cap I_3$, and $I_3 \cap I_1$. Hence $\rho(\gamma)$ as an element of $\mathrm{GL}(3, \mathbf{Z})$ is the identity on $T^3$. For the converse, it is important to observe that the image of any element under $\rho$ that does not lie in $\ker \rho$ acts effectively on $\Delta^2$ by rotating three vertices of $\Delta^2$. Thus, if $\rho(\gamma) = \mathrm{Id}$, then $\gamma$ acts trivially on $\Delta^2$. This argument also implies that $\ker \rho$ is cyclic, since there exists a circle orbit (or a vertex of $\Delta^2$) preserved by all the elements of $\ker \rho$.

Since the induced action of $\pi_1(M)$ on $\tilde{M}^*$ is pseudo-free by Lemma 2.1 (2), the fixed point set of $\gamma$-action on $\tilde{M}^*$ is isolated. Let $p_0$ denote an isolated fixed point in the fixed point set $F_0$ of $\gamma$. Then observe that for any $x \in \tilde{M}$ whose image under the projection map $\pi : \tilde{M} \to \tilde{M}^*$ is $p_0$, there exists an element $t_0 \in T^1$ such that $t_0\gamma(x) = x$. Here $t_0$ does not depend on the choice of $x \in \tilde{M}$ with $\pi(x) = p_0$, since the set $\{p_0\}$ is connected. Thus the fixed point set of $t_0\gamma$ in $T^3 \times_\rho \pi_1(M)$ contains a 1-dimensional totally geodesic submanifold $F$ in $\tilde{M}$ which maps to $p_0$ under the map $\pi$. By the choice of $\gamma$ (or $\rho(\gamma) = \mathrm{Id}$) and $F$, the totally geodesic submanifold $F$ is $T^3$-invariant. Since the torus subgroup $T^1$ as in Lemma 2.1 (1) acts freely on $\tilde{M}$, it also acts freely on $F$. Thus $T^3$ should have the torus subgroup $C = T^2$ of $T^3$ as the isotropy group.

Since $\ker \rho$ is a normal cyclic subgroup of $\pi_1(M)$, the cyclic subgroup $\langle\gamma\rangle \subset \ker \rho \subset \pi_1(M)$ is also a normal subgroup. Then we can claim that $\pi_1(M)$ acts on the isolated fixed point set $F_0$ and so on the preimage, still denoted $F$, of $F_0$ under the map $\pi$. To see it, since $\langle\gamma\rangle$ is a normal subgroup of $\pi_1(M)$, for any $\tau \in \pi_1(M)$ we have $\gamma(\tau(p_0)) = \tau(p_0)$. Thus $\tau(p_0)$ is in the fixed point set $F_0$ of $\gamma$. This implies that $\tau$ maps $F_0$ to $F_0$ and so $F$ to $F$. On the other hand, since the isotropy group of $\tau(F)$ is $\rho(\tau)(C)$, we have $\rho(\tau)(C) = C$. Thus $\rho(\tau)$ must have 1 as an eigenvalue with an eigenvector which generates a 1-dimensional torus $H = T^1 \subset C$. It will be important to note that $H$ acts on $\tilde{M}$ in such a way that $H$ commutes with the $\pi_1(M)$-action.

Furthermore, we can show that $H$ acts freely on $\tilde{M}$. Indeed, if $H$ does not act freely on $\tilde{M}$, then there exists a non-trivial isotropy group $K \subset H$ such that the fixed point set $\tilde{M}^K$ is non-empty. Then $\tilde{M}^K$ is either a circle or a 3-dimensional totally geodesic sphere. Since $K$ is a normal subgroup of $T^3$, $T^3$ acts on $\tilde{M}^K$ and has one or two circle orbits by Theorem 4.4 in [6] (or see Theorem 10.12 on page 164 of [1]). Since $\pi_1(M)$ commutes with the

$H$-action, $\pi_1(M)$ preserves $\tilde{M}^K$ and so preserves one or two vertices of the orbit space $\Delta^2$. Since $\pi_1(M)$ rotates three vertices of $\Delta^2$ as noted earlier, at least one vertex of $\Delta^2$ is fixed by $\pi_1(M)$. This implies that any element $\tau \in \pi_1(M)$ lies in the $\ker \rho$, which contradicts to the assumption that $\rho$ is non-trivial. Thus we can conclude that $H$ acts freely on $\tilde{M}$.

However, it is also true that by construction the fixed point set of $H$ contains the 1-dimensional totally geodesic submanifold $F$ of $\tilde{M}$ which is clearly non-empty. Therefore, $\rho$ cannot be non-trivial. This completes the proof of Theorem 2.2. $\square$

Finally we are ready to prove our main result (Theorem 1.1).

**Corollary 2.3.** *Let $M$ be a closed oriented Riemannian manifold of dimension 5 with positive sectional curvature. If $M$ admits a $\pi_1$-invariant isometric $T^3$-action, then $\pi_1(M)$ is always cyclic.*

*Proof.* If $\pi_1(M)$ is isomorphic to $\mathbf{Z}_3$, we are done. Otherwise, it follows from Theorem 2.2 that $M$ is diffeomorphic to a lens space, so that $\pi_1(M)$ is also cyclic even in this case. This completes the proof. $\square$

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Department of Mathematical Sciences, Korea Advanced Institute of Science and Technology, Yusong-gu, Daejon 305–701, Republic of Korea
*E-mail address*: `jinkim@kaist.ac.kr`