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Asymptotic Expansions for Currents Caused by a Small Interface Changes of an Electromagnetic Inclusion *

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Abstract

We consider solutions to the Helmholtz equations in two dimensions. The aim of this paper is to advance the development of high-order terms in the asymptotic expansions of the boundary perturbations of currents caused by small perturbations of the shape of an inhomogeneity with $\mathcal{C}^2$-boundary. Our derivation is rigorous derived by the field expansion (FE) method (*formal derivation*) and proved by layer potential techniques. It extends those already derived for small volume inhomogeneities for developing effective algorithms for determining certain properties of the shape of an inhomogeneity based on boundary measurements.

Mathematics Subject Classification (MSC2000): 35R30, 35B30

Keywords: Small perturbations, interface problem, Helmholtz equation, full-asymptotic expansions, boundary integral method

1 Introduction

The field of inverse shape problems has been an active research area for several decades. Several related problems belong to the electromagnetic field and are governed by Helmholtz equations, which are further subdivided by assumptions on the underlying media and on the boundary conditions. The main objective of this paper is to present a schematic way to derive high-order asymptotic expansions for boundary perturbations in the currents of the Helmholtz equations resulting from small perturbations of the shape of an inhomogeneity with $\mathcal{C}^2$-boundary. We adopt the field expansion (FE) method to derive formal asymptotic expansions and thanks to layer potential techniques we prove rigorously those asymptotic expansions. We then use these formulae to design algorithms to recover certain properties on the perturbations of the shape.

Suppose that an electromagnetic medium occupies a bounded domain Ω in $\mathbb{R}^2$, with a connected Lipschitz boundary $\partial\Omega$. Let μ_0 and ε_0 denote the permeability and the permittivity of the background medium Ω, and assume that $\mu_0 > 0$ and $\varepsilon_0 > 0$ are positive constants. Consider a bounded domain $D \subset\subset \Omega$ with $\mathcal{C}^2$-boundary, a permeability $0 < \mu \neq \mu_0 < \infty$, and a permittivity $0 < \varepsilon \neq \varepsilon_0 < \infty$. Let μ_* and ε_* be the constitutive parameters of the

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inhomogeneity defined by $\mu_* := \mu\chi_D + \mu_0\chi_{\Omega\backslash\overline{D}}$ and $\varepsilon_* := \varepsilon\chi_D + \varepsilon_0\chi_{\Omega\backslash\overline{D}}$, where χ_D is the indicator function of D. Let u denote the solution to the Helmholtz equation

$$\begin{cases} \nabla\cdot\big(\frac{1}{\mu_*}\nabla u\big) + \omega^2\varepsilon_* u = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = f \in W^2_{\frac{1}{2}}(\partial\Omega), \end{cases} \tag{1.1}$$

where $\omega > 0$ is a given frequency.

Suppose D_α is an α-perturbation of D, *i.e.*, there is a function $h \in \mathcal{C}^1(\partial D)$, such that

$$\partial D_\alpha = \{\tilde{x} : \tilde{x} = x + \alpha h(x)\nu(x) | x \in \partial D,\ 0 < \alpha << 1\},$$

Let u_α be the solution to the Helmholtz equation in the presence of D_α

$$\begin{cases} \nabla\cdot\big(\frac{1}{\mu_\alpha}\nabla u_\alpha\big) + \omega^2\varepsilon_\alpha u_\alpha = 0 & \text{in } \Omega, \\ u_\alpha|_{\partial\Omega} = f, \end{cases} \tag{1.2}$$

where the piecewise-constant magnetic permeability μ_α is given by

$$\mu_\alpha(x) = \begin{cases} \mu_0, & x \in \Omega\backslash\overline{D}_\alpha, \\ \mu, & x \in D_\alpha, \end{cases}$$

and the function $\varepsilon_\alpha(x)$ is defined analogously.

In order to ensure well-posedness we shall assume that $\omega^2\mu_0\varepsilon_0$ is not an eigenvalue for the operator $-\Delta$ in $L^2(\Omega)$ with the Dirichlet boundary condition.

To the best of our knowledge, this is the first work to rigorously investigate Helmholtz interface problem in two dimensional and derive high-order terms in the asymptotic expansion of $(\frac{\partial u_\alpha}{\partial\nu} - \frac{\partial u}{\partial\nu})|_{\partial\Omega}$ when $\alpha \to 0$. However, by the same method, one can derive asymptotic formula for the Neumann problem as well. In this paper, assuming that the unknown D_α boundary is a small perturbation of a circle, we determine a relationship between Fourier coefficients of the perturbation of the shape and boundary measurements. Our formula may also be extended those already derived for small volume inhomogeneities in [2, 3] for developing effective algorithms for determining certain properties of the shape of an inhomogeneity based on boundary measurements. In connection with this, we refer to recent works in the context of interface problems [1, 4, 5, 8, 10, 12, 15, 17].

Our general approach can be extended to other equations such as the anisotropic conductivity problem, Stokes, the Maxwell and the Lamé systems.

This paper is organized as follows. In the next section we introduce some notations for small perturbations of an interface of $\mathcal{C}^2$, review some basic facts on the layer potentials and give representation formulas. In section 3, we derive formal asymptotic expansion for currents by using FE method. In the section 4, we prove that formal expansion by layer potentials techniques. In the last section we present reconstruction formula of the shape deformation h from measurements on $\partial\Omega$ based on the expansion.

2 Definitions and Preliminary results

2.1 Small perturbation of an interface

Let $a, b \in \mathbb{R}$, with $a < b$, and let $X(t) : [a, b] \to \mathbb{R}^2$ be the arclength parametrization of ∂D, namely, X is a $\mathcal{C}^2$-function satisfying $|X'(t)| = 1$ for all $t \in [a, b]$ and

$$\partial D := \{x = X(t), t \in [a, b]\}.$$

Then the outward unit normal to ∂D, $\nu(x)$, is given by $\nu(x) = R_{-\frac{\pi}{2}} X'(t)$, where $R_{-\frac{\pi}{2}}$ is the rotation by $-\pi/2$, the tangential vector at x, $T(x) = X'(t)$, and $X'(t) \perp X''(t)$. Set the curvature $\tau(x)$ to be defined by

$$X''(t) = \tau(x)\nu(x).$$

We will sometimes use $h(t)$ for $h(X(t))$ and $h'(t)$ for the tangential derivative of $h(x)$.

Then, $\tilde{x} = \tilde{X}(t) = X(t) + \alpha h(t)\nu(x) = X(t) + \alpha h(t) R_{-\frac{\pi}{2}} X'(t)$ is a parametrization of ∂D_α. By $\tilde{\nu}(x) := \nu(\tilde{x})$, we denote the outward unit normal to ∂D_α at $\tilde{x}$. Then, it is proved in [4] that

$$\begin{aligned}
\tilde{\nu}(x) &= \frac{R_{-\frac{\pi}{2}} \tilde{X}'(t)}{|\tilde{X}'(t)|} \\
&= \frac{\Big(1 - \alpha h(t)\tau(x)\Big)\nu(x) - \alpha h'(t) X'(t)}{\sqrt{\alpha^2 h'(t)^2 + \Big(1 - \alpha h(t)\tau(x)\Big)^2}} \\
&= \frac{\Big(1 - \alpha h(t)\tau(x)\Big)\nu(x) - \alpha h'(t) T(x)}{\sqrt{\alpha^2 h'(t)^2 + \Big(1 - \alpha h(t)\tau(x)\Big)^2}},
\end{aligned}$$

and hence $\tilde{\nu}(x)$ can be expanded uniformly as

$$\tilde{\nu}(x) = \sum_{n=0}^{\infty} \alpha^n \nu_n(x), \quad x \in \partial D, \tag{2.1}$$

where the vector-valued functions ν_n are uniformly bounded regardless of n. In particular,

$$\nu_0(x) = \nu(x), \quad \nu_1(x) = -h'(t)T(x), \quad x \in \partial D. \tag{2.2}$$

Likewise, denote by $d\tilde{\sigma}(x) := d\sigma(\tilde{x})$ the length element to ∂D_α at $\tilde{x}$ which has an uniformly expansion [4]

$$d\tilde{\sigma}(x) = |\tilde{X}'(t)|dt = \sqrt{(1 - \alpha\tau(t)h(t))^2 + \alpha^2 h'^2(t)}dt = \sum_{n=0}^{\infty} \alpha^n \sigma_n(x) d\sigma(x), \quad x \in \partial D, \tag{2.3}$$

where σ_n are functions bounded regardless of n, with

$$\sigma_0(x) = 1, \quad \sigma_1(x) = -\tau(x)h(x), \quad x \in \partial D. \tag{2.4}$$

Denote by $\frac{d}{dt}$ the tangential derivative in the direction of $T(x) = X'(t)$. Let $x \in \partial D$ and $\phi(x) \in \mathcal{C}^2([a, b])$, we have

$$\frac{d\phi}{dt}(x) = \nabla\phi(x) \cdot X'(t) = \frac{\partial\phi}{\partial T}(x),$$

and

$$\begin{aligned}
\Big(\frac{d}{dt}\Big)^2 \phi(x) &= \frac{d}{dt}\Big(\nabla\phi(x) \cdot X'(t)\Big) = \langle D^2\phi(x) X'(t), X'(t)\rangle + \nabla\phi(x) \cdot X''(t) \\
&= \frac{\partial^2\phi}{\partial T^2}(x) + \tau \frac{\partial\phi}{\partial\nu}(x).
\end{aligned}$$

As consequence, the Laplacian in the local coordinates can be expressed as follows

$$\Delta = \frac{\partial^2}{\partial \nu^2} + \frac{\partial^2}{\partial T^2} = \frac{\partial^2}{\partial \nu^2} - \tau \frac{\partial}{\partial \nu} + \Big(\frac{d}{dt}\Big)^2 \quad \text{on } \partial D. \tag{2.5}$$

2.2 Representation of solutions

We start to review some basic facts in the theory of layer potentials. Denote $k_0 := \omega\sqrt{\mu_0\varepsilon_0}$ and $k := \omega\sqrt{\mu\varepsilon}$. Let $\Gamma_k(x)$ be the fundamental solution for $\Delta + k^2$ in $\mathbb{R}^2$, that is for $x \neq 0$,

$$\Gamma_k(x) = -\frac{i}{4} H_0^1(k|x|),$$

where H_0^1 is the Hankel function of the first kind of order 0. We have

$$-\frac{i}{4} H_0^1(k|x|) = \frac{1}{2\pi} \sum_{n=0}^{+\infty} (-1)^n \frac{k^{2n}}{2^{2n}(n!)^2} |x|^{2n} \Big(ln(|x|) + ln(k\gamma) - \sum_{j=1}^{n} \frac{1}{j} \Big) \quad \text{as } |x| \to 0, \tag{2.6}$$

where $2\gamma = e^{\tilde{\gamma} - i\pi/2}$, and $\tilde{\gamma}$ is Euler's constant.

For a bounded domain D in $\mathbb{R}^2$ and $k > 0$ let $\mathcal{S}_D^k$ and $\mathcal{D}_D^k$ be the single and double layer potentials defined by Γ_k, that is,

$$\mathcal{S}_D^k \phi(x) := \int_{\partial D} \Gamma_k(x-y)\phi(y) d\sigma(y), \quad x \in \mathbb{R}^2, \tag{2.7}$$

$$\mathcal{D}_D^k \phi(x) := \int_{\partial D} \frac{\partial \Gamma_k(x-y)}{\partial \nu(y)} \phi(y) d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \partial D, \tag{2.8}$$

for $\phi \in L^2(\partial D)$. The proof of the following traces formulas can be found in [6]

$$\frac{\partial(\mathcal{S}_D^k \phi)}{\partial \nu}\Big|_{\pm}(x) = \Big(\pm \frac{1}{2} I + (\mathcal{K}_D^k)^* \Big) \phi(x), \quad x \in \partial D, \tag{2.9}$$

$$(\mathcal{D}_D^k \phi)\Big|_{\pm} = \Big(\mp \frac{1}{2} I + \mathcal{K}_D^k \Big) \phi(x), \quad x \in \partial D, \tag{2.10}$$

where $\mathcal{K}_D^k$ is the operator defined by

$$\mathcal{K}_D^k \phi(x) = p.v. \int_{\partial D} \frac{\partial \Gamma_k(x-y)}{\partial \nu(y)} \phi(y) d\sigma(y),$$

and $(\mathcal{K}_D^k)^*$ is the L^2-adjoint of $\mathcal{K}_D^k$. Here *p.v.* denotes the cauchy principal value. The operator $\mathcal{K}_D^k$ is known to be bounded on $L^2(\partial D)$ [8].
Recall that $W_1^2(\partial D) = \{\phi : \phi \in L^2(\partial D), \frac{\partial \phi}{\partial T} \in L^2(\partial D)\}$. The following theorem is of importance to us. For proof, see [2].

Theorem 2.1 *Suppose that k_0^2 is not a Dirichlet eigenvalue for $-\Delta$ on D. For each $(F, G) \in W_1^2(\partial D) \times L^2(\partial D)$, there exists a unique solution $(\xi, \zeta) \in L^2(\partial D) \times L^2(\partial D)$ to the system of integral equations*

$$\begin{cases} \mathcal{S}_D^k \xi - \mathcal{S}_D^{k_0} \zeta = F \quad \text{on } \partial D, \\ \dfrac{1}{\mu} \dfrac{\partial(\mathcal{S}_D^k \xi)}{\partial \nu}\Big|_{-} - \dfrac{1}{\mu_0} \dfrac{\partial(\mathcal{S}_D^{k_0} \zeta)}{\partial \nu}\Big|_{+} = G \quad \text{on } \partial D. \end{cases} \tag{2.11}$$

There exists a constant C independent of F and G such that

$$\|\xi\|_{L^2(\partial D)} + \|\zeta\|_{L^2(\partial D)} \leq C\big(\|F\|_{W_1^2(\partial D)} + \|G\|_{L^2(\partial D)}\big). \tag{2.12}$$

For more details on the following representations of solutions, see [3, 2]. Suppose that k_0^2 is not a Dirichlet eigenvalue for $-\Delta$ on D_α. Define

$$H_\alpha(x) := -\mathcal{S}_\Omega^{k_0}\big(\frac{\partial u_\alpha}{\partial \nu}\big|_{\partial\Omega}\big)(x) + \mathcal{D}_\Omega^{k_0}\big(f\big)(x), \quad x \in \mathbb{R}^2\backslash\partial\Omega, \tag{2.13}$$

and $(\varphi_\alpha, \psi_\alpha) \in L^2(\partial D_\alpha) \times L^2(\partial D_\alpha)$ be the unique solution of

$$\begin{cases} \mathcal{S}_{D_\alpha}^k \varphi_\alpha - \mathcal{S}_{D_\alpha}^{k_0} \psi_\alpha = H_\alpha \quad \text{on } \partial D_\alpha, \\ \dfrac{1}{\mu}\dfrac{\partial \mathcal{S}_{D_\alpha}^k \varphi_\alpha}{\partial \nu}\Big|_- - \dfrac{1}{\mu_0}\dfrac{\partial \mathcal{S}_{D_\alpha}^{k_0} \psi_\alpha}{\partial \nu}\Big|_+ = \dfrac{1}{\mu_0}\dfrac{\partial H_\alpha}{\partial \nu} \quad \text{on } \partial D_\alpha. \end{cases} \tag{2.14}$$

Then u_α solution to (1.2) can be represented as

$$u_\alpha(x) = \begin{cases} H_\alpha(x) + \mathcal{S}_{D_\alpha}^{k_0} \psi_\alpha(x), \quad x \in \Omega\backslash\overline{D}_\alpha, \\ \mathcal{S}_{D_\alpha}^k \varphi_\alpha(x), \quad x \in D_\alpha. \end{cases} \tag{2.15}$$

Similarly, suppose that k_0^2 is not a Dirichlet eigenvalue for $-\Delta$ on D. Define

$$H(x) := -\mathcal{S}_\Omega^{k_0}\big(\frac{\partial u}{\partial \nu}\big|_{\partial\Omega}\big)(x) + \mathcal{D}_\Omega^{k_0}\big(f\big)(x), \quad x \in \mathbb{R}^2\backslash\partial\Omega, \tag{2.16}$$

and $(\varphi, \psi) \in L^2(\partial D) \times L^2(\partial D)$ be the unique solution of

$$\begin{cases} \mathcal{S}_D^k \varphi - \mathcal{S}_D^{k_0} \psi = H \quad \text{on } \partial D, \\ \dfrac{1}{\mu}\dfrac{\partial \mathcal{S}_D^k \varphi}{\partial \nu}\Big|_- - \dfrac{1}{\mu_0}\dfrac{\partial \mathcal{S}_D^{k_0} \psi}{\partial \nu}\Big|_+ = \dfrac{1}{\mu_0}\dfrac{\partial H}{\partial \nu} \quad \text{on } \partial D. \end{cases} \tag{2.17}$$

Then u solution to (1.1) has the following representation

$$u(x) = \begin{cases} H(x) + \mathcal{S}_D^{k_0} \psi(x), \quad x \in \Omega\backslash\overline{D}, \\ \mathcal{S}_D^k \varphi(x), \quad x \in D. \end{cases} \tag{2.18}$$

Let $G_{k_0}(x, y)$ be the Dirichlet Green's function for $\Delta + k_0^2$ in Ω, i.e., for each $y \in \Omega$, G is the solution of

$$\begin{cases} (\Delta + k_0^2) G_{k_0}(x, y) = \delta_y(x), \quad x \in \Omega, \\ G_{k_0}(x, y) = 0, \quad x \in \partial\Omega. \end{cases} \tag{2.19}$$

Let $G_D^{k_0}$ be defined by

$$G_D^{k_0} \phi(x) := \int_{\partial D} G_{k_0}(x, y)\phi(y) d\sigma(y), \quad x \in \overline{\Omega},$$

for $\phi \in L^2(\partial D)$. The following identity holds

$$\int_{\partial\Omega} \Gamma_{k_0}(x - y) \frac{\partial G_{k_0}(z, y)}{\partial \nu(y)}\Big|_{\partial\Omega} d\sigma(y) = \Gamma_{k_0}(x - z), \quad x \in \mathbb{R}^2\backslash\Omega, \quad z \in \Omega. \tag{2.20}$$

As a consequence of (2.20), it follows from (2.9) that

$$\Big(\frac{1}{2}I + (\mathcal{K}_\Omega^{k_0})^*\Big)\Big(\frac{\partial G_{k_0}(z,\cdot)}{\partial \nu(\cdot)}\Big|_{\partial\Omega}\Big)(x) = \frac{\partial \Gamma_{k_0}(x-z)}{\partial \nu(x)}, \quad x\in\partial\Omega, \quad z\in\Omega. \tag{2.21}$$

The following lemma holds, see [6, 2].

Lemma 2.2 *If k_0^2 is not a Dirichlet eigenvalue for $-\Delta$ on Ω, then $\frac{1}{2}I + (\mathcal{K}_\Omega^{k_0})^* : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is injective.*

Define the background voltage potential, U, to be the unique solution to

$$\begin{cases} (\Delta + k_0^2)U = 0 \quad \text{in } \Omega, \\ U = f \quad \text{on } \partial\Omega. \end{cases}$$

Let u_α be the solution to (1.2). The following representation can be proved easily from (2.21) and Lemma 2.2, see also [2].

$$\frac{\partial u_\alpha}{\partial \nu}(x) = \frac{\partial U}{\partial \nu}(x) + \frac{\partial (G_{D_\alpha}^{k_0}\psi_\alpha)}{\partial \nu}(x), \quad x\in\partial\Omega, \tag{2.22}$$

where $\psi_\alpha \in L^2(\partial D_\alpha)$ is the solution to (2.15).
Likewise, the solution u to (1.1) has the following representation

$$\frac{\partial u}{\partial \nu}(x) = \frac{\partial U}{\partial \nu}(x) + \frac{\partial (G_{D}^{k_0}\psi)}{\partial \nu}(x), \quad x\in\partial\Omega, \tag{2.23}$$

where $\psi \in L^2(\partial D)$ is the solution of (2.18). Subtracting (2.23) from (2.22), we get

$$\frac{\partial u_\alpha}{\partial \nu}(x) - \frac{\partial u}{\partial \nu}(x) = \frac{\partial (G_{D_\alpha}^{k_0}\psi_\alpha)}{\partial \nu}(x) - \frac{\partial (G_{D}^{k_0}\psi)}{\partial \nu}(x), \quad x\in\partial\Omega. \tag{2.24}$$

The integral equation (2.24) will be investigated by layer potential techniques to prove formal high-order terms in the asymptotic expansions of $(\frac{\partial u_\alpha}{\partial\nu} - \frac{\partial u}{\partial\nu})|_{\partial\Omega}$ as α tends to 0, derived by the field expansion (FE) method.

3 Formal derivations: field expansion (FE) method

Recall that u_α is the solution to the following problem

$$(\Delta + \omega^2\varepsilon_0\mu_0)u_\alpha = 0 \quad \text{in } \Omega\backslash\overline{D}_\alpha, \tag{3.1}$$

$$(\Delta + \omega^2\varepsilon\mu)u_\alpha = 0 \quad \text{in } D_\alpha, \tag{3.2}$$

$$\frac{1}{\mu_0}\frac{\partial u_\alpha}{\partial \nu}\Big|_+ - \frac{1}{\mu}\frac{\partial u_\alpha}{\partial \nu}\Big|_- = 0 \quad \text{on } \partial D_\alpha, \tag{3.3}$$

$$u_\alpha|_+ - u_\alpha|_- = 0 \quad \text{on } \partial D_\alpha, \tag{3.4}$$

$$u_\alpha = f \quad \text{on } \partial\Omega. \tag{3.5}$$

To derive the formal asymptotic formula of u_α to order an integer N, we apply the field expansion (FE) method, see [8, 12]. Firstly, we expand u_α in powers of α, *i.e.*

$$u_\alpha(x) = u_0(x) + \alpha u_1(x) + \alpha^2 u_2(x) + \cdots + \alpha^N u_N(x) + O(\alpha^{N+1}), \quad x \in \Omega,$$

where u_n is well defined in $\Omega \backslash \partial D$ and

$$(\Delta + \omega^2 \varepsilon_0 \mu_0) u_n = 0 \quad \text{in } \Omega \backslash \overline{D}, \quad (\Delta + \omega^2 \varepsilon \mu) u_n = 0 \quad \text{in } D, \quad u_n = f \delta_{0n} \quad \text{on } \partial\Omega, \tag{3.6}$$

for $n = 0, \cdots, N$. Here δ_{0n} is the Kronecker symbol.

Let $x \in \partial D$, then $\tilde{x} = x + \alpha h(x) \nu(x) \in \partial D_\alpha$. By Taylor expansion, we get

$$\nabla u_\alpha(\tilde{x}) \cdot \nu(\tilde{x}) \Big|_\pm = \sum_{n=0}^{N} \alpha^n \sum_{m=0}^{n} \sum_{s=0}^{m} \sum_{|i|=s} \frac{(h(x))^s}{i!} \partial^i \nabla u_{m-s}(x) (\nu(x))^i \cdot \nu_{n-m}(x) \Big|_\pm + O(\alpha^{N+1}). \tag{3.7}$$

If we substitute the expansions in (3.7) into (3.3), we formally get

$$\begin{aligned} \frac{1}{\mu_0} \frac{\partial u_n}{\partial \nu} \Big|_+ - \frac{1}{\mu} \frac{\partial u_n}{\partial \nu} \Big|_- &= \sum_{m=0}^{n-1} \sum_{s=0}^{m} \sum_{|i|=s} \frac{h^s}{i!} \Big(\frac{1}{\mu} \partial^i \nabla u_{m-s} \nu^i \cdot \nu_{n-m} \Big|_- - \frac{1}{\mu_0} \partial^i \nabla u_{m-s} \nu^i \cdot \nu_{n-m} \Big|_+ \Big) \\ &\quad + \sum_{s=1}^{n} \sum_{|i|=s} \frac{h^s}{i!} \Big(\frac{1}{\mu} \partial^i \nabla u_{n-s} \nu^i \cdot \nu_0 \Big|_- - \frac{1}{\mu_0} \partial^i \nabla u_{n-s} \nu^i \cdot \nu_0 \Big|_+ \Big) \quad \text{on } \partial D. \end{aligned} \tag{3.8}$$

In particular

$$\frac{1}{\mu_0} \frac{\partial u_0}{\partial \nu} \Big|_+ - \frac{1}{\mu} \frac{\partial u_0}{\partial \nu} \Big|_- = 0 \quad \text{on } \partial D,$$

$$\begin{aligned} \frac{1}{\mu_0} \frac{\partial u_1}{\partial \nu} \Big|_+ - \frac{1}{\mu} \frac{\partial u_1}{\partial \nu} \Big|_- &= h \Big(\frac{1}{\mu} \frac{\partial^2 u_0}{\partial \nu^2} \Big|_- - \frac{1}{\mu_0} \frac{\partial^2 u_0}{\partial \nu^2} \Big|_+ \Big) + \Big(\frac{1}{\mu} \nabla u_0 \cdot \nu_1 \big|_- - \frac{1}{\mu_0} \nabla u_0 \cdot \nu_1 \big|_+ \Big) \\ &= (\frac{1}{\mu_0} - \frac{1}{\mu}) \frac{d}{dt} \Big(h \frac{du_0}{dt} \Big) + h \omega^2 (\varepsilon_0 - \varepsilon) u_0 \quad \text{on } \partial D. \end{aligned} \tag{3.9}$$

The last equality in (3.9) is proved by using the representation of Laplacian in the local coordinates in (2.5) and (2.2).

For $\tilde{x} = x + \alpha h(x) \nu(x) \in \partial D_\alpha$, we write

$$u_\alpha(\tilde{x}) \Big|_\pm = \sum_{n=0}^{N} \alpha^n \sum_{m=0}^{n} \sum_{|i|=m} \frac{(h(x))^m}{i!} \partial^i u_{n-m}(x) \big(\nu(x)\big)^i \Big|_\pm + O(\alpha^{N+1}). \tag{3.10}$$

Inserting the expansions in (3.10) into (3.4), formally leads to

$$u_n|_+ - u_n|_- = \sum_{m=1}^{n} \sum_{|i|=m} \frac{h^m}{i!} \Big(\partial^i u_{n-m} \nu^i |_- - \partial^i u_{n-m} \nu^i |_+ \Big) \quad \text{on } \partial D. \tag{3.11}$$

In particular

$$u_0|_+ - u_0|_- = 0 \quad \text{on } \partial D,$$

and

$$u_1|_+ - u_1|_- = h\Big(\frac{\partial u_0}{\partial \nu}\Big|_- - \frac{\partial u_0}{\partial \nu}\Big|_+\Big) = h\big(1 - \frac{\mu_0}{\mu}\big)\frac{\partial u_0}{\partial \nu}\Big|_- \quad \text{on } \partial D.$$

Note that $u_0 = u$.

One can see easily that $O(\alpha^{N+1})$ depends on N, Ω, $dist(D, \partial\Omega)$, k_0, k, $\mathcal{C}^1$-norm of h, and $\mathcal{C}^2$-norm of X. We formally obtain the following theorem and give the proof in Subsection 4.2.

Theorem 3.1 *The following asymptotic formula formally holds*

$$\frac{\partial u_\alpha}{\partial \nu}(x) = \frac{\partial u}{\partial \nu}(x) + \sum_{n=1}^{N} \alpha^n \frac{\partial u_n}{\partial \nu}(x) + O(\alpha^{N+1}), \quad x \in \partial\Omega, \tag{3.12}$$

where u_n, $n = 1, \cdots, N$, can be determined uniquely by (3.6), (3.8), and (3.11), and the remainder $O(\alpha^{N+1})$ depends on N, Ω, k_0, k, the $\mathcal{C}^2$-norm of X, the $\mathcal{C}^1$-norm of h, and $dist(D, \partial\Omega)$.

4 Layer potential techniques method

4.1 Asymptotic formulas of layer potentials

Let $\tilde{x}, \tilde{y} \in \partial D_\alpha$, that is,

$$\tilde{x} = x + \alpha h(x)\nu(x), \qquad \tilde{y} = y + \alpha h(y)\nu(y).$$

Hence

$$|\tilde{x} - \tilde{y}|^2 = |x - y|^2 \Big(1 + 2\alpha \frac{\langle x - y, h(x)\nu(x) - h(y)\nu(y)\rangle}{|x - y|^2} + \alpha^2 \frac{|h(x)\nu(x) - h(y)\nu(y)|^2}{|x - y|^2}\Big).$$

Denote by

$$F(x, y) := \frac{\langle x - y, h(x)\nu(x) - h(y)\nu(y)\rangle}{|x - y|^2}, \quad G(x, y) := \frac{|h(x)\nu(x) - h(y)\nu(y)|^2}{|x - y|^2}.$$

Since ∂D is of class $\mathcal{C}^2$. One can easily see that

$$|F(x, y)| + |G(x, y)|^{\frac{1}{2}} \le C\|X\|_{\mathcal{C}^2(\partial D)}\|h\|_{\mathcal{C}^1(\partial D)} \quad \text{for all } x, y \in \partial D,$$

and hence

$$|\tilde{x} - \tilde{y}| = |x - y|\sqrt{1 + 2\alpha F(x, y) + \alpha^2 G(x, y)} := |x - y| \sum_{n=0}^{\infty} \alpha^n L_n(x, y), \tag{4.1}$$

where the series converges absolutely and uniformly. In particular, we can see that

$$L_0(x, y) = 1, \quad L_1(x, y) = F(x, y).$$

According to Leibniz's rule, the p'th derivative of $r^{2n}ln(r)$ is given by

$$(r^{2n}ln(r))^{(p)} = \sum_{l=0}^{p} C_p^l (r^{2n})^{(l)} (ln(r))^{(p-l)} = (r^{2n})^{(p)} ln(r) + \sum_{l=0}^{p-1} C_p^l (r^{2n})^{(l)} \big(\frac{1}{r}\big)^{(p-l-1)},$$

where C_p^l is a binomial coefficient, and then, it follows from (2.6) that

$$-\frac{ik^p}{4} H_0^{1\,(p)}(kr) r^p \text{ is continuous at zero for } \quad p > 0. \tag{4.2}$$

From (4.1), (4.2), Taylor expansion of $-\frac{i}{4}H_0^1(k|\tilde{x}-\tilde{y}|)$ at $k|x-y|$, and (2.3), we write

$$\begin{aligned} -\frac{i}{4} H_0^1(k|\tilde{x}-\tilde{y}|) d\sigma(\tilde{y}) = & -\frac{i}{4} \sum_{n=0}^{N} \alpha^n \Big(H_0^1(k|x-y|)\sigma_n(y) + \mathbb{R}_n(x,y) \Big) d\sigma(y) \\ & + O(\alpha^{N+1}) \Big(H_0^1(k|x-y|) + 1 \Big), \end{aligned}$$

with

$$|\mathbb{R}_n(x,y)| \leq C \|X\|_{\mathcal{C}^2(\partial D)} \|h\|_{\mathcal{C}^1(\partial D)} \quad \text{for all } x, y \in \partial D.$$

In particular

$$\mathbb{R}_0(x,y) = 0, \quad \mathbb{R}_1(x,y) = kH_0^{1'}(k|x-y|) \frac{\langle x-y, h(x)\nu(x) - h(y)\nu(y)\rangle}{|x-y|}.$$

Introduce a sequence of integral operators $(\mathcal{S}_D^{k,n})_{n\in\mathbb{N}}$, defined for any $\phi \in L^2(\partial D)$ by:

$$\mathcal{S}_D^{k,n}\phi(x) = -\frac{i}{4} \int_{\partial D} \Big(H_0^1(k|x-y|)\sigma_n(y) + \mathbb{R}_n(x,y) \Big) \phi(y) d\sigma(y) \quad \text{for } n \geq 0,$$

where $\mathcal{S}_D^{k,0} = \mathcal{S}_D^k$ and

$$\begin{aligned} \mathcal{S}_D^{k,1}\phi(x)|_{\pm} &= -\mathcal{S}_D^k(\tau h\phi)(x) + h(\mathcal{K}_D^k)^*(\phi)(x) + \mathcal{K}_D^k(h\phi)(x) \\ &= -\mathcal{S}_D^k(\tau h\phi)(x) + \Big(h\frac{\partial(\mathcal{S}_D^k\phi)}{\partial\nu} + \mathcal{D}_D^k(h\phi) \Big)\Big|_{\pm}(x), \quad x \in \partial D. \end{aligned} \tag{4.3}$$

It is easily to check by using the celebrated theorem of Coifman-McIntosh-Meyer, see [7], that the operators $\mathcal{S}_D^{k,n}$ and $\frac{\partial \mathcal{S}_D^{k,n}}{\partial T}$ are bounded in $L^2(\partial D)$ for $n \geq 0$.

Now let us investigate the following term

$$\Big(kH_0^{1'}(k|\tilde{x}-\tilde{y}|)|\tilde{x}-\tilde{y}| \Big) \frac{\langle \tilde{x}-\tilde{y}, \nu(\tilde{x})\rangle}{|\tilde{x}-\tilde{y}|^2} d\sigma(\tilde{y}).$$

It is proved in [4] that

$$\frac{\langle \tilde{x}-\tilde{y}, \nu(\tilde{x})\rangle}{|\tilde{x}-\tilde{y}|^2} d\sigma(\tilde{y}) = \sum_{n=0}^{\infty} \alpha^n \mathbb{M}_n(x,y) d\sigma(y),$$

where the series converges absolutely and uniformly. In particular

$$\mathbb{M}_0(x,y) = \frac{\langle x-y, \nu(x)\rangle}{|x-y|^2},$$

and

$$\begin{aligned}\mathbb{M}_1(x,y) = &\left(-2\frac{\langle x-y, h(x)\nu(x)-h(y)\nu(y)\rangle}{|x-y|^2} + \tau(x)h(x) - \tau(y)h(y)\right)\frac{\langle x-y,\nu(x)\rangle}{|x-y|^2}\\ &+ \frac{\langle h(x)\nu(x)-h(y)\nu(y), \nu(x)\rangle}{|x-y|^2} - \frac{\langle x-y, \tau(x)h(x)\nu(x)+h'(t)T(x)\rangle}{|x-y|^2}.\end{aligned}$$

In other hand, by Taylor expansion, we write

$$kH_0^{1'}(k|\tilde{x}-\tilde{y}|)|\tilde{x}-\tilde{y}| = \sum_{n=0}^{\infty}\alpha^n \mathbb{H}_n(x,y), \tag{4.4}$$

According to (4.2), the series in (4.4) converges absolutely and uniformly. Note that

$$\mathbb{H}_0(x,y) = kH_0^{1'}(k|x-y|)|x-y|,$$

and

$$\mathbb{H}_1(x,y) = \left[k^2 H_0^{1''}(k|x-y|)|x-y| + kH_0^{1'}(k|x-y|)\right]\frac{\langle x-y, h(x)\nu(x)-h(y)\nu(y)\rangle}{|x-y|}.$$

Finally, we get

$$kH_0^{1'}(k|\tilde{x}-\tilde{y}|)\frac{\langle \tilde{x}-\tilde{y}, \nu(\tilde{x})\rangle}{|\tilde{x}-\tilde{y}|}d\sigma(\tilde{y}) = \sum_{n=0}^{\infty}\alpha^n \underbrace{\sum_{m=0}^{n}\mathbb{M}_m(x,y)\mathbb{H}_{n-m}(x,y)}_{:=\mathbb{k}_n(x,y)}d\sigma(y),$$

with

$$\mathbb{k}_0(x,y) = kH_0^{1'}(k|x-y|)|\frac{\langle x-y, \nu(x)\rangle}{|x-y|},$$

and

$$\begin{aligned}\mathbb{k}_1(x,y) =& k\Big[\big(\tau(x)h(x)-\tau(y)h(y)\big)\frac{\langle x-y,\nu(x)\rangle}{|x-y|} - \frac{\langle x-y, h(x)\nu(x)-h(y)\nu(y)\rangle\langle x-y,\nu(x)\rangle}{|x-y|^3}\\ &+ \frac{\langle h(x)\nu(x)-h(y)\nu(y),\nu(x)\rangle}{|x-y|} - \frac{\langle x-y, \tau h(x)\nu(x)+h'(t)T(x)\rangle}{|x-y|}\Big]H_0^{1'}(k|x-y|)\\ &+ k^2 H_0^{1''}(k|x-y|)\frac{\langle x-y, h(x)\nu(x)-h(y)\nu(y)\rangle\langle x-y,\nu(x)\rangle}{|x-y|^2}.\end{aligned}$$

Introduce a sequence of integral operators $(\mathcal{K}_D^{k,n})_{n\in\mathbb{N}}$, defined for any $\phi \in L^2(\partial D)$ by:

$$\mathcal{K}_D^{k,n}\phi(x) = -\frac{i}{4}\int_{\partial D}\mathbb{k}_n(x,y)\phi(y)d\sigma(y) \quad \text{for } n \geq 0.$$

Note that $\mathcal{K}_D^{k,0} = (\mathcal{K}_D^k)^*$. It is easily to prove that the operator $\mathcal{K}_D^{k,n}$ for $n \geq 0$ is bounded in $L^2(\partial D)$. In fact, it is an immediate consequence of the celebrate theorem of Coifman-McIntosh-Meyer, see [7]. In particular

$$
\begin{aligned}
&\mathcal{K}_D^{k,1}\phi(x) = \tau(x)h(x)(\mathcal{K}_D^k)^*(\phi)(x) - (\mathcal{K}_D^k)^*(\tau h\phi)(x)\\
&+ \frac{ik}{4}\Big[h(x)\int_{\partial D} {H_0^1}'(k|x-y|)|\frac{(\langle x-y,\nu(x)\rangle)^2}{|x-y|^3}\phi(y)d\sigma(y) - h(x)\int_{\partial D} {H_0^1}'(k|x-y|)|\frac{\phi(y)}{|x-y|}d\sigma(y)\\
&\qquad + \int_{\partial D} {H_0^1}'(k|x-y|)|\frac{\langle x-y,\tau(x)h(x)\nu(x)+h'(t)T(x)\rangle}{|x-y|}\phi(y)d\sigma(y)\\
&\qquad - kh(x)\int_{\partial D} {H_0^1}''(k|x-y|)|\frac{(\langle x-y,\nu(x)\rangle)^2}{|x-y|^2}\phi(y)d\sigma(y)\Big]\\
&- \frac{ik}{4}\Big[\int_{\partial D} {H_0^1}'(k|x-y|)|\frac{\langle x-y,\nu(x)\rangle\langle x-y,\nu(y)\rangle}{|x-y|^3}h(y)\phi(y)d\sigma(y)\\
&\qquad - \int_{\partial D} {H_0^1}'(k|x-y|)|\frac{\langle \nu(x),\nu(y)\rangle}{|x-y|}h(y)\phi(y)d\sigma(y)\\
&\qquad - k\int_{\partial D} {H_0^1}''(k|x-y|)|\frac{\langle x-y,\nu(x)\rangle\langle x-y,\nu(y)\rangle}{|x-y|^2}h(y)\phi(y)d\sigma(y)\Big]\\
&= \tau(x)h(x)(\mathcal{K}_D^k)^*(\phi)(x) - (\mathcal{K}_D^k)^*(\tau h\phi)(x)\\
&\quad + \frac{\partial\mathcal{D}_D^k(h\phi)}{\partial\nu}(x) + h(x)\Big[\frac{\partial^2\mathcal{S}_D^k(\phi)}{\partial\nu^2}(x) - \tau\frac{\partial\mathcal{S}_D^k(\phi)}{\partial\nu}(x)\Big] - h'(t)\frac{d\mathcal{S}_D^k(\phi)}{dt}(x), \quad x \in \partial D.
\end{aligned}
$$

Recall that $\frac{\partial^2\mathcal{S}_D^k(\phi)}{\partial\nu^2}$ and $\frac{\partial\mathcal{S}_D^k(\phi)}{\partial\nu}$ are not continuous on ∂D, but $\frac{\partial^2\mathcal{S}_D^k(\phi)}{\partial\nu^2} - \tau\frac{\partial\mathcal{S}_D^k(\phi)}{\partial\nu}$ is continuous and given by

$$
\frac{\partial^2\mathcal{S}_D^k(\phi)}{\partial\nu^2} - \tau\frac{\partial\mathcal{S}_D^k(\phi)}{\partial\nu} = -\Big(\frac{d}{dt}\Big)^2\mathcal{S}_D^k(\phi) - k^2\mathcal{S}_D^k(\phi) \quad \text{on } \partial D.
$$

To justify the last equality, we use the Laplacian in the local coordinates in (2.5)

$$
0 = \Delta + k^2 = \frac{\partial^2}{\partial\nu^2} - \tau\frac{\partial}{\partial\nu} + \Big(\frac{d}{dt}\Big)^2 + k^2 \quad \text{on } \partial D.
$$

Since

$$
\begin{aligned}
\mathcal{K}_D^{k,1}\phi(x) &= \tau(x)h(x)(\mathcal{K}_D^k)^*(\phi)(x) - (\mathcal{K}_D^k)^*(\tau h\phi)(x)\\
&\quad + \frac{\partial\mathcal{D}_D^k(h\phi)}{\partial\nu}(x) - \frac{d}{dt}\Big(h\frac{d\mathcal{S}_D^k(\phi)}{dt}\Big)(x) - k^2h(x)\mathcal{S}_D^k(\phi)(x)\\
&= \Big(\tau h\frac{\partial(\mathcal{S}_D^k\phi)}{\partial\nu} - \frac{\partial\big(\mathcal{S}_D^k(\tau h\phi)\big)}{\partial\nu}\Big)\Big|_{\pm}(x) + \frac{\partial\mathcal{D}_D^k(h\phi)}{\partial\nu}(x)\\
&\quad - \frac{d}{dt}\big(h\frac{d\mathcal{S}_D^k(\phi)}{dt}\big)(x) - k^2h(x)\mathcal{S}_D^k(\phi)(x), \quad x \in \partial D. \qquad (4.5)
\end{aligned}
$$

Let Ψ_α be the diffeomorphism from ∂D to ∂D_α given by $\Psi_\alpha(x) = x + \alpha h(t)\nu(x)$, where $x = X(t)$. The following theorem holds.

Theorem 4.1 *Let $N \in \mathbb{N}$. There exists C depending only on N, k, $\|X\|_{\mathcal{C}^2}$, and $\|h\|_{\mathcal{C}^1}$ such that for any $\phi \in L^2(\partial D_\alpha)$, we have*

$$\Big\| \mathcal{S}^k_{D_\alpha}[\phi] o \Psi_\alpha - \mathcal{S}^k_D[\tilde{\phi}] - \sum_{n=1}^{N} \alpha^n \mathcal{S}^{k,n}_D[\tilde{\phi}] \Big\|_{L^2(\partial D)} \leq C \alpha^{N+1} \|\tilde{\phi}\|_{L^2(\partial D)}, \tag{4.6}$$

and

$$\Big\| (\mathcal{K}^k_{D_\alpha})^*[\phi] o \Psi_\alpha - (\mathcal{K}^k_D)^*[\tilde{\phi}] - \sum_{n=1}^{N} \alpha^n \mathcal{K}^{k,n}_D[\tilde{\phi}] \Big\|_{L^2(\partial D)} \leq C \alpha^{N+1} \|\tilde{\phi}\|_{L^2(\partial D)}, \tag{4.7}$$

where $\tilde{\phi} := \phi o \Psi_\alpha$.

We need the following lemma.

Lemma 4.2 *Suppose that k_0^2 is not a Dirichlet eigenvalue for $-\Delta$ on D. For each $(F, G) \in W^2_1(\partial D) \times L^2(\partial D)$ and for α small enough, there exists a unique solution $(\xi, \zeta) \in L^2(\partial D) \times L^2(\partial D)$ to the system of integral equations*

$$\begin{cases} \displaystyle\sum_{n=0}^{N} \alpha^n \mathcal{S}^{k,n}_D \xi - \sum_{n=0}^{N} \alpha^n \mathcal{S}^{k_0,n}_D \zeta = F \quad \text{on } \partial D, \\ \displaystyle\frac{1}{\mu} \sum_{n=0}^{N} \alpha^n \Big(-\frac{\delta_{0n}}{2} I + \mathcal{K}^{k,n}_D \Big) \xi - \frac{1}{\mu_0} \sum_{n=0}^{N} \alpha^n \Big(\frac{\delta_{0n}}{2} I + \mathcal{K}^{k_0,n}_D \Big) \zeta = G \quad \text{on } \partial D. \end{cases} \tag{4.8}$$

Furthermore, there exists a constant C independent of α, F, and G such that

$$\|\xi\|_{L^2(\partial D)} + \|\zeta\|_{L^2(\partial D)} \leq C \big(\|F\|_{W^2_1(\partial D)} + \|G\|_{L^2(\partial D)} \big). \tag{4.9}$$

Proof. Let $X := L^2(\partial D) \times L^2(\partial D)$ and $Y := W^2_1(\partial D) \times L^2(\partial D)$. For $n \in \mathbb{N}$, define the operator $T_n : X \to Y$ by

$$T_n(f, g) := \Big(\mathcal{S}^{k,n}_D \xi - \mathcal{S}^{k_0,n}_D \zeta, \frac{1}{\mu} \big(-\frac{\delta_{0n}}{2} I + \mathcal{K}^{k,n}_D \big) \xi - \frac{1}{\mu_0} \big(\frac{\delta_{0n}}{2} I + \mathcal{K}^{k_0,n}_D \big) \zeta \Big).$$

Since T_n is bounded. In Particular

$$T_0(\xi, \zeta) := \Big(\mathcal{S}^k_D \xi - \mathcal{S}^{k_0}_D \zeta, \frac{1}{\mu} \frac{\partial (\mathcal{S}^k_D \xi)}{\partial \nu} \Big|_- - \frac{1}{\mu_0} \frac{\partial (\mathcal{S}^{k_0}_D \zeta)}{\partial \nu} \Big|_+ \Big).$$

Define the operator T_N by

$$T_N = T_0 + \sum_{n=1}^{N} \alpha^n T_n.$$

It is proved in [3] that if k_0^2 is not a Dirichlet eigenvalue for $-\Delta$ on D, then T_0 is invertible. For α small enough, it follows from Theorem 1.16, section 4 of [11], that the operator T_N is invertible. This completes the proof of solvability of (4.8). The estimate (4.9) is a consequence of solvability and the closed graph theorem. ■

4.2 Proof of Theorem 3.1

Fix an integer N and write

$$H_\alpha(x) - H(x) = -\mathcal{S}_\Omega^{k_0}\big(\frac{\partial u_\alpha}{\partial \nu}\big|_{\partial\Omega} - \frac{\partial u}{\partial \nu}\big|_{\partial\Omega}\big)(x) := \sum_{n=1}^{N} \alpha^n H_n(x) + O(\alpha^{N+1}), \quad x \in \mathbb{R}^2, \tag{4.10}$$

with $H_n|_- - H_n|_+ = 0$ on $\partial\Omega$, $(\Delta + k_0^2)H_n = 0$ in $\mathbb{R}^2 \backslash \partial\Omega$, and

$$\big|\big|\partial^i\big(H_\alpha - H - \sum_{n=1}^{N} \alpha^n H_n\big)\big|\big|_{\mathcal{C}^l(\Omega')} \leq C\alpha^{N+1}, \quad i \in \mathbb{N}^2, l \in \mathbb{N},$$

for any $D \subset \Omega' \subset\subset \Omega$, where C depends only on $\mathrm{dist}(\Omega', \partial\Omega), \Omega, N, i$, and l.

Denote $H_0 := H$. By Taylor expansion, we have

$$\begin{aligned} H_\alpha(x + \alpha h(x)\nu(x)) &= \sum_{n=0}^{N} \alpha^n \sum_{m=0}^{n} \sum_{|i|=m} \frac{(h(x))^m}{i!} \partial^i H_{n-m}(x)\nu(x)^i + O(\alpha^{N+1}) \\ &:= \sum_{n=0}^{N} \alpha^n F_n(x) + O(\alpha^{N+1}), \quad x \in \partial D, \end{aligned} \tag{4.11}$$

where

$$F_0(x) = H(x), \quad F_1(x) = H_1(x) + h(x)\frac{\partial H}{\partial \nu}(x), \quad x \in \partial D.$$

Similarly, by Taylor expansion and (2.1), we obtain

$$\begin{aligned} \nabla H_\alpha(\tilde{x}) \cdot \nu(\tilde{x}) &= \sum_{n=0}^{N} \alpha^n \sum_{m=0}^{n} \sum_{s=0}^{m} \sum_{|i|=s} \frac{(h(x))^s}{i!} \partial^i \nabla H_{m-s}(x)(\nu(x))^i \cdot \nu_{n-m}(x) + O(\alpha^{N+1}) \\ &:= \sum_{n=0}^{N} \alpha^n G_n(x) + O(\alpha^{N+1}), \quad x \in \partial D, \end{aligned} \tag{4.12}$$

where

$$G_0(x) = \frac{\partial H}{\partial \nu}(x), \quad G_1(x) = \frac{\partial H_1}{\partial \nu}(x) + h(x)\frac{\partial^2 H}{\partial \nu^2}(x) - h'(t)\frac{\partial H}{\partial T}(x), \quad x \in \partial D.$$

Define $(\varphi^{(n)}, \psi^{(n)})$ as solution of the following system

$$\mathcal{S}_D^k \varphi^{(n)} - \mathcal{S}_D^{k_0} \psi^{(n)} = F_n + \sum_{m=0}^{n-1} \mathcal{S}_D^{k_0, n-m} \psi^{(m)} - \mathcal{S}_D^{k, n-m} \varphi^{(m)} \quad \text{on } \partial D, \tag{4.13}$$

$$\frac{1}{\mu}\frac{\partial(\mathcal{S}_D^k \varphi^{(n)})}{\partial \nu}\Big|_- - \frac{1}{\mu_0}\frac{\partial(\mathcal{S}_D^{k_0} \psi^{(n)})}{\partial \nu}\Big|_+ = \frac{1}{\mu_0} G_n + \sum_{m=0}^{n-1} \frac{1}{\mu_0}\mathcal{K}_D^{k_0, n-m} \psi^{(m)} - \frac{1}{\mu}\mathcal{K}_D^{k, n-m} \varphi^{(m)} \quad \text{on } \partial D. \tag{4.14}$$

Thanks to Theorem 2.1, one can see recursively the existence and the uniqueness of $(\varphi^{(n)}, \psi^{(n)})$.

Define

$$\varphi^N := \sum_{n=0}^{N} \alpha^n \varphi^{(n)}, \quad \psi^N := \sum_{n=0}^{N} \alpha^n \psi^{(n)}, \quad F^N := \sum_{n=0}^{N} \alpha^n F_n, \quad G^N := \sum_{n=0}^{N} \alpha^n G_n \quad \text{on } \partial D.$$

It follows from the theorem 4.1, (4.13), and (4.14) that

$$\begin{cases} \displaystyle\sum_{n=0}^{N} \alpha^n \mathcal{S}_D^{k,n}\big(\tilde{\varphi} - \varphi^N\big) - \sum_{n=0}^{N} \alpha^n \mathcal{S}_D^{k_0,n}\big(\tilde{\psi} - \psi^N\big) = H_\alpha o \Psi_\alpha - F^N + O(\alpha^{N+1}) \quad \text{on } \partial D, \\ \displaystyle\sum_{n=0}^{N} \alpha^n \frac{1}{\mu}\big(-\frac{\delta_{0n}}{2} I + \mathcal{K}_D^{k,n}\big)\big(\tilde{\varphi} - \varphi^N\big) - \sum_{n=0}^{N} \alpha^n \frac{1}{\mu_0}\big(\frac{\delta_{0n}}{2} I + \mathcal{K}_D^{k_0,n}\big)\big(\tilde{\psi} - \psi^N\big) \\ \qquad\qquad\qquad\qquad \displaystyle = \frac{1}{\mu_0}\frac{\partial H_\alpha}{\partial \nu} o \Psi_\alpha - \frac{1}{\mu_0} G^N + O(\alpha^{N+1}) \quad \text{on } \partial D, \end{cases}$$

where $\tilde{\varphi} = \varphi_\alpha o \Psi_\alpha$ and $\tilde{\psi} = \psi_\alpha o \Psi_\alpha$.

The following lemma follows immediately from (4.11), (4.12), and the estimate in (4.9).

Lemma 4.3 *Let $N \in \mathbb{N}$. For α small enough, there exists C depending only on N, k, k_0, the $\mathcal{C}^2$-norm of X, and the $\mathcal{C}^1$-norm of h such that*

$$\Big\|\varphi_\alpha o \Psi_\alpha - \sum_{n=0}^{N} \alpha^n \varphi^{(n)}\Big\|_{L^2(\partial D)} + \Big\|\psi_\alpha o \Psi_\alpha - \sum_{n=0}^{N} \alpha^n \psi^{(n)}\Big\|_{L^2(\partial D)} \leq C\alpha^{N+1},$$

where $(\phi^{(n)}, \psi^{(n)})$ is defined by recursive relations (4.13) and (4.14).

Recall that the domain D is separated apart from the boundary $\partial\Omega$, then

$$\sup_{x\in\partial\Omega, y\in\partial D} \Big|\partial^i G_{k_0}(x,y)\Big| \leq C, \quad i \in \mathbb{N}^2.$$

For some constant C depending on $dist(D, \partial\Omega)$. Let $x \in \partial\Omega$, we get

$$\frac{\partial (G_{D_\alpha}^{k_0}\psi_\alpha)}{\partial \nu}(x) = \int_{\partial D} \Big(\sum_{n=0}^{N}\sum_{|i|=n} \alpha^n \frac{h(y)^n}{i!}\frac{\partial \partial^i G_{k_0}(x,y)}{\partial \nu(x)}\nu^i(y)\Big) \times \Big(\sum_{n=0}^{N} \alpha^n \psi^{(n)}(y)\Big)\Big(\sum_{n=0}^{N} \alpha^n \sigma_n(y)\Big) d\sigma(y) + O(\alpha^{N+1}). \tag{4.15}$$

For $n, m \in \mathbb{N}$ and $x \in \partial\Omega$. Define

$$G_{D,m}^{k_0,n}\psi^{(n-m)}(x) := \sum_{s=0}^{m}\sum_{|i|=s} \int_{\partial D} \frac{h(y)^s}{i!}\partial^i G_{k_0}(x,y)\nu^i(y)\sigma_{m-s}(y)\psi^{(n-m)}(y) d\sigma(y). \tag{4.16}$$

Note that $G_{D,0}^{k_0,0}\psi^{(0)} = G_D^{k_0}\psi$. In view of (4.15), we obtain the following theorem.

Theorem 4.4 *For α small enough. The following formula holds uniformly for $x \in \partial\Omega$:*

$$\frac{\partial(G^{k_0}_{D_\alpha}\psi_\alpha)}{\partial\nu}(x) = \frac{\partial(G^{k_0}_{D}\psi)}{\partial\nu}(x) + \sum_{n=1}^{N}\alpha^n\sum_{m=0}^{n}\frac{\partial(G^{k_0,n}_{D,m}\psi^{(n-m)})}{\partial\nu}(x) + O(\alpha^{N+1}), \tag{4.17}$$

where $G^{k_0,n}_{D,m}\psi^{(n-m)}$ is defined by (4.16). The remainder $O(\alpha^{N+1})$ depends on N, Ω, k_0, k, the $\mathcal{C}^2$-norm of X, the $\mathcal{C}^1$-norm of h, and dist$(D, \partial\Omega)$.

As a consequence of (4.17), it follows from (2.24) and (4.10) that

$$H_n(x) = -\sum_{m=0}^{n}\mathcal{S}^{k_0}_{\Omega}\Big(\frac{\partial(G^{k_0,n}_{D,m}\psi^{(n-m)})}{\partial\nu}\Big|_{\partial\Omega}\Big)(x), \quad x \in \mathbb{R}^2.$$

Now to prove Theorem 3.1. Define the operator $\mathcal{S}^{k_0,n}_{D,m}$ as the $G^{k_0,n}_{D,m}$, only we replace the kernel G_{k_0} by Γ_{k_0}. Thanks to (4.13) and (4.14), one can check that u_n given by

$$u_n(x) = \begin{cases} H_n(x) + \displaystyle\sum_{m=0}^{n}\mathcal{S}^{k_0,n}_{D,m}\psi^{(n-m)}(x), & x \in \Omega\backslash\overline{D}, \\ \displaystyle\sum_{m=0}^{n}\mathcal{S}^{k,n}_{D,m}\varphi^{(n-m)}(x), & x \in D. \end{cases} \tag{4.18}$$

satisfies (3.6), (3.11), and (3.8). Therefore, it follows from lemma 2.2 that

$$\frac{\partial u_n}{\partial\nu}(x) = \sum_{m=0}^{n}\frac{\partial(G^{k_0,n}_{D,m}\psi^{(n-m)})}{\partial\nu}(x), \quad x \in \partial\Omega,$$

for $n = 1, \cdots, N$, and then, Theorem 3.1 can be proved from (2.24) and Theorem 4.4, as desired.

In particular

$$u_1(x) = \begin{cases} H_1(x) + \mathcal{S}^{k_0}_D\psi^{(1)}(x) - \mathcal{S}^{k_0}_D(\tau h\psi^{(0)})(x) + \mathcal{D}^{k_0}_D(h\psi^{(0)})(x), & x \in \Omega\backslash\overline{D}, \\ \mathcal{S}^{k}_D\varphi^{(1)}(x) - \mathcal{S}^{k}_D(\tau h\varphi^{(0)})(x) + \mathcal{D}^{k}_D(h\varphi^{(0)})(x), & x \in D. \end{cases} \tag{4.19}$$

In fact, it follows from (4.3) and (4.13) that

$$\begin{aligned} u_1|_- - u_1|_+ =& h\frac{\partial H}{\partial\nu} + \Big(\mathcal{S}^{k_0,1}_D\psi^{(0)} - \mathcal{S}^{k,1}_D\varphi^{(0)}\Big) + \mathcal{S}^{k_0}_D(\tau h\psi^{(0)}) - \mathcal{S}^{k}_D(\tau h\varphi^{(0)}) \\ &+ \mathcal{D}^{k}_D(h\varphi^{(0)})|_- - \mathcal{D}^{k_0}_D(h\psi^{(0)})|_+ \\ =& h\Big(\frac{\partial H}{\partial\nu} + \frac{\partial(\mathcal{S}^{k_0}_D\psi^{(0)})}{\partial\nu}\Big|_+\Big) - h\frac{\partial(\mathcal{S}^{k_0}_D\varphi^{(0)})}{\partial\nu}\Big|_- \\ =& h\Big(\frac{\mu_0}{\mu} - 1\Big)\frac{\partial u}{\partial\nu}\Big|_- \quad \text{on } \partial D. \end{aligned}$$

By using (4.14), we have

$$\frac{1}{\mu}\frac{\partial u_1}{\partial \nu}\Big|_- - \frac{1}{\mu_0}\frac{\partial u_1}{\partial \nu}\Big|_+ = \frac{1}{\mu_0}\mathcal{K}_D^{k_0,1}\psi^{(0)}\Big|_+ - \frac{1}{\mu}\mathcal{K}_D^{k,1}\varphi^{(0)}\Big|_- + \frac{1}{\mu_0}\Big(h\frac{\partial^2 H}{\partial \nu^2} - h'\frac{\partial H}{\partial T}\Big)$$
$$+ \Big(\frac{1}{\mu_0}\frac{\partial \mathcal{S}_D^{k_0}(\tau h\psi^{(0)})}{\partial \nu}\Big|_+ - \frac{1}{\mu}\frac{\partial \mathcal{S}_D^{k}(\tau h\varphi^{(0)})}{\partial \nu}\Big|_-\Big)$$
$$+ \Big(\frac{1}{\mu}\frac{\partial \mathcal{D}_D^{k}(h\varphi^{(0)})}{\partial \nu}\Big|_- - \frac{1}{\mu_0}\frac{\partial \mathcal{D}_D^{k_0}(h\psi^{(0)})}{\partial \nu}\Big|_+\Big) \quad \text{on } \partial D.$$

Since $(\Delta + k_0^2)H = 0$ in Ω. According to (2.5), we have

$$h\frac{\partial^2 H}{\partial \nu^2} - h'\frac{\partial H}{\partial T} = \tau h\frac{\partial H}{\partial \nu} - \frac{d}{dt}\Big(h\frac{dH}{dt}\Big) - hk_0^2 H \quad \text{on } \partial D.$$

It then follows from (4.5) that u_1 satisfies (3.9).

In order to prove the boundary condition $u_1 = 0$ on $\partial\Omega$. Let

$$w(x) := H_1(x) + \mathcal{S}_D^{k_0}\psi^{(1)}(x) - \mathcal{S}_D^{k_0}(\tau h\psi^{(0)})(x) + \mathcal{D}_D^{k_0}(h\psi^{(0)})(x), \quad x \in \mathbb{R}^2\backslash\overline{\Omega}.$$

Since w satisfies

$$\begin{cases} (\Delta + k_0^2)w = 0 \quad \text{in } \mathbb{R}^2\backslash\overline{\Omega}, \\ \Big|\dfrac{\partial w}{\partial |x|} - ik_0 w\Big| = O(|x|^{-\frac{3}{2}}), \quad |x| \to \infty, \\ \dfrac{\partial w}{\partial \nu}\Big|_{\partial\Omega} = 0, \end{cases} \tag{4.20}$$

and hence $w \equiv 0$ in $\mathbb{R}^2\backslash\overline{\Omega}$ (see [3], pp. 187). This completes the proof.

5 Reconstruction of the interface deformation

Let v be the solution of the following problem

$$\begin{cases} \nabla \cdot \big(\frac{1}{\mu_*}\nabla v\big) + \omega^2 \varepsilon_* v = 0 \quad \text{in } \Omega, \\ v|_{\partial\Omega} = g \in W^2_{\frac{1}{2}}(\partial\Omega), \end{cases} \tag{5.1}$$

Integrating by parts over $\partial\Omega$ yields

$$\int_{\partial\Omega} g\Big(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial u}{\partial \nu}\Big)d\sigma = \alpha\int_{\partial\Omega} g\frac{\partial u_1}{\partial \nu}d\sigma + O(\alpha^2) \tag{5.2}$$
$$= \alpha\int_{\partial D} h\left[\Big(\frac{\mu_0}{\mu} - 1\Big)\Big(\frac{\partial v}{\partial T}\frac{\partial u}{\partial T} + \frac{\mu_0}{\mu}\frac{\partial v}{\partial \nu}\Big|_-\frac{\partial u}{\partial \nu}\Big|_-\Big) + k_0^2\Big(1 - \frac{\varepsilon}{\varepsilon_0}\Big)vu\right]d\sigma + O(\alpha^2).$$

Our goal is to use the formula (5.2) to determine the Fourier coefficients h_p from a finite number of measurements provided that the order of magnitude of h_p is much larger than α.

To illustrate this, we consider Ω to be the unit disk centered at the origin, and D to be the disk centered at the origin with radius ρ. Set $f(\theta) = e^{ip\theta}$, then

$$u(r,\theta) = \begin{cases} \Big(a_p J_{|p|}(k_0 r) + b_p Y_{|p|}(k_0 r)\Big)e^{ip\theta}, \quad \rho \le r \le 1, \\ c_p J_{|p|}(kr)e^{ip\theta}, \quad 0 \le r \le \rho. \end{cases}$$

where $J_{|p|}$ and $Y_{|p|}$ are the Bessel functions of the first and second kind, respectively, and a_p, b_p and c_p are constants and can be computed from the following system

$$\begin{cases} a_p J_{|p|}(k_0) + b_p Y_{|p|}(k_0) = 1, \\ a_p J_{|p|}(k_0\rho) + b_p Y_{|p|}(k_0\rho) - c_p J_{|p|}(k\rho) = 0, \\ a_p J'_{|p|}(k_0\rho) + b_p Y'_{|p|}(k_0\rho) - c_p \sqrt{\frac{\mu_0\varepsilon}{\mu\varepsilon_0}} J'_{|p|}(k\rho) = 0. \end{cases}$$

Let $g(\theta) = e^{iq\theta}$. Define

$$\mathcal{C}_{p,q} := c_p c_q \left(\frac{\rho k^2 \mu_0}{\mu} (\frac{\mu_0}{\mu} - 1) J'_{|p|}(k\rho) J'_{|q|}(k\rho) + J_{|p|}(k\rho) J_{|q|}(k\rho) \Big[\rho k_0^2 (1 - \frac{\varepsilon}{\varepsilon_0}) - (\frac{\mu_0}{\mu} - 1) \frac{pq}{\rho} \Big] \right).$$

It follows from (5.2) that

$$\int_{\partial\Omega} g(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial u}{\partial \nu}) d\sigma = \alpha \mathcal{C}_{p,q} \int_{\partial D} h e^{i(p+q)\theta} d\theta + (\alpha^2). \tag{5.3}$$

The formula (5.3) implies that the Fourier coefficients h_p can be determined from measurements of $\big(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial u}{\partial \nu}\big)$ on $\partial\Omega$ provided that the order of magnitude of h_p is much larger than α. To reconstruct Fourier coefficients more accurately the first ones, the high-order asymptotic expansions should be used.

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