Multiwindow dual Gabor frames with compact support of rational sampling

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Abstract

For positive integers $N, M \in \mathbb{N}$, let $g_n, \widetilde{g}_n \in L^2(R)$ with $\mathrm{supp}(D_b g_n) \cup \mathrm{supp}(D_b \widetilde{g}_n) \subset [0, M]$, where $D_b g(t) := b^{-1/2} g(t/b),\ b > 0$ and $n = 1, 2, ..., N$. We give another set of necessary and sufficient conditions for the multiwindow Gabor system $(G_N; a, b) := \{g_{n;k,l}(t) := g_n(t - ka)e^{2\pi ilbt}\}_{n=1,...,N;\ k,l \in Z}$ and the corresponding Gabor system $(\widetilde{G}_N; a, b)$ to form a pair of dual frames for the rational sampling $ab = P/Q$ with $P, Q \in \mathbb{N}$, in addition to the Zibulski-Zeevi and Janssen conditions. The conditions come from the back transform of Zibulski-Zeevi condition to the time domain but are more informative for the applications. As applications, we show that a multiwindow Gabor system $(G_N; 1, 1)$ forms an orthonormal basis if and only if $N = 1$ and $|g_1(t)| = \sum_{m=0}^{M-1} \chi_{m+E_m}(t)$ a.e. where $\{E_m\}_{m=0,...,M-1}$ forms a Lebesgue measurable partition of the unit interval $[0, 1)$. Our criteria also provide a rich family of multiwindow dual Gabor frames and multiwindow tight Gabor frames for $L^2(\mathbb{R})$ for the particular choices of $P, Q, N, M \in \mathbb{N}$.

1 Introduction

We are concerned with multiwindow Gabor systems with compact support of rational sampling in $L^2(\mathbb{R})$. Throughout the paper we assume $ab = P/Q$, $P, Q \in \mathbb{N}$, $\gcd(P, Q) = 1$, and $N, M \in \mathbb{N}$. Let $G_N := \{g_n \in L^2(\mathbb{R}) \mid n = 1, ..., N\}$, $\widetilde{G}_N := \{\widetilde{g}_n \in L^2(\mathbb{R}) \mid n = 1, ..., N\}$ and consider the multiwindow Gabor system $(G_N; a, b) := \{g_{n;k,l}(t) := g_n(t - ka)e^{2\pi ilbt} \mid k, l \in \mathbb{Z}\}$ and the corresponding multiwindow Gabor system $(\widetilde{G}_N; a, b)$. We recall that the Gabor systems $(G_N; a, b)$ and $(\widetilde{G}_N; a, b)$ form a pair of dual frames if and only if they are Bessel sequences and

$$< f, h >= \sum_{k,l \in \mathbb{Z}} \sum_{n=1}^{N} < f, \widetilde{g}_{n;ka,lb} >< g_{n;ka,lb}, h >, \quad {}^{\forall} f, {}^{\forall} h \in L^2(\mathbb{R}), \tag{1.1}$$

and that the Gabor system $(G_N; a, b)$ is a tight frame with bound B if and only if

$$\sum_{k,l \in \mathbb{Z}} \sum_{n=1}^{N} |< f, g_{n;ka,lb} >|^2 = B||f||^2, \quad {}^{\forall} f \in L^2(R). \tag{1.2}$$

We only consider compactly supported generators in order to avoid cumbersome infinite dimensional matrices and to construct concrete examples of Gabor frames with compact support. We also restrict ourselves to the rational sampling because we rely on the characterizations of

Zibulski-Zeevi [15] in terms of Zak transform, which is only applicable to the rational sampling. The Zak transform is defined as

$$Zg(t,w) := \sum_{k\in\mathbb{Z}} g\,(t+k)\, e^{-2\pi i k w}. \tag{1.3}$$

For functions $f \in C_c(R)$, it is defined pointwise, but for general functions in $L^2(R)$, the series defining Zf converges in $L^2(I^2)$ where $I := [0,1)$ for all $f \in L^2(R)$. It is well known that the Zak transform Z is a unitary operator from $L^2(R)$ onto $L^2(I^2)$. For the basic facts about the Zak transform we refer to [4, 10, 15]. We define the dilation operator $D_c f(t) := c^{-1/2} f(t/c),\ c > 0$, which is also a unitary operator on $L^2(\mathbb{R})$.

The main interest in the Gabor system has been to find the criteria on a single window Gabor system or a multiwindow Gabor system to form a tight frame [8, 9, 10, 11, 12, 14, 15], and to construct dual pairs of Gabor systems [5, 6, 7].

In this paper, we derive another set of necessary and sufficient conditions (Theorem 2.1) for the Gabor systems $(G_N; a, b)$ and $(\widetilde{G}_N; a, b)$ to form a pair of dual frames in addition to the Zibulski-Zeevi condition and Janssen condition. Our criteria have many interesting consequences and provide a rich family of multiwindow dual Gabor frames and multiwindow tight frames (Examples 4.1 $\sim$ 4.4) In Section 2, we state the main theorem and its consequences. The proof of the main theorem will be given in Section 3. The illustrations and examples will be given in Section 4.

2 Main result and consequences

In this section, we transform back the characterization of Zibulski-Zeevi in the Zak transform domain to the time domain and obtain the following main theorem, whose proof will be given in Section 3. Before stating the main theorem, we note from Proposition 8.2.4 in [4] that the multiwindow Gabor systems $(G_N; a, b)$ and $(\widetilde{G}_N; a, b)$ are dual to each other if and only if the dilated systems $(D_c G_N; ac, \frac{b}{c})$ and $(D_c \widetilde{G}_N; ac, \frac{b}{c})$ are dual to each other, where $D_c G_N := \{D_c g_n \mid g_n \in G_N\}$ for a given $c > 0$.

Theorem 2.1 *Let $ab = P/Q,\ P, Q \in \mathbb{N}$, $\gcd(P,Q) = 1$ and let $g_n, \widetilde{g}_n \in L^2(R)$ with $\operatorname{supp}(D_b g_n) \cup \operatorname{supp}(\widetilde{D}_b g_n) \subset [0, M]$, $n = 1, 2, ..., N$. Then the followings are equivalent.*
(a) *The Gabor systems $(G_N; a, b)$ and $(\widetilde{G}_N; a, b)$ are dual to each other.*
(b) *(Zibulski-Zeevi condition [15]) For $\tilde{p}, p \in \{0, ..., P-1\}$,*

$$\frac{1}{P}\sum_{n=1}^{N}\sum_{q=0}^{Q-1} \overline{ZD_b\widetilde{g}_n\left(t - \frac{qP}{Q}, w + \frac{\widetilde{p}}{P}\right)} ZD_b g_n\left(t - \frac{qP}{Q}, w + \frac{p}{P}\right) = \delta_{\widetilde{p},p}, \tag{2.1}$$

for a.e $(t,w) \in I \times \frac{1}{P} I$.
(c) *For $\tilde{p}, p \in \{0, ..., P-1\}$ and for $m \in \{0, \pm1, \pm2, ..., \pm\check{M}\}$ with $\check{M} := \lfloor M + (Q-1)P/Q \rfloor$,*

$$\frac{1}{P}\sum_{k-j=m} a_{k,j}^{\tilde{p},p}(t) = \delta_{\tilde{p},p}\delta_{0,m}, \qquad a.e.\ \ t \in I, \tag{2.2}$$

where

$$\left[a_{k,j}^{\widetilde{p},p}(t)\right]_{k,j=1,..,\check{M}+1} := \sum_{q=0}^{Q-1} \widetilde{\mathbf{M}}_{\widetilde{p}}^{*}\, \mathbf{M}_p \left(t - \frac{qP}{Q}\right), \text{ for } \widetilde{p}, p = 0, ..., P-1, \tag{2.3}$$

$$\mathbf{M}_p := \left(\mathbf{g}_b(t), \mathbf{g}_b(t+1)e^{-2\pi i \frac{p}{P}}, ..., \mathbf{g}_b(t+\check{M})e^{-2\pi i \check{M} \frac{p}{P}}\right)_{N\times\check{M}},$$

$$\mathbf{g}_b(t) := \left(D_b g_1(t), ..., D_b g_N(t)\right)^T,$$

and $\widetilde{\mathbf{M}}_{\widetilde{p}}$ *is defined in the same way with* $\widetilde{\mathbf{g}}_b(t) = (D_b\widetilde{g}_1(t), ..., D_b\widetilde{g}_N(t))^T$.
(d) *For* $p \in \{0, ..., P-1\}$, $m \in \{0, \pm 1, ..., \pm\check{M}\}$,

$$\sum_{\mu=0}^{\lfloor \check{M}/P \rfloor} \sum_{q=0}^{Q-1} \overline{\widetilde{\mathbf{g}}_b \left(t - \frac{qP}{Q} + (\mu P + p + m)\right)} \cdot \mathbf{g}_b\left(t - \frac{qP}{Q} + (\mu P + p)\right) = \delta_{m,0}, \tag{2.4}$$

for a.e. $t \in I$.
(e) *For* $p \in \{0, ..., P-1\}$, $m \in \{0, \pm 1, ..., \pm\check{M}\}$,

$$\sum_{\mu=0}^{\lfloor \check{M}/P \rfloor} \sum_{q=0}^{Q-1} \overline{\widetilde{\mathbf{g}}_b \left(t + \frac{p}{Q} + \frac{qP}{Q} + \mu P + m\right)} \cdot \mathbf{g}_b\left(t + \frac{p}{Q} + \frac{qP}{Q} + \mu P\right) = \delta_{m,0}, \tag{2.5}$$

for a.e. $t \in \frac{1}{Q} I$.
(f) *(Jassen condition [3])*

$$\frac{1}{b}\sum_{n=1}^{N}\sum_{k\in\mathbb{Z}} \overline{\widetilde{g}_n\left(t + \frac{m}{b} + ka\right)} \cdot g_n(t+ka) = \delta_{m,0} \quad a.e. \quad t \in \mathbb{R}. \tag{2.6}$$

Remark 2.2 The condition (b) is the dual form of the Zibulski-Zeevi condition for the tight Gabor system in [15]. The condition (c) means, in the block matirx

$$\left[\sum_{q=0}^{Q-1} \widetilde{\mathbf{M}}_{\widetilde{p}}^{*}\, \mathbf{M}_p\right]_{\widetilde{p},p=0,...,P-1}, \tag{2.7}$$

that the diagonal blocks have the trace P and off-diagonals sum to 0 and that the off-diagonal blocks have on-and-off diagonals sum to 0. Condition (d) and (e) mean the biorthogonality of certain "block" vectors. Condition(e) is not a restriction of condition (d) in a short interval and is very useful to construct examples (See section 4). The Janssen condition (f) is true even for irrational sampling rate $ab < 1$.

Some particular cases of Theorem 2.1 are also of interest and will be stated as corollaries without proof, where the corresponding conditions for (c) and (e) of Theorem 2.1 are given.

Corollary 2.3 *($Q = 1$) Let* $ab = P \in \mathbb{N}$. *Then the followings are equivalent.*
(a) $(G_N; a, b)$ *and* $(\widetilde{G}_N; a, b)$ *are dual to each other.*
(b) *For* $\tilde{p}, p \in \{0, ..., P-1\}$ *and for* $m \in \{0, \pm 1, \pm 2, ..., \pm M\}$,

$$\frac{1}{P}\sum_{k-j=m} a_{k,j}^{\tilde{p},p}(t) = \delta_{\tilde{p},p}\delta_{0,m} \quad a.e. \quad t \in I \tag{2.8}$$

with corresponding notation as in Theorem 2.1.
(c) *For* $p \in \{0, ..., P-1\}$, $m \in \{0, \pm 1, \pm 2, ..., \pm M\}$,

$$\sum_{\mu=0}^{\lfloor M/P \rfloor} \overline{\widetilde{\mathbf{g}}_b (t+p+\mu P+m)} \cdot \mathbf{g}_b (t+p+\mu P) = \delta_{m,0}, \quad a.e. \quad t \in I.$$

Corollary 2.4 *($P = 1$) Let* $ab = 1/Q$ *with* $Q \in \mathbb{N}$*. Then the followings are equivalent.*
(a) $(G_N; a, b)$ *and* $(\widetilde{G}_N; a, b)$ *are dual to each other.*
(b) *For* $m \in \{0, \pm 1, \pm 2, ..., \pm M\}$,

$$\sum_{k-j=m} a_{k,j}(t) = \delta_{0,m} \quad a.e. \ t \in I. \tag{2.9}$$

with the matrix $\left[a_{i,j}(t) := a_{i,j}^{0,0}(t)\right]_{(M+1)\times(M+1)}$ *as in Theorem 2.1.*
(c) *For* $m \in \{0, \pm 1, ..., \pm M\}$,

$$\sum_{\mu=0}^{M} \sum_{q=0}^{Q-1} \overline{\widetilde{\mathbf{g}}_b \left(t + \frac{q}{Q} + \mu + m\right)} \cdot \mathbf{g}_b \left(t + \frac{q}{Q} + \mu\right) = \delta_{m,0}, \quad a.e. \quad t \in \frac{1}{Q} I.$$

Corollary 2.5 *(Tight frame) Let* $ab = P/Q$ *with* $P, Q \in \mathbb{N}$*, and* $\gcd(P, Q) = 1$*. Then the followings are equivalent.*
(a) $(G_N; a, b)$ *is a tight frame for* $L^2(R)$ *with bound* B*.*
(b) *For* $\tilde{p}, p \in \{0, ..., P-1\}$ *and for* $m \in \{0, \pm 1, \pm 2, ..., \pm \check{M}\}$,

$$\frac{1}{P} \sum_{k-j=m} a_{k,j}^{\tilde{p},p}(t) = B \delta_{\tilde{p},p} \delta_{0,m} \quad a.e. \quad t \in I \tag{2.10}$$

with corresponding notation as in Theorem 2.1.
(c) *For* $p \in \{0, ..., P-1\}$, $m \in \{0, \pm 1, ..., \pm \check{M}\}$,

$$\sum_{\mu=0}^{\lfloor \check{M}/P \rfloor} \sum_{q=0}^{Q-1} \overline{\mathbf{g}_b \left(t + \frac{p}{Q} + \frac{qP}{Q} + \mu P + m\right)} \cdot \mathbf{g}_b \left(t + \frac{p}{Q} + \frac{qP}{Q} + \mu P\right) = B \delta_{m,0},$$

for a.e. $t \in \frac{1}{Q} I$.

Corollary 2.6 *(Tight frame for the case a=b=1) The followings are equivalent.*
(a) $(G_N; 1, 1)$ *is a tight frame for* $L^2(R)$ *with bound* B*.*
(b) *For* $m \in \{0, \pm 1, \pm 2, ..., \pm (M-1)\}$,

$$\sum_{k-j=m} a_{k,j}(t) = B \delta_{0,m} \quad a.e. \quad t \in I \tag{2.11}$$

with the matrix $\left[a_{k,j}(t) := a_{k,j}^{0,0}(t) = \mathbf{M}_0^* \ \mathbf{M}_0 \ (t)\right]_{M \times M}$ *as in Theorem 2.1.*
(c) *For* $m \in \{0, \pm 1, \pm 2, ..., \pm (M-1)\}$,

$$\sum_{\mu=0}^{M-1} \overline{\mathbf{g} (t + \mu + m)} \cdot \mathbf{g} (t + \mu) = B \delta_{m,0}, \quad a.e. \quad t \in I.$$

where $\mathbf{g} := \mathbf{g}_1$.

As an application of Corollary 2.6, we completely determine when a multiwindow Gabor system $(G_N;1,1)$ forms an orthonormal basis for $L^2(R)$.

Theorem 2.7 *(Orthonormal basis for the case a=b=1) Let $g_n \in L^2(R)$ with* $\mathrm{supp}(g_n) \subset [0,M]$ *for $n=1,...,N$. Then Gabor system $(G_N;1,1)$ is an orthonormal basis for $L^2(R)$ if and only if $N=1$ and $|g_1(t)| = \sum_{m=0}^{M-1}\chi_{m+E_m}(t)$ a.e. where $\{E_m\}_{m=0,...,M-1}$ is a Lebesgue measurable partition of* $[0,1]$.

Proof. Suppose $\{g_{n;k,l} := g_n(t-k)e^{2\pi ilt}\}_{n=1,...,N;\ k,l\in Z}$ is an orthonormal basis for $L^2(R)$. Then, in addition to the condition (2.11), we need two more conditions; $||g_n||_{L^2(R)} = 1$ for all $n=1,...,N$ and the frame bound $B=1$. Setting $k-j=0$ in (2.11), we have for a.e. $t\in I$,

$$1 = B = \sum_{k-j=0} a_{k,j}(t) = \sum_{m'=0}^{M-1} ||\mathbf{g}(t+m')||^2 = \sum_{n=1}^{N}\sum_{m'=0}^{M-1} |g_n(t+m')|^2, \tag{2.12}$$

which implies

$$1 = \int_0^1 \sum_{n=1}^{N}\sum_{m'=0}^{M-1} |g_n(t+m')|^2 dt = \sum_{n=1}^{N}\int_0^M |g_n(t)|^2 dt = N.$$

Since $P=Q=1$ and $N=1$, the matrix in (2.11) has the form

$$\begin{pmatrix} \overline{g(t)} \\ \overline{g(t+1)} \\ \vdots \\ \overline{g(t+M-1)} \end{pmatrix} \begin{pmatrix} g(t), & g(t+1), & \cdots, & g(t+M-1) \end{pmatrix} \tag{2.13}$$

with

$$\sum_{k-j=m'} a_{k,j}(t) = B\delta_{0,m'} \text{ for a.e. } t\in I.$$

First, we take $m' = 1-M$ then

$$a_{1,M} = \overline{g(t)}g(t+M-1) = 0 \text{ for a.e. } t\in I,$$

which is equivalent to $|E_0\cap E_{M-1}| = 0$, where $E_m := \mathrm{supp}(g)\cap[m,m+1)-m$ and $|\cdot|$ denote the Lebesgue measure. Next, we take $m' = 2-M$ then

$$a_{1,M-1} + a_{2,M} = \overline{g(t)}g(t+M-2) + \overline{g(t+1)}g(t+M-1) = 0 \text{ for a.e. } t\in I,$$

which, together with $|E_0\cap E_{M-1}| = 0$, implies $|E_0\cap E_{M-2}| = 0$ and $|E_1\cap E_{M-1}| = 0$. By continuing this process until $m' = -1$, we see

$$|E_m \cap E_{\widetilde{m}}| = 0, \quad \text{if } m \neq \widetilde{m}. \tag{2.14}$$

Now, we take $m'=0$ (i.e., $k=j$) then

$$\sum_{k=0}^{M-1} a_{k,k} = \sum_{k=0}^{M-1} |g(t+k)|^2 = B, \quad \text{a.e. } t\in I, \tag{2.15}$$

which means

$$\sum_{m=0}^{M-1} |g(t+m)|^2 \ \chi_{E_m}(t) = B\chi_I(t), \quad \text{a.e. } t \in I. \tag{2.16}$$

Hence

$$\left| \bigcup_{m=0}^{M-1} E_m \ \triangle \ I \right| = 0, \tag{2.17}$$

where $A \triangle B$ is the symmetric difference. From (2.14) and (2.17), we see $\{E_m\}_{m=0,\dots,M-1}$ forms a Lebesgue measurable partition of $[0,1]$. Therefore, (2.16) means that

$$|g(t)| = \sqrt{B} \sum_{m=0}^{M-1} \chi_{m+E_m}(t) \ \text{ a.e. } t \in I.$$

where $\{E_m\}_{m=0,\dots,M-1}$ a Lebesgue measurable partition of $[0,1]$.

The converse is trivial.

3 Proof of the main theorem

For the proof of the main theorem, we need several lemmas which are of interest in itself.

Lemma 3.1 *Let $2 \le P \in \mathbb{N}$ and $\omega := e^{2\pi i/P}$ be the P-th primitive root of unity. For the Vandemonde matrix*

$$\mathbf{A} := \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{(P-1)} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(P-1)} \\ & & & \vdots & \\ 1 & \omega^{(P-1)} & \omega^{2(P-1)} & \cdots & \omega^{(P-1)\cdot(P-1)} \end{pmatrix}_{P\times P},$$

let $\mathbf{A}_{l,k}$ be the $(P-1)\times(P-1)$ matrix obtained from $\mathbf{A}$ by removing l-th row and k-th column. Then

$$\det(\mathbf{A}_{l,k}) = (-1)^{(l-l')+(k-k')} \ \omega^{(l-l')+(k-k')-(lk-l'k')} \ \det\left(\mathbf{A}_{l',k'}\right).$$

Proof. It suffice to show that

$$\det(\mathbf{A}_{l,k}) = (-1)^{(l+k)} \ \omega^{(l+k)-lk} \det(\mathbf{A}_{P-1,P-1}), \quad {}^{\forall}l, \ {}^{\forall}k. \tag{3.1}$$

The determinant $\det(\mathbf{A}_{l,k})$ has the form

$$\det(\mathbf{A}_{l,k}) := \begin{vmatrix} 1 & 1 & & 1 & 1 & & 1 \\ 1 & \omega & & \omega^{(k-2)} & \omega^{k} & & \omega^{(P-1)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & \omega^{(l-2)} & \cdots & \omega^{(k-2)(l-2)} & \omega^{k(l-2)} & \cdots & \omega^{(P-1)(l-2)} \\ 1 & \omega^{l} & & \omega^{(k-2)l} & \omega^{kl} & & \omega^{(P-1)l} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & \omega^{(P-1)} & & \omega^{(k-2)(P-1)} & \omega^{k(P-1)} & & \omega^{(P-1)(P-1)} \end{vmatrix}. \tag{3.2}$$

We factor out all the factors in the k-th column of $\mathbf{A}_{l,k}$ to have $\vec{1}$ in the column and then replace all negative powers to the equivalent positive powers of ω to have $\det(\mathbf{A}_{l,k}) =$

$$\omega^{\frac{(P-1)P}{2}k-(l-1)k} \begin{vmatrix} 1 & 1 & & 1 & 1 & & 1 \\ \omega^{P-k} & \omega^{(P-k+1)} & & \omega^{P-2} & 1 & & \omega^{(P-k-1)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \omega^{(P-k)(l-2)} & \omega^{(P-k+1)(l-2)} & \dots & \omega^{(P-2)(l-2)} & 1 & \dots & \omega^{(P-k-1)(l-2)} \\ \omega^{(P-k)l} & \omega^{(P-k+1)l} & & \omega^{(P-2)l} & 1 & & \omega^{(P-k-1)l} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \omega^{(P-k)(P-1)} & \omega^{(P-k+1)(P-1)} & & \omega^{(P-2)(P-1)} & 1 & & \omega^{(P-k-1)(P-1)} \end{vmatrix} \tag{3.3}$$

By interchanging columns to have increasing powers of ω in each row, (3.3) can be made to be

$$(-1)^{(k-1)(P-k)}\ \omega^{\frac{(P-1)P}{2}k-(l-1)k} \begin{vmatrix} 1 & 1 & & 1 \\ 1 & \omega & & \omega^{(P-2)} \\ \vdots & \vdots & & \vdots \\ 1 & \omega^{(l-2)} & \dots & \omega^{(P-2)(l-2)} \\ 1 & \omega^{l} & & \omega^{(P-2)l} \\ \vdots & \vdots & & \vdots \\ 1 & \omega^{(P-1)} & & \omega^{(P-2)(P-1)} \end{vmatrix}. \tag{3.4}$$

We repeat the process with rows as in columns to have

$$\begin{aligned} \det(\mathbf{A}_{l,k}) &= (-1)^{(k-1)(P-k)}\ \omega^{\frac{(P-1)P}{2}k-(l-1)k} \times (-1)^{(l-1)(P-l)}\ \omega^{\frac{(P-2)(P-1)}{2}l}\det(\mathbf{A}_{P-1,P-1}) \\ &= \left((-1)^{P(l+k)}\ \omega^{\frac{(P-1)P}{2}(k+l)}\right)\ \omega^{(k+l)-lk}\det(\mathbf{A}_{P-1,P-1}). \end{aligned} \tag{3.5}$$

We observe that if P is odd then $\omega^{P(P-1)/2} = 1$; hence

$$(-1)^{P(l+k)}\ \omega^{\frac{(P-1)P}{2}(k+l)} = (-1)^{P(l+k)} = (-1)^{(l+k)}.$$

and if P is even then $\omega^{(P/2)(P-1)} = (-1)^{P-1} = -1$; hence

$$(-1)^{P(l+k)}\ \omega^{\frac{(P-1)P}{2}(k+l)} = (-1)^{(l+k)}.$$

Therefore we have

$$\det(\mathbf{A}_{l,k}) = (-1)^{(l+k)}\ \omega^{(k+l)-lk}\det(\mathbf{A}_{P-1,P-1}). \tag{3.6}$$

Lemma 3.2 *Let $P \in \mathbb{N}$, $r_p \in \mathbb{R}$ for $p = 0, ..., P-1$, and let ω be the P-th primitive root of unity. If*

$$\sum_{p=0}^{P-1} r_p = c \in \mathbb{R}, \quad \sum_{p=0}^{P-1} r_p\omega^p = 0, \quad \cdots \quad , \quad \sum_{p=0}^{P-1} r_p\omega^{p(P-1)} = 0.$$

then $r_0 = r_1 = \cdots = r_{P-1} = \frac{c}{P}$.

Proof. We can rewrite the given condition in the matrix form as follows.

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{(P-1)} \\ & & & \vdots & \\ 1 & \omega^{(P-1)} & \omega^{2(P-1)} & \cdots & \omega^{(P-1)\cdot(P-1)} \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_{P-1} \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

By the Cramer's rule and Lemma 3.1, for $k = 1, ..., P-1$,

$$\begin{aligned} r_k &= c(-1)^{1+k+1}\det\left(\mathbf{A}_{1,k+1}\right)/\det\left(\mathbf{A}\right) \\ &= c(-1)^{1+k+1}\cdot(-1)\det\left(\mathbf{A}_{1,k}\right)/\det\left(\mathbf{A}\right) \\ &= c(-1)^{1+k}\det\left(\mathbf{A}_{1,k}\right)/\det\left(\mathbf{A}\right) \\ &= r_{k-1}, \end{aligned}$$

where $\mathbf{A}$ and $\mathbf{A}_{l,k}$ are defined as in Theorem 3.1. Since $\sum_{p=0}^{P-1} r_p = c$, the conclusion follows.

Lemma 3.3 *Let* $P, Q \in \mathbb{N}$, $\gcd(P,Q) = 1$. *For* $t \in [0, P/Q) \cap I$, *define* $\tau := t\%\frac{1}{Q}$ *and* $t_p := (t+p)\%\frac{P}{Q}$ *for* $p = 0, ..., P-1$, *where* $\alpha\%\beta := \alpha - \lfloor\frac{\alpha}{\beta}\rfloor\beta$ *for* $\alpha, \beta > 0$. *Then*

$$\begin{aligned} &(a) \quad \left|t_{p_1} - t_{p_2}\right|\%\frac{1}{Q} = 0, \\ &(b) \quad t_{p_1} \neq t_{p_2} \text{ if } p_1 \neq p_2, \text{ and} \\ &(c) \quad \{t_p \ : \ p = 0, ..., P-1\} = \left\{\frac{p}{Q} + \tau \ : \ p = 0, ..., P-1\right\}. \end{aligned}$$

Proof. We note $\tau \in [0, 1/Q)$ and $t_p \in [0, P/Q)$. For (a), we observe that

$$t_p\%\frac{1}{Q} = \left\{(t+p)\%\frac{P}{Q}\right\}\%\frac{1}{Q} = (t+p)\%\frac{1}{Q} = \tau, \quad {}^\forall p = 0, ..., P-1.$$

This implies that there exists $n_1, n_2 \in \{0, ..., P-1\}$ such that

$$t_{p_1} = \frac{n_1}{Q} + \tau \text{ and } t_{p_2} = \frac{n_2}{Q} + \tau.$$

Therefore,

$$\left|t_{p_1} - t_{p_2}\right|\%\frac{1}{Q} = \left|\frac{n_1}{Q} - \frac{n_2}{Q}\right|\%\frac{1}{Q} = 0.$$

For (b), suppose $t_{p_1} = t_{p_2}$ for some $p_1 \neq p_2$. Then we have

$$0 = \left|t + p_1 - t - p_2\right|\%\frac{P}{Q} = \left|p_1 - p_2\right|\%\frac{P}{Q},$$

and this is equivalent to

$$0 = \left|p_1 - p_2\right|Q\%P.$$

That is, $|p_1-p_2|Q$ is a multiple of P. Since $\gcd(P,Q)=1$, $|p_1-p_2|$ is a multiple of P. This contradicts to $|p_1-p_2|<P$. The contradiction shows (b). Then (c) follows from (a) and (b).

Now we present the proof of Theorem 2.1.

Proof of **Theorem 2.1.**
$(a) \Longleftrightarrow (b)$ is the dual form of the Zibulski-Zeevi characterization for the tight Gabor frame with rational sampling given in [15].
$(b) \Longrightarrow (c)$: We observe that for a.e. $t \in I$ and $q \in \{0,...,Q-1\}$,

$$D_b\widetilde{g}_n\left(t-\frac{qP}{Q}+m\right)=0=D_bg_n\left(t-\frac{qP}{Q}+m\right) \quad \text{if } m \notin \{0,...,\check{M}\}, \tag{3.7}$$

where $\check{M}=\lfloor M+(Q-1)P/Q\rfloor$.
For a.e. $(t,w)\in I\times\frac{1}{P}I$,

$$P\delta_{\widetilde{p},p}=\sum_{n=1}^{N}\sum_{q=0}^{Q-1}\overline{ZD_b\widetilde{g_n}\left(t-\frac{qP}{Q},w+\frac{\widetilde{p}}{P}\right)}ZD_bg_n\left(t-\frac{qP}{Q},w+\frac{p}{P}\right) \tag{3.8}$$

$$=\sum_{n=1}^{N}\sum_{q=0}^{Q-1}\overline{\left(\sum_{\widetilde{k}=0}^{\check{M}}D_b\widetilde{g_n}\left(t-\frac{qP}{Q}+\widetilde{k}\right)e^{-2\pi i\widetilde{k}(w+\frac{\widetilde{p}}{P})}\right)}\left(\sum_{k=0}^{\check{M}}D_bg_n\left(t-\frac{qP}{Q}+k\right)e^{-2\pi ik(w+\frac{p}{P})}\right)$$

$$=\sum_{n=1}^{N}\sum_{q=0}^{Q-1}\sum_{\widetilde{k}=0}^{\check{M}}\sum_{k=0}^{\check{M}}\left(\overline{D_b\widetilde{g_n}\left(t-\frac{qP}{Q}+\widetilde{k}\right)}D_bg_n\left(t-\frac{qP}{Q}+k\right)\right)e^{2\pi i\frac{(\widetilde{k}\widetilde{p}-kp)}{P}}e^{-2\pi i(k-\widetilde{k})w}$$

$$=\sum_{q=0}^{Q-1}\sum_{\widetilde{k}=0}^{\check{M}}\sum_{m=-\widetilde{k}}^{\check{M}-\widetilde{k}}\left(\sum_{n=1}^{N}\overline{D_b\widetilde{g_n}\left(t-\frac{qP}{Q}+\widetilde{k}\right)}D_bg_n\left(t-\frac{qP}{Q}+(\widetilde{k}+m)\right)\right)e^{2\pi i\frac{(\widetilde{k}\widetilde{p}-(\widetilde{k}+m)p)}{P}}e^{-2\pi imw}$$

$$=\sum_{q=0}^{Q-1}\sum_{\widetilde{k}=0}^{\check{M}}\sum_{m=-\check{M}}^{\check{M}}\left(\sum_{n=1}^{N}\overline{D_b\widetilde{g_n}\left(t-\frac{qP}{Q}+\widetilde{k}\right)}D_bg_n\left(t-\frac{qP}{Q}+(\widetilde{k}+m)\right)\right)e^{2\pi i\frac{(\widetilde{k}\widetilde{p}-(\widetilde{k}+m)p)}{P}}e^{-2\pi imw}$$

$$=\sum_{m=-\check{M}}^{\check{M}}\sum_{\widetilde{k}=0}^{\check{M}}\left(\sum_{q=0}^{Q-1}\overline{\widetilde{\mathbf{g}}_b\left(t-\frac{qP}{Q}+\widetilde{k}\right)e^{2\pi i\widetilde{k}\frac{\widetilde{p}}{P}}}\cdot\mathbf{g}_b\left(t-\frac{qP}{Q}+(\widetilde{k}+m)\right)e^{-2\pi i\frac{(\widetilde{k}+m)p}{P}}\right)e^{-2\pi imw}. \tag{3.9}$$

For the second last equality, we use (3.7). We observe that

$$\widetilde{\mathbf{M}}_{\widetilde{p}}^{*}\mathbf{M}_p\,(t)=\begin{pmatrix}\overline{\widetilde{\mathbf{g}}_b(t)}^T\\ \overline{\widetilde{\mathbf{g}}_b(t+1)}^T e^{2\pi i\frac{\widetilde{p}}{P}}\\ \vdots\\ \overline{\widetilde{\mathbf{g}}_b(t+\check{M})}^T e^{2\pi i\check{M}\frac{\widetilde{p}}{P}}\end{pmatrix}\begin{pmatrix}\mathbf{g}_b(t) & \mathbf{g}_b(t+1)e^{-2\pi i\frac{p}{P}} & \cdots & \mathbf{g}_b(t+\check{M})e^{-2\pi i\check{M}\frac{p}{P}}\end{pmatrix},$$

and that

$$a_{k,j}^{\widetilde{p},p}(t)=\left[\sum_{q=0}^{Q-1}\widetilde{\mathbf{M}}_{\widetilde{p}}^{*}\mathbf{M}_p\left(t-\frac{qP}{Q}\right)\right]_{k,j}, \qquad k,j\in\{1,...,\check{M}+1\}.$$

Defining $a_{k,j}^{\widetilde{p},p}(t)=0$ for $k,j\notin\{1,...,\check{M}+1\}$, (3.9) becomes

$$\sum_{m=-\check{M}}^{\check{M}}\sum_{k=1}^{\check{M}+1}\left[\sum_{q=0}^{Q-1}\widetilde{\mathbf{M}}_{\widetilde{p}}^{*}\mathbf{M}_{p}\left(t-\frac{qP}{Q}\right)\right]_{k,k+m} e^{-2\pi imw}$$
$$=\sum_{m\in\mathbb{Z}}\left(\sum_{j-k=m}a_{k,j}^{\widetilde{p},p}(t)\right)e^{-2\pi imw}. \tag{3.10}$$

Combining (3.10) and the left hand side of (3.8), we have

$$P\delta_{\widetilde{p},p}=\sum_{m\in\mathbb{Z}}\left(\sum_{j-k=m}a_{k,j}^{\widetilde{p},p}(t)\right)e^{-2\pi imw}.$$

Therefore, by the uniqueness of Fourier series expansion,

$$P\delta_{\widetilde{p},p}\delta_{m,0}=\sum_{j-k=m}a_{k,j}^{\widetilde{p},p}(t).$$

$(c)\implies(d)$: The (p_1,p_2)-th block of (2.7) is the the following matrix with trace $P\delta_{p_1,p_2}$ and off-diagonal sum 0.

$$\begin{bmatrix}
\overline{\widetilde{\mathbf{g}}}\cdot\mathbf{g}(0,0) & \overline{\widetilde{\mathbf{g}}}\cdot\mathbf{g}(0,1)\ e^{2\pi i(-p_2)/P} & \cdots & \overline{\widetilde{\mathbf{g}}}\cdot\mathbf{g}(0,\check{M})\ e^{2\pi i(-\check{M}p_2)/P}\\
\overline{\widetilde{\mathbf{g}}}\cdot\mathbf{g}(1,0)\ e^{2\pi i\frac{p_1}{P}} & \overline{\widetilde{\mathbf{g}}}\cdot\mathbf{g}(1,1)\ e^{2\pi i\frac{(p_1-p_2)}{P}} & \cdots & \overline{\widetilde{\mathbf{g}}}\cdot\mathbf{g}(1,\check{M})\ e^{2\pi i\frac{(p_1-\check{M}p_2)}{P}}\\
\vdots & & \ddots & \vdots\\
\overline{\widetilde{\mathbf{g}}}\cdot\mathbf{g}(\check{M},0)\ e^{2\pi i\check{M}p_1/P} & \overline{\widetilde{\mathbf{g}}}\cdot\mathbf{g}(\check{M},1)\ e^{2\pi i(\check{M}p_1-p_2)/P} & \cdots & \overline{\widetilde{\mathbf{g}}}\cdot\mathbf{g}(\check{M},\check{M})\ e^{2\pi i\frac{\check{M}(p_1-p_2)}{P}}
\end{bmatrix} \tag{3.11}$$

where $\overline{\widetilde{\mathbf{g}}}\cdot\mathbf{g}(m_1,m_2):=\sum_{q=0}^{Q-1}\overline{\widetilde{\mathbf{g}}_b\left(t-\frac{qP}{Q}+m_1\right)}\cdot\mathbf{g}_b(t-\frac{qP}{Q}+m_2)$.

First, the trace (i.e., $m_1=m_2$) of each block is

$$\sum_{m=0}^{\check{M}}\overline{\widetilde{\mathbf{g}}}\cdot\mathbf{g}(m,m)e^{2\pi imp'/P}=P\delta_{0,p'},\quad p'=0,\pm1...,\pm(P-1), \tag{3.12}$$

where $p'=p_1-p_2$. We note that we may assume $p'\in\{0,...,P-1\}$ in (3.12).
We write $m=\mu P+p$ where $\mu:=\lfloor m/P\rfloor$ and $p:=m\%P$, and then rearrange the summation to have

$$\sum_{p=0}^{P-1}\sum_{\mu=0}^{\lfloor\check{M}/P\rfloor}\overline{\widetilde{\mathbf{g}}}\cdot\mathbf{g}(\mu P+p,\ \mu P+p)e^{2\pi ip\cdot p'/P}=P\delta_{0,p'},\quad p'=0,..,P-1. \tag{3.13}$$

We define

$$r_p:=\sum_{\mu=0}^{\lfloor\check{M}/P\rfloor}\overline{\widetilde{\mathbf{g}}}\cdot\mathbf{g}(\mu P+p,\ \mu P+p)\qquad\text{and}\qquad\omega:=e^{2\pi i\frac{1}{P}},$$

for each $p \in \{0, ..., P-1\}$, then (3.13) reduces to

$$\sum_{p=0}^{P-1} r_p \omega^{p \cdot p'} = P\delta_{0,p'}, \qquad p' = 0, .., P-1.$$

By Lemma 3.2 with $c = P$, we have for $p = 0, ..., P-1$,

$$\begin{aligned} 1 &= r_p = \sum_{\mu=0}^{\lfloor \check{M}/P \rfloor} \overline{\widetilde{\mathbf{g}}} \cdot \mathbf{g}(\mu P + p,\ \mu P + p) \\ &= \sum_{\mu=0}^{\lfloor \check{M}/P \rfloor} \sum_{q=0}^{Q-1} \overline{\widetilde{\mathbf{g}}_b \left(t - \frac{qP}{Q} + (\mu P + p) \right)} \cdot \mathbf{g}_b(t - \frac{qP}{Q} + (\mu P + p)). \end{aligned} \tag{3.14}$$

Now, let us consider the off-diagonal (i.e., $m_1 \neq m_2$) sum of each block. We define $m' := m_1 - m_2$ and then choose m' and fix it. Since

$$\overline{\widetilde{\mathbf{g}}} \cdot \mathbf{g}(m_1, m_2) = 0, \text{ if } m_1 \text{ or } m_2 \notin \{0, ..., \check{M}\},$$

we can write the m'-th off diagonal sum of each (p_1, p_2)-th block as

$$\begin{aligned} 0 &= \sum_{m_1 - m_2 = m'} \overline{\widetilde{\mathbf{g}}} \cdot \mathbf{g}(m_1, m_2) e^{2\pi i (m_1 p_1 - m_2 p_2)/P} \\ &= \sum_{m_2=0}^{\check{M}} \overline{\widetilde{\mathbf{g}}} \cdot \mathbf{g}(m_2 + m', m_2) e^{2\pi i ((m_2 + m')(p_2 + p') - m_2 p_2)/P} \\ &= \sum_{m_2=0}^{\check{M}} \overline{\widetilde{\mathbf{g}}} \cdot \mathbf{g}(m_2 + m', m_2) e^{2\pi i (m_2 p' + m'(p_2 + p'))/P}, \quad p' = 0, \pm 1, ..., \pm(P-1), \end{aligned}$$

which is equivalent to

$$0 = \sum_{m=0}^{\check{M}} \overline{\widetilde{\mathbf{g}}} \cdot \mathbf{g}(m + m', m) e^{2\pi i m p'/P}, \quad p' = 0, \pm 1 ..., \pm(P-1). \tag{3.15}$$

Again, we note that we may assume $p' \in \{0, ..., P-1\}$ in (3.15). Using the similar argument used in the diagonal case, we write $m = \mu P + p$ and rearrange the summation to have

$$\sum_{p=0}^{P-1} \sum_{\mu=0}^{\lfloor \check{M}/P \rfloor} \overline{\widetilde{\mathbf{g}}} \cdot \mathbf{g}(\mu P + p + m',\ \mu P + p) e^{2\pi i p \cdot p'/P} = 0, \tag{3.16}$$

for $p' = 0, ..., P-1$. We redefine

$$r_p := \sum_{\mu=0}^{\lfloor \check{M}/P \rfloor} \overline{\widetilde{\mathbf{g}}} \cdot \mathbf{g}(\mu P + p + m',\ \mu P + p) \qquad \text{and} \qquad \omega := e^{2\pi i \frac{1}{P}}.$$

for each $p \in \{0, ..., P-1\}$. Then (3.16) reduces to

$$\sum_{p=0}^{P-1} r_p w^{p \cdot p'} = 0, \qquad p' = 0, .., P-1.$$

By Lemma 3.2 again with $c = 0$, we have

$$\begin{aligned} 0 &= r_p = \sum_{\mu=0}^{\lfloor \check{M}/P \rfloor} \overline{\widetilde{\mathbf{g}}} \cdot \mathbf{g}(\mu P + p + m',\ \mu P + p) \\ &= \sum_{\mu=0}^{\lfloor \check{M}/P \rfloor} \sum_{q=0}^{Q-1} \overline{\widetilde{\mathbf{g}}_b \left(t - \frac{qP}{Q} + (\mu P + p + m') \right)} \cdot \mathbf{g}_b \left(t - \frac{qP}{Q} + (\mu P + p) \right), \end{aligned} \tag{3.17}$$

for $p = 0, ..., P-1$, and $m' = \pm 1, ..., \pm \check{M}$. In fact, (3.17) holds for all $|m'| \neq 0$ since

$$\overline{\widetilde{\mathbf{g}}} \cdot \mathbf{g}(m_1, m_2) = 0, \text{ if } |m_1 - m_2 = m'| > \check{M}.$$

Therefore, the condition (d) follows from (3.14) and (3.17).
$(d) \Longrightarrow (e)$: For $t \in I$ and $p \in \{0, 1, ..., P-1\}$, we can write

$$t + p = q_0 \frac{P}{Q} + t_p$$

where $q_0 := \lfloor (t+p)/\frac{P}{Q} \rfloor$ and $t_p := (t+p) \% \frac{P}{Q}$. We define a P-periodic discrete set $\Omega(t) := \{t + \mu P \mid \mu \in \mathbb{Z}\}$ and observe that

$$\begin{aligned} & \bigcup_{q=0}^{Q-1} \Omega \left(t + p - \frac{qP}{Q} \right) \\ = & \bigcup_{q=0}^{q_0} \Omega \left(t + p - \frac{qP}{Q} \right) \ \cup \ \bigcup_{q=1}^{Q-(q_0+1)} \Omega \left(t + p + \frac{qP}{Q} \right) \\ = & \bigcup_{q=0}^{q_0} \Omega \left(t_p + \frac{qP}{Q} \right) \ \cup \ \bigcup_{q=q_0+1}^{Q-1} \Omega \left(t_p + \frac{qP}{Q} \right) \\ = & \bigcup_{q=0}^{Q-1} \Omega \left(t_p + \frac{qP}{Q} \right). \end{aligned}$$

We recall from Lemma 3.3 (c), defining $\tau := t \% \frac{1}{Q}$,

$$\left\{ t_p + \frac{qP}{Q} \ ; \ p = 0, ..., P-1 \right\} = \left\{ \tau + \frac{p}{Q} + \frac{qP}{Q} \ ; \ p = 0, ..., P-1 \right\}.$$

Hence

$$\bigcup_{p=0}^{P-1} \bigcup_{q=0}^{Q-1} \Omega \left(t_p + \frac{qP}{Q} \right) = \bigcup_{p=0}^{P-1} \bigcup_{q=0}^{Q-1} \Omega \left(\tau + \frac{p}{Q} + \frac{qP}{Q} \right). \tag{3.18}$$

Summing according to the above partition in the following, we see, for $t \in I$ satisfying the condition (d),

$$\begin{aligned}
\delta_{m,0} &= \sum_{\mu=0}^{\lfloor \check{M}/P \rfloor} \sum_{q=0}^{Q-1} \overline{\widetilde{\mathbf{g}}_b\left(t+p-\frac{qP}{Q}+\mu P+m\right)} \cdot \mathbf{g}_b\left(t+p-\frac{qP}{Q}+\mu P\right) \\
&= \sum_{\mu\in\mathbb{Z}} \sum_{q=0}^{q_0} \overline{\widetilde{\mathbf{g}}_b\left(t+p-\frac{qP}{Q}+\mu P+m\right)} \cdot \mathbf{g}_b\left(t+p-\frac{qP}{Q}+\mu P\right) \\
&\quad + \sum_{\mu\in\mathbb{Z}} \sum_{q=1}^{Q-(q_0+1)} \overline{\widetilde{\mathbf{g}}_b\left(t+p+\frac{qP}{Q}+(\mu-1)P+m\right)} \cdot \mathbf{g}_b\left(t+p+\frac{qP}{Q}+(\mu-1)P\right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\mu\in\mathbb{Z}} \sum_{q=0}^{q_0} \overline{\widetilde{\mathbf{g}}_b\left(t_p+\frac{qP}{Q}+\mu P+m\right)} \cdot \mathbf{g}_b\left(t_p+\frac{qP}{Q}+\mu P\right) \\
&\quad + \sum_{\mu\in\mathbb{Z}} \sum_{q=q_0+1}^{Q-1} \overline{\widetilde{\mathbf{g}}_b\left(t_p+\frac{qP}{Q}+\mu P+m\right)} \cdot \mathbf{g}_b\left(t_p+\frac{qP}{Q}+\mu P\right) \\
&= \sum_{\mu=0}^{\lfloor \check{M}/P \rfloor} \sum_{q=0}^{Q-1} \overline{\widetilde{\mathbf{g}}_b\left(t_p+\frac{qP}{Q}+\mu P+m\right)} \cdot \mathbf{g}_b\left(t_p+\frac{qP}{Q}+\mu P\right). \qquad (3.19)
\end{aligned}$$

By the same reasoning with (3.18), we see (3.19) for whole $0 \le p \le P-1$ with $t_p \in [0, P/Q)$, is equivalent to the following (3.20) for whole $0 \le p \le P-1$ with $\tau \in [0, 1/Q)$.

$$\delta_{m,0} = \sum_{\mu=0}^{\lfloor \check{M}/P \rfloor} \sum_{q=0}^{Q-1} \overline{\widetilde{\mathbf{g}}_b\left(\tau+\frac{p}{Q}+\frac{qP}{Q}+\mu P+m\right)} \cdot \mathbf{g}_b\left(\tau+\frac{p}{Q}+\frac{qP}{Q}+\mu P\right) \qquad (3.20)$$

which is the condition (e).
$(e) \Longrightarrow (f)$: We define $t := \tau + p/Q$ for each $\tau \in [0, 1/Q)$ and p satisfying the condition (e);

$$\delta_{m,0} = \sum_{\mu=0}^{\lfloor \check{M}/P \rfloor} \sum_{q=0}^{Q-1} \overline{\widetilde{\mathbf{g}}_b\left(\tau+\frac{p}{Q}+\frac{qP}{Q}+\mu P+m\right)} \cdot \mathbf{g}_b\left(\tau+\frac{p}{Q}+\frac{qP}{Q}+\mu P\right).$$

Then t is defined for a.e. $[0, P/Q)$ and satisfies the following

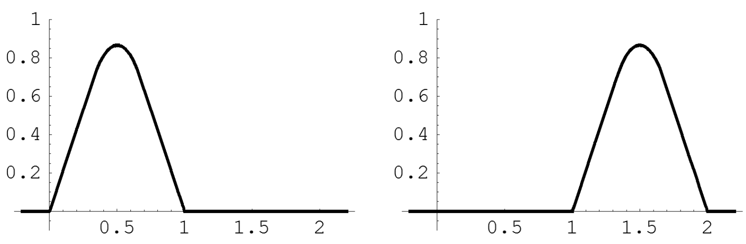
$$
\begin{aligned}
\delta_{m,0} &= \sum_{\mu=0}^{\lfloor \check{M}/P \rfloor} \sum_{q=0}^{Q-1} \overline{\widetilde{\mathbf{g}}_b \left(t + \frac{qP}{Q} + \mu P + m \right)} \cdot \mathbf{g}_b \left(t + \frac{qP}{Q} + \mu P \right), \quad \text{a.e. } t \in \left[0, \frac{P}{Q} \right), \\
&= \sum_{k \in \mathbb{Z}} \overline{\widetilde{\mathbf{g}}_b \left(t + k\frac{P}{Q} + m \right)} \cdot \mathbf{g}_b \left(t + k\frac{P}{Q} \right), \quad \text{a.e. } t \in \left[0, \frac{P}{Q} \right), \\
&= \sum_{k \in \mathbb{Z}} \overline{\widetilde{\mathbf{g}}_b \left(t + k\frac{P}{Q} + m \right)} \cdot \mathbf{g}_b \left(t + k\frac{P}{Q} \right), \quad \text{a.e. } t \in \mathbb{R}, \\
&= \frac{1}{b} \sum_{n=1}^{N} \sum_{k \in \mathbb{Z}} \overline{\widetilde{g}_n \left(\frac{t}{b} + \frac{m}{b} + k\frac{P}{Qb} \right)} \cdot g_n \left(\frac{t}{b} + k\frac{P}{Qb} \right), \quad \text{a.e. } t \in \mathbb{R}, \\
&= \frac{1}{b} \sum_{n=1}^{N} \sum_{k \in \mathbb{Z}} \overline{\widetilde{g}_n \left(t + \frac{m}{b} + ka \right)} \cdot g_n \left(t + ka \right), \quad \text{a.e. } t \in \mathbb{R}, \text{ where } ab = \frac{P}{Q},
\end{aligned}
$$

which is the condition (f).
$(f) \iff (a)$: The condition (f) is the dual form of Theorem 3.2 in [3] with $ab \in \mathbb{Q}$, which is a characterization for the tight Gabor frame.

4 Illustrations and examples

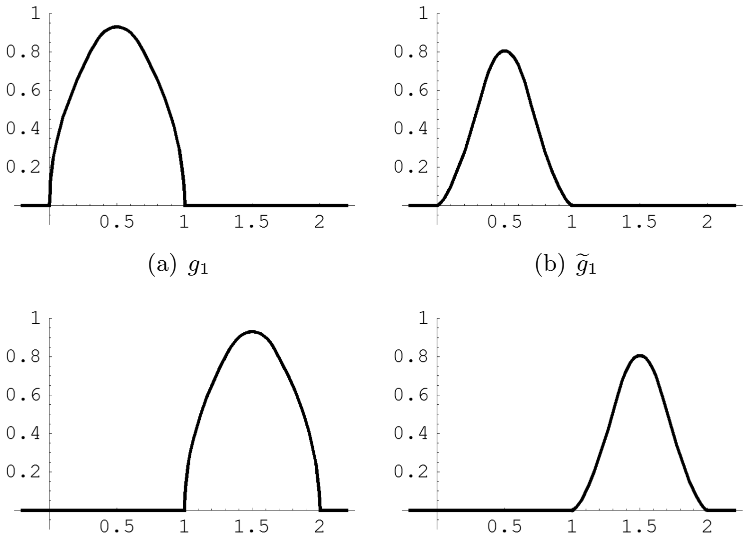
We illustrate the use of Theorem 2.1 for the construction of multiwindow dual Gabor frames in Example 4.1 and give more examples for the particular choices of P, Q, N and M. We only consider real window functions and the case $b = 1$ for simplicity. First, we exploit the condition (2.5) in Theorem 2.1.

Example 4.1 *(Tight or dual pair, P=2,Q=3,M=2, and N=2)* We consider tight Gabor frame



(a) g_1 (b) g_2

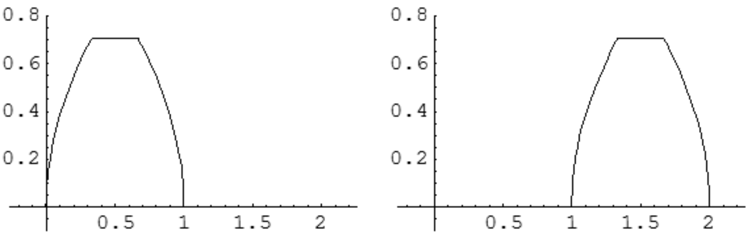
Figure 1: g_1 and g_2 in (4.2). $(G_2, 2/3, 1)$ forms a tight Gabor frame with bound 1.



(a) g_1 (b) $\widetilde{g}_1$

(c) g_2 (d) $\widetilde{g}_2$

Figure 2: g_1, $\widetilde{g}_1$ in (4.3) with $\alpha = 1/4$, and g_2, $\widetilde{g}_2$ in (4.4). $(G_2, 2/3, 1)$ and $(\widetilde{G}_2, 2/3, 1)$ form dual Gabor frames.



(a) g_1 (b) g_2

Figure 3: g_1,g_2 in (4.5),(4.6). $(G_2, 2/3, 1)$ forms a tight Gabor frame with bound 1.

with bound 1. In this case, the condition (2.5) means that, for *a.e.* $t \in [0, 1/3)$,

$$\begin{aligned}
\|\mathbf{g}(t)\|^2 + \left\|\mathbf{g}\left(t + \frac{2}{3}\right)\right\|^2 + \left\|\mathbf{g}\left(t + \frac{4}{3}\right)\right\|^2 &= 1, \\
\left\|\mathbf{g}\left(t + \frac{1}{3}\right)\right\|^2 + \|\mathbf{g}(t+1)\|^2 + \left\|\mathbf{g}\left(t + \frac{5}{3}\right)\right\|^2 &= 1, \\
\mathbf{g}(t+1) \cdot \mathbf{g}(t) + \mathbf{g}\left(t + \frac{5}{3}\right) \cdot \mathbf{g}\left(t + \frac{2}{3}\right) &= 0, \\
\mathbf{g}\left(t + \frac{4}{3}\right) \cdot \mathbf{g}\left(t + \frac{1}{3}\right) &= 0.
\end{aligned}$$

Therefore, any $G_2 = \{g_1, g_2\}$ satisfying the following two conditions forms a tight Gabor frame

$(G_2; 2/3, 1)$ with bound 1.

$$\sum_{q=0}^{2} \left| g_1 \left(t + \frac{q}{3} \right) \right|^2 = \frac{1}{2}, \quad t \in \left[0, \frac{1}{3} \right), \tag{4.1}$$
$$g_2(t) = g_1(t-1).$$

For example, we can take

$$g_1(t) = \sqrt{\frac{1}{2} B_3(3t)} \quad \text{and} \quad g_2(t) = g_1(t-1), \tag{4.2}$$

where B_3 is the B-spline of order 3 with support $[0, 3]$. Then $(G_2, 2/3, 1)$ forms a tight Gabor frame with bound 1 (Figure 1). By splitting the B-spline function as a product of two function g_1 and $\widetilde{g}_1$ as

$$g_1(t) = \left(\tfrac{1}{2} B_3(3t) \right)^{\alpha} \quad \text{and} \quad \widetilde{g}_1(t) = \left(\tfrac{1}{2} B_3(3t) \right)^{1-\alpha}, \tag{4.3}$$

where $\alpha \in (0, 1)$, and taking g_2 and $\widetilde{g}_2$ as

$$g_2(t) = g_1(t-1) \quad \text{and} \quad \widetilde{g}_2(t) = \widetilde{g}_1(t-1), \tag{4.4}$$

we get a dual pair of Gabor frames $(G_2; 2/3, 1)$ and $(\widetilde{G}_2; 2/3, 1)$ (Figure 2). This kind of construction can be easily extended to the cases with general P, Q.
For another example of g_1 and g_2 satisfying (4.1), we can take

$$g_1(t) = \sqrt{\frac{3}{2} t \chi_{[0, \frac{1}{3})}(t) + \frac{1}{2} \chi_{[\frac{1}{3}, \frac{2}{3})}(t) + \left(\frac{3}{2} - \frac{3t}{2} \right) \chi_{[\frac{2}{3}, 1)}(t)} \quad \text{and} \tag{4.5}$$
$$g_2(t) = g_1(t-1), \tag{4.6}$$

so that $(G_2, 2/3, 1)$ forms a tight Gabor frame with bound 1 (Figure 3).

In the following examples, we consider tight Gabor frames $(G_N; P/Q, 1)$ in the cases $(P, Q) \in \{(2, 1), (1, 1), (1, 2)\}$. We note $\check{M} = M$ in these cases. We use the following notation for the block matrix

$$\left(\left[a_{i,j}^{\widetilde{p},p}(t) \right]_{N_1} \right)_{N_2} := \left(\left[a_{i,j}^{\widetilde{p},p}(t) \right]_{i,j=1,\ldots,N_1} \right)_{p,\widetilde{p}=1,\ldots,N_2}.$$

Example 4.2 *(M=1)* For $P = 2$ and $Q = 1$, the block matrix in (2.10) has the form

$$\left(\left[a_{i,j}^{\widetilde{p},p}(t) \right]_2 \right)_2 = \|\mathbf{g}(t)\|^2 \begin{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{pmatrix}. \tag{4.7}$$

Hence the condition (2.10) can never be satisfied. This result reveals that there is no tight Gabor frame $(G_N; 2, 1)$ with $\text{supp}(g_n) \subset [0, 1]$.
For $P = 1$ and $Q = 1$, the matrix in (2.11) has the form

$$\left([a_{i,j}(t)]_1 \right)_1 = \|\mathbf{g}(t)\|^2.$$

Hence the condition (2.11) is satisfied if and only if

$$\sum_{n=1}^{N} |g_n(t)|^2 = B \text{ for a.e. } t \in I.$$

For example, we can take

$$g_n(t) := \sqrt{\binom{N}{n}}(\cos\alpha\pi t)^{N-n}(\sin\alpha\pi t)^n \chi_{[0,1)}(t),$$

for $n = 0, ..., N$ with real $\alpha \neq 0$, then (2.10) is satisfied with $B = 1$ since

$$\begin{aligned}\mathbf{M}^*\mathbf{M} &= \sum_{n=0}^{N} \binom{N}{n}(\cos^2\alpha\pi t)^{N-n}(\sin^2\alpha\pi t)^n \chi_{[0,1)}(t) \\ &= (\cos^2\alpha\pi t + \sin^2\alpha\pi t)^N \chi_{[0,1)}(t) = 1.\end{aligned}$$

Therefore, $(G_{N+1}; 1, 1)$ forms a tight Gabor frame with bound 1.
For $P = 1$ and $Q = 2$, the matrix in (2.10) has the form

$$\left([a_{i,j}(t)]_2\right)_1 = \left(\begin{bmatrix} \|\mathbf{g}(t)\|^2 + \left\|\mathbf{g}(t-\frac{1}{2})\right\|^2 & 0 \\ 0 & \|\mathbf{g}(t+1)\|^2 + \left\|\mathbf{g}(t+\frac{1}{2})\right\|^2 \end{bmatrix}\right).$$

Hence (2.10) is satisfied if and only if

$$\left\|\mathbf{g}(t-\frac{1}{2})\right\|^2 + \|\mathbf{g}(t)\|^2 + \left\|\mathbf{g}(t+\frac{1}{2})\right\|^2 = B, \quad \text{a.e. } t \in I.$$

Or equivalently,

$$\|\mathbf{g}(t)\|^2 + \left\|\mathbf{g}\left(t+\frac{1}{2}\right)\right\|^2 = B, \quad \text{a.e. } t \in \left[0, \frac{1}{2}\right).$$

For example, we can take

$$g_n(t) := \sqrt{\binom{2N}{n}}(\cos\pi t)^{2N-n}(\sin\pi t)^n \chi_{[0,1)}(t), \quad n = 0, ..., N-1,$$

and take

$$g_N(t) := \sqrt{\frac{1}{2}\binom{2N}{N}}(\cos\pi t)^N(\sin\pi t)^N \chi_{[0,1)}(t),$$

so that (2.10) is satisfied with $B = 1$. Then $(G_{N+1}; 1/2, 1)$ forms a tight Gabor frame with bound 1.

Example 4.3 *($N = 2$ and $M = 2$)* For $P = 2$ and $Q = 1$, the block matrix in (2.10) has the form

$$\left(\left[a_{i,j}^{\widetilde{p},p}(t)\right]_3\right)_2$$

$$= \begin{pmatrix} \begin{bmatrix} \|\mathbf{g}(t)\|^2 & \mathbf{g}(t)\cdot\mathbf{g}(t+1) & 0 \\ \mathbf{g}(t)\cdot\mathbf{g}(t+1) & \|\mathbf{g}(t+1)\|^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \|\mathbf{g}(t)\|^2 & -\mathbf{g}(t)\cdot\mathbf{g}(t+1) & 0 \\ \mathbf{g}(t)\cdot\mathbf{g}(t+1) & -\|\mathbf{g}(t+1)\|^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \|\mathbf{g}(t)\|^2 & \mathbf{g}(t)\cdot\mathbf{g}(t+1) & 0 \\ -\mathbf{g}(t)\cdot\mathbf{g}(t+1) & -\|\mathbf{g}(t+1)\|^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \|\mathbf{g}(t)\|^2 & -\mathbf{g}(t)\cdot\mathbf{g}(t+1) & 0 \\ -\mathbf{g}(t)\cdot\mathbf{g}(t+1) & \|\mathbf{g}(t+1)\|^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{pmatrix}.$$

Hence (2.10) is satisfied if and only if, for a.e. $t \in I$,

$$\begin{aligned} &\|\mathbf{g}(t)\|^2 = \|\mathbf{g}(t+1)\|^2 = B, \text{ and} \\ &\mathbf{g}(t)\cdot\mathbf{g}(t+1) = 0. \end{aligned}$$

For example, we can take

$$\begin{aligned} g_1(t) &= \cos\frac{\pi}{2}t\chi_{[0,2)}(t), \text{ and} \\ g_2(t) &= \sin\frac{\pi}{2}t\chi_{[0,2)}(t) \end{aligned}$$

so that (2.10) is satisfied with $B = 2$. Then $(G_2; 2, 1)$ forms a tight Gabor frame with bound 2. For $P = 1$ and $Q = 1$, the matrix in (2.11) has the form

$$\left([a_{i,j}(t)]_2\right)_1 = \left(\begin{bmatrix} \|\mathbf{g}(t)\|^2 & \mathbf{g}(t)\cdot\mathbf{g}(t+1) \\ \mathbf{g}(t)\cdot\mathbf{g}(t+1) & \|\mathbf{g}(t+1)\|^2 \end{bmatrix}\right).$$

Hence the condition (2.11) is satisfied if and only if, for a.e. $t \in I$,

$$\begin{aligned} &\|\mathbf{g}(t)\|^2 + \|\mathbf{g}(t+1)\|^2 = B, \text{ and} \\ &\mathbf{g}(t)\cdot\mathbf{g}(t+1) = 0. \end{aligned}$$

For example, we can take

$$\begin{aligned} g_1(t) &= \alpha_1\cos\frac{\pi}{2}t\chi_{[0,2)}(t), \text{ and} \\ g_2(t) &= \alpha_2\sin\frac{\pi}{2}t\chi_{[0,2)}(t), \end{aligned}$$

with real $\alpha_1 \neq 0 \neq \alpha_2$, so that (2.10) is satisfied with $B = \alpha_1^2 + \alpha_2^2$. Then $(G_2; 1, 1)$ forms a tight Gabor frame with bound $\alpha_1^2 + \alpha_2^2$.
For $P = 1$ and $Q = 2$, then the matrix in (2.10) has the form
$\left([a_{i,j}(t)]_3\right)_1 =$

$$\left(\begin{bmatrix} \|\mathbf{g}\left(t-\frac{1}{2}\right)\|^2 + \|\mathbf{g}(t)\|^2 & , & \mathbf{g}\left(t-\frac{1}{2}\right)\cdot\mathbf{g}\left(t+\frac{1}{2}\right) + \mathbf{g}(t)\cdot\mathbf{g}(t+1) & , & 0 \\ \mathbf{g}\left(t+\frac{1}{2}\right)\cdot\mathbf{g}\left(t-\frac{1}{2}\right) + \mathbf{g}(t+1)\cdot\mathbf{g}(t) & , & \|\mathbf{g}\left(t+\frac{1}{2}\right)\|^2 + \|\mathbf{g}(t+1)\|^2 & , & \mathbf{g}\left(t+\frac{1}{2}\right)\cdot\mathbf{g}\left(t+\frac{3}{2}\right) \\ 0 & , & \mathbf{g}\left(t+\frac{3}{2}\right)\cdot\mathbf{g}\left(t+\frac{1}{2}\right) & , & \|\mathbf{g}\left(t+\frac{3}{2}\right)\|^2 \end{bmatrix}\right).$$

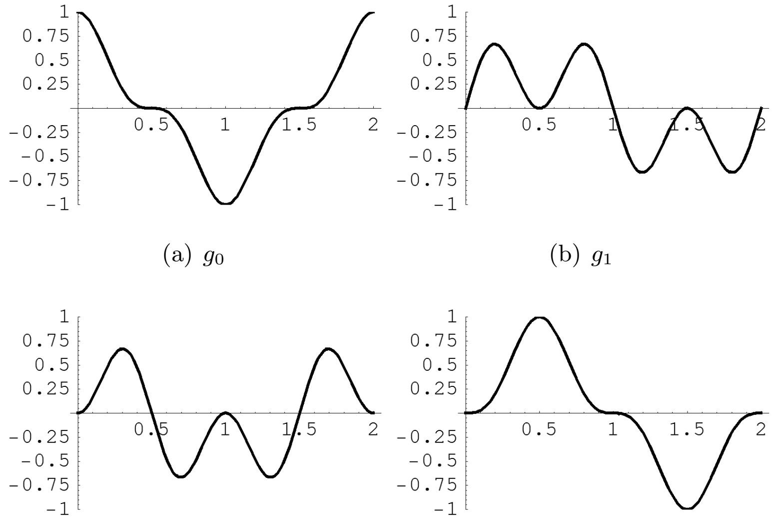
Hence (2.10) is satisfied if and only if, for a.e. $t \in [0, 1/2)$,

$$\begin{aligned} &\|\mathbf{g}(t)\|^2 + \left\|\mathbf{g}\left(t+\frac{1}{2}\right)\right\|^2 + \|\mathbf{g}(t+1)\|^2 + \left\|\mathbf{g}\left(t+\frac{3}{2}\right)\right\|^2 = B, \\ &\mathbf{g}(t)\cdot\mathbf{g}(t+1) + \mathbf{g}\left(t+\frac{1}{2}\right)\cdot\mathbf{g}\left(t+\frac{3}{2}\right) = 0. \end{aligned}$$

The second condition means the orthogonality of two vectors $\left(\mathbf{g}(t), \mathbf{g}\left(t+\frac{1}{2}\right)\right)^T$ and $\left(\mathbf{g}(t+1), \mathbf{g}\left(t+\frac{3}{2}\right)\right)^T$. Hence we can take

$$\begin{aligned} g_1(t) &= \cos\pi t\chi_{[0,\frac{3}{2})}(t) - \cos\pi t\chi_{[\frac{3}{2},2)}(t), \text{ and} \\ g_2(t) &= \sin\pi t\chi_{[0,\frac{3}{2})}(t) - \sin\pi t\chi_{[\frac{3}{2},2)}(t), \end{aligned}$$

so that (2.10) is satisfied with $B = 4$. Then $(G_2; 1/2, 1)$ forms a tight Gabor frame with bound 4.



(a) g_0 (b) g_1

(c) g_2 (d) g_3

Figure 4: UEP type example; $g_n = \sqrt{\binom{3}{n}}(\cos\pi t)^{3-n}(\sin\pi t)^n\chi_{[0,2)}(t)$, $n = 0, ..., 3$. $(G_4, 1, 1)$ forms a tight Gabor frame with bound 2.

The following example is interesting since the masks satisfying the condition for unitary extension principle(UEP) generates a tight Gabor system [13].

Example 4.4 *($M = 2$)* Suppose $m_0(t), ..., m_N(t)$ are 2π-periodic L^∞ functions satisfying the following UEP condition ([13], Chapter 14 in [4]):

$$\begin{pmatrix} m_0(t) & m_2(t) & \cdots & m_N(t) \\ m_0(t+\pi) & m_2(t+\pi) & \cdots & m_N(t+\pi) \end{pmatrix} \begin{pmatrix} m_0(t) & m_0(t+\pi) \\ m_1(t) & m_1(t+\pi) \\ \vdots & \vdots \\ m_N(t) & m_N(t+\pi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then multiwindow Gabor system $(G_{N+1}; 1, 1)$ with

$$g_n(t) := m_n(\pi t)\chi_{[0,2)}(t), \quad n = 0, ..., N,$$

forms a tight Gabor frame with bound 2. For example, we can take

$$g_n(t) := \sqrt{\binom{N}{n}}(\cos\pi t)^{N-n}(\sin\pi t)^n\chi_{[0,2)}(t), \quad n = 0, ..., N.$$

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