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Abstract

Consider a bounded function g supported on $[-1, 1]$ and a modulation parameter $b \in]1/2, 1[$ for which the Gabor system $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is a frame. We show that such a frame always has a compactly supported dual window. More precisely, we show that if $b < \frac{N}{N+1}$ for some $N \in \mathbb{N}$, it is possible to find a dual window supported on $[-N, N]$. Under the additional assumption that g is continuous and only has a finite number of zeros on $]-1, 1[$, we characterize the frame property of $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$. As a consequence we obtain easily verifiable criteria for a function g to generate a Gabor frame with a dual window having compact support of prescribed size.

Keywords: Gabor system; Gabor frame; Dual frame; Dual window.

1 Introduction

Let $g \in L^2(\mathbb{R})$ be a function with $\operatorname{supp} g \subseteq [-1, 1]$. It is well known that for modulation parameters $b \leq 1/2$, the Gabor system $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ given by

$$E_{mb}T_n g(x) := e^{2\pi imbx} g(x - n), \; x \in \mathbb{R},$$

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forms a frame for $L^2(\mathbb{R})$ if and only if there exist two positive constants A, B such that

$$A \leq \sum_{n \in \mathbb{Z}} |g(x-n)|^2 \leq B, \; a.e. \, x \in \mathbb{R}.$$

If $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is a frame and $b \leq 1/2$, it is also known that the canonical dual generator is supported on $[-1, 1]$.

The purpose of this paper is to investigate the properties of the dual frames of $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ for $b \in]1/2, 1[$. In particular, we show that a frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ for which g is supported on $[-1, 1]$ always has a dual Gabor frame generated by a compactly supported function. More precisely, we show that if $b < \frac{N}{N+1}$ for some $N \in \mathbb{N}$, it is possible to find a dual window supported on $[-N, N]$.

Under the additional assumptions that g is continuous and only has a finite number of zeros on [-1,1] we are able to characterize the frame property for $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$. It turns out that a continuous and compactly supported dual window always exists in this case. As a special case of the general result we are thus able to derive easily verifiable conditions for a function g to generate a Gabor frame having a continuous dual window with a specified size of the support.

In a sense, our results complement the results by Bölcskei and Janssen in [1]. For any Gabor frame $\{E_{mb}T_{na} g\}_{m,n\in\mathbb{Z}}$ for which g is compactly supported and $ab \in \mathbb{Q}$, the results in [1] characterize the existence of a dual frame generator with compact support in terms of the rank of the Zibulski-Zeevi matrix. If $ab = p/q$ with $\gcd(p, q) = 1$, the Zibulski-Zeevi matrix is of the size $p \times q$, so even for the quite simple functions g considered in the current paper, it is difficult to check the rank condition directly. Furthermore, our results apply to the general case, not just to the case of rational oversampling. On the other hand, we only consider functions g supported on $[-1, 1]$, a restriction that does not appear in [1].

We also note that Laugesen recently obtained constructions of dual pairs of spline windows supported on $[-1, 1]$, see [9]. Most of his windows and dual windows are with knots at the points $x = -1, 0, 1$ and are constructed so that the functions become continuous, or even smooth up to a certain order. The constructions are made by counting the number of constraints (in the duality conditions presented below, and on the points where continuity/differentiability is required) and then search for polynomials on $[-1, 0]$ and on $[0, 1]$ of a matching degree; the coefficients in the polynomials are found by Mathematica. The drawback of the method is that one can not be completely sure in advance that it actually yields a solution. The results presented here shed light on the conditions that are necessary for Laugesen's approach to work.

We finally note that our motivation stems from recent results, showing that small modulation parameters b in Gabor frames $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ yield amazing flexibility in the choice of dual window. For example, one of the main results in [4] shows that functions of the type

$$g(x) = \left(\sum_{k=0}^{N-1} c_k x^k\right) \chi_I(x),$$

considered for sufficiently large intervals I, usually lead to Gabor frames having B-spline dual windows for small values of b; and for functions g with support on $[0, N]$, for which the integer-translates form a partition of unity, one can find dual windows of the type

$$h(x) = \sum_{n=-N+1}^{N-1} a_n g(x+n) \tag{1.1}$$

for appropriate choices of the coefficients a_n. Unfortunately, the results in the current paper show that we do not have the same freedom in the choice of "nice dual windows" for larger values of b.

The paper is organized as follows. In Section 2 we state the results. All proofs are collected in Section 3. In the rest of the introduction we state a few key results and definitions.

Recall that $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is a *frame* for $L^2(\mathbb{R})$ if there exist constants $A, B > 0$ such that

$$A\,||f||^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_n g\rangle|^2 \leq B\,||f||^2, \ \forall \in L^2(\mathbb{R}).$$

If at least the upper frame condition is satisfied, $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is a *Bessel sequence.*

Given a frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$, a Bessel sequence $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ is a *dual frame* if

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_n h\rangle E_{mb}T_n g, \quad \forall \in L^2(\mathbb{R}).$$

The function g generating the frame is called the *window* and h is called the *dual window.* For more information we refer to, e.g., [2] or [5].

The starting point is the duality conditions for two Gabor systems, due to Ron and Shen [10]. We will apply the version presented by Janssen [8]:

Theorem 1.1 *Two Bessel sequences* $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ *form dual frames for* $L^2(\mathbb{R})$ *if and only if*

$$\sum_{k\in\mathbb{Z}} \overline{g(x-n/b+k)}h(x+k) \;=\; b\delta_{n,0},\ a.e.\ x\in[0,1]. \tag{1.2}$$

We will only consider bounded and compactly supported candidates for the functions g and h, so $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ are automatically Bessel sequences. Due to the compact support of g and h, the condition in (1.2) is automatically satisfied whenever $|n|$ is sufficiently large. By specifying the support of g and h we can identify the relevant equations in (1.2). Note also that the infinite sum appearing in (1.2) is periodic; thus, for a given value of n the condition can be checked by looking at *any* interval of length 1. These observations immediately lead to the following consequence of Theorem 1.1:

Corollary 1.2 *Let* $N\in\mathbb{N}\setminus\{1\}$ *and* $b\in]0,\frac{N}{N+1}[$. *Assume that* g *and* h *are bounded and real-valued functions with* $\text{supp}\ g\subseteq[-1,1]$ *and* $\text{supp}\ h\subseteq[-N,N]$, *and that*

$$\sum_{k\in\mathbb{Z}} g(x+k)h(x+k)=b,\ a.e.\ x\in[0,1].$$

Then the conditions (i) – (ii) below are equivalent:

(i) $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ *form dual frames for* $L^2(\mathbb{R})$*;*

(ii) *For* $n=\pm1,\pm2,\cdots,\pm(N-1)$,

$$g(x-\frac{n}{b})h(x)+g(x-\frac{n}{b}+1)h(x+1)=0,\ a.e.\ x\in[\frac{n}{b}-1,\frac{n}{b}].$$

2 The main results and examples

Consider a Gabor frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ for which the window g is supported on $[-1,1]$. We will show that $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ has a dual frame, generated by a compactly supported function h. As explained in the introduction we focus on the range $b \in]1/2, 1[$. The result exhibits a relationship between the modulation parameter b and the size of the support of the dual window:

Theorem 2.1 *Let $b \in [1/2, 1[$, and choose $N \in \mathbb{N}$ such that $\frac{N-1}{N} \leq b < \frac{N}{N+1}$. Assume that $g \in L^2(\mathbb{R})$ is supported on $[-1, 1]$ and that $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. Then $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ has a dual $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$, generated by a function $h \in L^2(\mathbb{R})$ with* $\operatorname{supp} h \subseteq [-N, N]$.

The proofs of Theorem 2.1 and all the following results are collected in Section 3. Even if the window g is continuous, the dual window h constructed in the proof of Theorem 2.1 will usually not be continuous. Under additional assumptions on g we will now show that continuous dual windows with compact support exist. We will consider windows belonging to the following subspace of $L^2(\mathbb{R})$:

$$V := \{f \in C(\mathbb{R}) \mid \operatorname{supp} f = [-1, 1],\ f \text{ has a finite number of zeros on } [-1, 1]\}. \tag{2.1}$$

We will actually characterize the frame property for windows $g \in V$. Note that for a given function $g \in V$, it is only possible for $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ to be a frame for $b \in]0, 1[$; in fact, the option $b = 1$ has to be excluded because a continuous function with compact support can't generate a Riesz basis, see [7] or [2].

In order to characterize the frame property we need to introduce a class of help functions. As in Theorem 2.1 we fix $b \in [\frac{N-1}{N}, \frac{N}{N+1}[$ for some $N \in \mathbb{N}$. Let $n_+ \in \{1, 2, \cdots, N-1\}$, and define the function R_{n_+} on (a subset of) $[0, n_+ - \frac{n_+}{b} + 1]$ by

$$R_{n_+}(y) := \begin{cases} \frac{1}{g(y)}, & \text{if } n_+ = 1; \\ \frac{\prod_{n=1}^{n_+-1} g(y+\frac{n}{b}-n-1)}{\prod_{n=0}^{n_+-1} g(y+\frac{n}{b}-n)}, & \text{if } n_+ = 2, \cdots, N-1. \end{cases}$$

Note that for $n = 0, 1, \ldots, n_+ - 1$,

$$\begin{aligned} y \in [0, n_+ - \frac{n_+}{b} + 1] \Rightarrow \frac{n}{b} - n &\leq y + \frac{n}{b} - n \leq n_+ - \frac{n_+}{b} + 1 + \frac{n}{b} - n \\ &= (n - n_+)(\frac{1}{b} - 1) + 1 < 1. \end{aligned}$$

This implies that R_{n_+} is defined on $[0, n_+ - \frac{n_+}{b} + 1]$, except maybe on a finite set of points.

Similarly, for $n_- \in \{1, 2, \cdots, N-1\}$, we define the function $L_{n_-}(y)$ on (a subset of) $[-n_- + \frac{n_-}{b} - 1, 0]$ by

$$L_{n_-}(y) := \begin{cases} \frac{1}{g(y)}, & \text{if } n_- = 1; \\ \frac{\prod_{n=1}^{n_- -1} g(y - \frac{n}{b} + n + 1)}{\prod_{n=0}^{n_- -1} g(y - \frac{n}{b} + n)}, & \text{if } n_- = 2, \cdots, N-1. \end{cases}$$

For functions $g \in V$ we now show that one can characterize the frame property of $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ in terms of the behavior of the functions R_{n_+} and L_{n_-} close to the zeros of g. In particular, the stated conditions lead to the existence of a continuous compactly supported dual window. Afterwards, we state easily verifiable sufficient conditions directly in terms of the zeros of g.

Theorem 2.2 *Let $N \in \mathbb{N} \setminus \{1\}$ and $b \in [\frac{N-1}{N}, \frac{N}{N+1}[$. Assume that $g \in V$. Then the following assertions are equivalent:*

(1) *The function g generates a Gabor frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$;*

(2) *There exists a continuous dual window h with $\operatorname{supp} h \subseteq [-N, N]$;*

(3) *The following four conditions are satisfied:*

(i) $|g(x)| + |g(x+1)| > 0, \ x \in [-1, 0]$;

(ii) *If there exist $n_+ \in \{1, 2, \cdots, N-1\}$ and $y_+ \in [0, n_+ - \frac{n_+}{b} + 1]$ such that $g(y_+) = 0$ and $\lim_{y \to y_+} |R_{n_+}(y)| = \infty$, then*

$$g(y_+ + \frac{n_+}{b} - n_+ - 1) \neq 0;$$

(iii) *If there exist $n_- \in \{1, 2, \cdots, N-1\}$ and $y_- \in [-n_- + \frac{n_-}{b} - 1, 0]$ such that $g(y_-) = 0$ and $\lim_{y \to y_-} |L_{n_-}(y)| = \infty$, then*

$$g(y_- - \frac{n_-}{b} + n_- + 1) \neq 0;$$

(iv) *For y_+, y_-, n_+, n_- as in* (ii) *and* (iii),

$$y_+ + \frac{n_+}{b} - n_+ \neq y_- - \frac{n_-}{b} + n_- + 1.$$

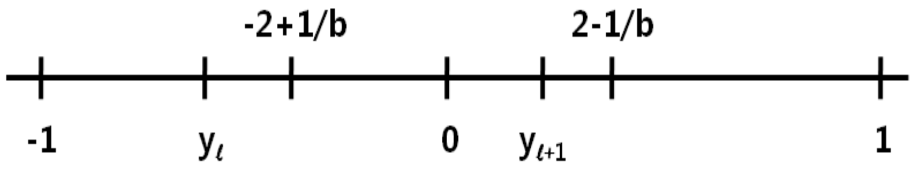

Figure 1: One possibility for the location of the points y_ℓ and $y_{\ell+1}$

Based on Theorem 2.2 we will now derive a sufficient condition for the existence of a continuous dual window supported on $[-N, N]$; this result, to be stated in Theorem 2.4, is formulated directly in terms of the zeros of the function g and does not involve the functions R_{n_+} and L_{n_-}. For a function $g \in V$, denote the zeros on $[-1, 1]$ by $Z(g) = \{y_i\}_{i=1}^n$, ordered as

$$-1 = y_1 < y_2 < \cdots < y_\ell < 0 < y_{\ell+1} < \cdots < y_n = 1. \tag{2.2}$$

Note that $\ell \in \mathbb{N}$ is chosen such that $y_\ell < 0 < y_{\ell+1}$. The results to follow depend on the exact location of the zeros, in particular, whether $y_\ell < \frac{1}{b} - 2$ or $y_\ell \geq \frac{1}{b} - 2$ (see Figure 1). For this reason we need the following definition:

Definition 2.3 *Let $N \in \mathbb{N} \setminus \{1\}$ and $b \in [\frac{N-1}{N}, \frac{N}{N+1}[$. Let $g \in V$, and denote the zeros of g in $[-1, 1]$ by $Z(g) = \{y_i\}_{i=1}^n$, as above.*

(1) *If $y_{\ell+1} \leq 2 - \frac{1}{b}$, let k_0 be the largest integer for which $0 < y_{\ell+k_0} \leq 2 - \frac{1}{b}$. For $k = 1, 2, \cdots, k_0$, let $n_k \in \{1, \cdots, N-1\}$ denote the largest integer for which*

$$y_{\ell+k} \in [0, n - \frac{n}{b} + 1], \quad n = 1, \cdots, n_k.$$

(2) *If $y_\ell \geq \frac{1}{b} - 2$, let k_1 be the largest integer for which $\frac{1}{b} - 2 \leq y_{\ell-k_1} < 0$. For $k = -k_1, -k_1 + 1, \cdots, 0$, let $n_k \in \{1, \cdots, N-1\}$ be the largest*

integer for which

$$y_{\ell+k} \in [-n + \frac{n}{b} - 1, 0], \quad n = 1, \cdots, n_k.$$

Theorem 2.4 *Let $N \in \mathbb{N} \setminus \{1\}$ and $b \in [\frac{N-1}{N}, \frac{N}{N+1}[$. Assume that $g \in V$, and denote the zeros by $Z(g) = \{y_i\}_{i=1}^n$, ordered as in* (2.2)*. Assume that*

$$|g(x)| + |g(x+1)| > 0, \; x \in [-1, 0]. \tag{2.3}$$

Consider the following (disjoint and exhausting) cases (a)–(d):

(a) $y_\ell < \frac{1}{b} - 2$ *and* $y_{\ell+1} > 2 - \frac{1}{b}$.

(b) $y_\ell < \frac{1}{b} - 2$ *and* $y_{\ell+1} \leq 2 - \frac{1}{b}$. *In this case, take $k_0 \geq 1$ and $n_k, \; k = 1, \ldots, k_0$ as in Definition 2.3(1), and assume that*

$$\bigcup_{n=1}^{n_k} \{y_{\ell+k} + \frac{n}{b} - n - 1\} \cap Z(g) = \emptyset, \; k = 1, 2, \cdots, k_0; \tag{2.4}$$

(c) $y_\ell \geq \frac{1}{b} - 2$ *and* $y_{\ell+1} > 2 - \frac{1}{b}$. *In this case, take $k_1 \geq 0$ and $n_k, \; k = -k_1, \ldots, 0$ as in Definition 2.3(2), and assume that*

$$\bigcup_{n=1}^{n_{-k}} \{y_{\ell-k} - \frac{n}{b} + n + 1\} \cap Z(g) = \emptyset, \; k = 0, 1, \cdots, k_1; \tag{2.5}$$

(d) $y_\ell \geq \frac{1}{b} - 2$ *and* $y_{\ell+1} \leq 2 - \frac{1}{b}$. *In this case, take k_0, k_1, and the associated numbers $n_k, \; k = -k_1, \ldots, k_0$ as in Definition 2.3, and assume* (2.4), (2.5) *and*

$$\left[\bigcup_{k=1}^{k_0} \bigcup_{n=1}^{n_k} \{y_{\ell+k} + \frac{n}{b} - n - 1\} \right] \cap \left[\bigcup_{j=0}^{k_1} \bigcup_{m=1}^{n_{-j}} \{y_{\ell-j} - \frac{m}{b} + m + 1\} \right] = \emptyset. \tag{2.6}$$

In any of the cases (a)–(d), the function g generates a Gabor frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$, having a continuous dual window supported on $[-N, N]$.

In case b is irrational and $Z(g)$ consists of rational numbers, it is clear that the relevant condition in (a),(b),(c) or (d) in Theorem 2.4 is satisfied. This observation leads to an interesting special case:

Corollary 2.5 *Let $N \in \mathbb{N}\backslash\{1\}$ and assume that $b \in [\frac{N-1}{N}, \frac{N}{N+1}[$ is irrational. Assume that $g \in V$, that all the zeros are rational numbers, and that*

$$|g(x)| + |g(x+1)| > 0, \; x \in [-1, 0]. \tag{2.7}$$

Then g generates a Gabor frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$, having a continuous dual window supported on $[-N, N]$.

For continuous functions g without zeros on $]-1, 1[$ we obtain the following immediate consequence of Theorem 2.4:

Corollary 2.6 *Let $N \in \mathbb{N} \setminus \{1\}$ and $b \in [\frac{N-1}{N}, \frac{N}{N+1}[$. Assume that $g \in V$ satisfies that*

$$g(x) > 0, \; x \in]-1, 1[.$$

Then g generates a Gabor frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$, having a continuous dual window supported on $[-N, N]$.

The conditions in Theorem 2.4 are very easy to verify. Let us demonstrate this in two examples:

Example 2.7 Let $b = 0.7$ and consider

$$g(x) = (x+1)(x-0.3)(x-1)\chi_{[-1,1]}(x).$$

Then $b \in [\frac{N-1}{N}, \frac{N}{N+1}[$ for $N = 3$. The zeros for g on $[-1, 1]$ are

$$y_1 = -1, \; y_2 = 0.3, \; y_3 = 1.$$

Thus $\ell = 1$, and

$$y_1 < \frac{1}{b} - 2, \; y_2 \leq 2 - \frac{1}{b} < y_3.$$

Thus $k_0 = 1$ and $n_{k_0} = 1$. Also,

$$\bigcup_{n=1}^{n_1} \{y_{\ell+1} + \frac{n}{b} - n - 1\} = \{y_2 + \frac{1}{b} - 2\} = \{-\frac{19}{70}\}.$$

Thus the condition (b) in Theorem 2.4 is satisfied. Hence g generates a Gabor frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$, having a continuous dual supported on $[-3, 3]$.

Example 2.8 Let $b = 0.7$ and consider

$$g(x) = (x+1)(x+0.2)(x-0.3)(x-0.6)(x-1)\chi_{[-1,1]}(x).$$

Then $b \in [\frac{N-1}{N}, \frac{N}{N+1}[$ for $N = 3$. The zeros for g on $[-1, 1]$ are

$$y_1 = -1,\ y_2 = -0.2,\ y_3 = 0.3,\ y_4 = 0.6,\ y_5 = 1,$$

so $\ell = 2$. The conditions $y_3 \le 2 - \frac{1}{b} < y_4$ and $y_1 < \frac{1}{b} - 2 \le y_2$ imply $k_0 = 1$ and $k_1 = 0$. Since $3 - \frac{2}{b} < y_{\ell+1} \le 2 - \frac{1}{b}$ and $-2 + \frac{1}{b} < y_\ell \le -3 + \frac{2}{b}$, choose $n_1 = 1$ and $n_0 = 1$. Then

$$\bigcup_{n=1}^{n_1} \{y_{\ell+1} + \frac{n}{b} - n - 1\} = \{y_3 + \frac{1}{b} - 2\} = \{-\frac{19}{70}\}$$
$$\bigcup_{n=1}^{n_0} \{y_\ell - \frac{n}{b} + n + 1\} = \{y_2 - \frac{1}{b} + 2\} = \{\frac{26}{70}\}$$

Thus the condition (a) in Theorem 2.4 is satisfied. Hence g generates a Gabor frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$, having a continuous dual window supported on $[-3, 3]$.

Unfortunately, the calculations leading to the results in the current paper show that we do not obtain the same amount of freedom in the choice of "nice dual windows" for large values of b as for small: for example, in general it is not possible to obtain dual windows of the form (1.1). The next example illustrates this:

Example 2.9 Let $b \in]1/2, 2/3[$ and consider the B-spline B_2, defined by

$$B_2(x) = \begin{cases} 1+x & \text{if } x \in [-1,0], \\ 1-x & \text{if } x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 2.6 implies that the Gabor frame $\{E_{mb}T_n B_2\}_{m,n\in\mathbb{Z}}$ has a continuous dual window supported on $[-2, 2]$. But easy direct calculations based on Corollary 1.2 show that no dual window of the form in (1.1) exists.

3 Proofs

3.1 Proof of Theorem 2.1

The proof of Theorem 2.1 uses the following elementary lemma:

Lemma 3.1 *Let $N \in \mathbb{N}$, and assume that $\frac{N-1}{N} \le b < \frac{N}{N+1}$. Then the intervals*

$$[\frac{k}{b}, k+1], \;\; k = 1, \dots, N-1$$

are nonempty and disjoint. In particular, the interval $[1, N]$ can be decomposed into two sets,

$$[1, N] = J \cup \widetilde{J},$$

where

$$J := \biguplus_{k=1}^{N-1} [k, \frac{k}{b}], \;\; \widetilde{J} := \biguplus_{k=1}^{N-1} [\frac{k}{b}, k+1]. \tag{3.1}$$

Here $\biguplus$ denotes a disjoint union. Furthermore, the sets J and $\widetilde{J}$ overlap only at the endpoints of the appearing intervals.

Proof of Theorem 2.1: Let $\tilde{h} \in L^2(\mathbb{R})$ be any function such that $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n \tilde{h}\}_{m,n\in\mathbb{Z}}$ are dual frames. Such a function always exist (for example, $\tilde{h} := S^{-1}g$, where S is the frame operator associated with $\{E_{mb}T_n g\}$). The function $\tilde{h}$ satisfies the duality condition (ii) in Corollary 1.2. Define h by $h(x) = \tilde{h}(x)\chi_I(x)$, where $I = (-\tilde{J}) \cup [-1, 1] \cup \tilde{J}$. We check that h also satisfies the duality conditions, i.e., that for $n = 0, \pm 1, \dots, \pm N$,

$$g(x - n/b)h(x) + g(x - n/b + 1)h(x+1) = b\delta_{n,0}, \;\; a.e. \; x \in [\frac{n}{b} - 1, \frac{n}{b}]. \tag{3.2}$$

We split into various cases:
(1) For $n = 0$, we note that $h(x) = \tilde{h}(x)$ for $x \in [-1, 1]$. So (3.2) follows immediately from the duality conditions for $\tilde{h}$.
(2) For $1 \le n \le N-1$, we check (3.2) for $x \in [n/b - 1, n/b]$ by splitting into the cases $x \in [n/b - 1, n]$ and $x \in [n, n/b]$.

(2a) For $x \in [n/b-1, n]$, (3.2) only involves $x \in [n/b-1, n]$ and $x+1 \in [n/b, n+1]$ for h. Note that $[\frac{n}{b}-1, n] \subset [\frac{n-1}{b}, n]$ because $b<1$; thus

$$[n/b-1, n] \cup [n/b, n+1] \subset [\frac{n-1}{b}, n] \cup [n/b, n+1] \subset I.$$

By definition, this implies that $h = \tilde{h}$ on $[n/b-1, n] \cup [n/b, n+1]$. So by the duality conditions for $\tilde{h}$, (3.2) is satisfied for *a.e.* $x \in [n/b-1, n]$.
(2b) For $x \in [n, n/b]$, (3.2) only involves $x \in [n, n/b]$ and $x+1 \in [n+1, n/b+1]$ for h. Since

$$[n, n/b] \cup [n+1, n/b+1] \subset [n, n/b] \cup [n+1, \frac{n+1}{b}] \subset J,$$

$h(x) = 0 = h(x+1)$ for $n = 1, 2, , \cdots, N-2$. For $x \in [N-1, \frac{N-1}{b}]$, $h(x) = 0 = h(x+1)$ since $[N-1. \frac{N-1}{b}] \cap I = \emptyset$ and $[N, \frac{N-1}{b}+1] \cap \operatorname{supp} h = \emptyset$; (3.2) follows.
(3) For $n \geq N$, supp $g(\cdot - n/b)$ and supp h are disjoint. In fact,

$$\operatorname{supp} g(\cdot - n/b) \subseteq [-1+n/b, 1+n/b],$$

and $-1+N/b > -1+N+1 = N$ for $b < \frac{N}{N+1}$. Thus (3.2) is satisfied.
(4) For $n < 0$, the proof of (3.2) is similar by the symmetry. □

3.2 Proof of Theorem 2.2

The proof of Theorem 2.2 is quite lengthy and requires some preparation. We use the sets J and $\widetilde{J}$ defined in (3.1).

First, we note that the duality condition and the chosen restrictions on the support and on the parameter b force a dual window to vanish on certain intervals.

Lemma 3.2 *Let* $N \in \mathbb{N} \setminus \{1\}$ *and* $b \in [\frac{N-1}{N}, \frac{N}{N+1}[$. *Assume that* g *is a bounded function on* $\mathbb{R}$ *and that* supp $g = [-1, 1]$. *Assume that* h *is supported in* $[-N, N]$, *and that for all* $n = \pm 1, \pm 2, \cdots, \pm(N-1)$,

$$g(x - \frac{n}{b})h(x) + g(x - \frac{n}{b} + 1)h(x+1) = 0, \ \ a.e. \ x \in [\frac{n}{b} - 1, \frac{n}{b}]. \qquad (3.3)$$

Then $h(x) = 0$, *a.e.* $x \in (-J) \cup J$.

Proof. Note that $b \in [\frac{N-1}{N}, \frac{N}{N+1}[$ implies that for $n = 1, 2, \ldots, N-1$, $b \geq \frac{n}{n+1}$; thus,

$$\frac{n}{b} - 1 \leq n < \frac{n}{b}, \tag{3.4}$$

which will be used at several instances in the proof.

We first show that $h(x) = 0$, *a.e.* on $[N-1, \frac{N-1}{b}]$ and induct on $[n, \frac{n}{b}]$ for $n = 1, 2, \cdots, N-2$ in reverse order.

We consider (3.3) for $n = N-1$. For *a.e.* $x \in [N-1, \frac{N-1}{b}]$, which by (3.4) is a subinterval of $[\frac{N-1}{b} - 1, \frac{N-1}{b}]$, we see that $h(x+1) = 0$ due to the support assumption on h. If we note that, by (3.4) with $n = N-1$,

$$[N-1, \frac{N-1}{b}] \subset [\frac{N-1}{b} - 1, \frac{N-1}{b} + 1] = \text{supp } g(\cdot - \frac{N-1}{b}),$$

then $g(x - \frac{N-1}{b}) \neq 0$ for *a.e.* $x \in [N-1, \frac{N-1}{b}]$. This together with (3.3) implies that

$$h(x) = 0, \ a.e. \ x \in [N-1, \frac{N-1}{b}].$$

Assuming $h(x) = 0$, *a.e.* $x \in [n_0, \frac{n_0}{b}]$ for some $n_0 \in \{2, 3, \cdots, N-1\}$, we will show that $h(x) = 0$, *a.e.* $x \in [n_0 - 1, \frac{n_0-1}{b}]$. An application of (3.4) shows that

$$[n_0 - 1, \frac{n_0 - 1}{b}] \subset [n_0 - 1, \frac{n_0}{b} - 1] \cap \text{supp } g(\cdot - \frac{n_0 - 1}{b}).$$

Then we have $g(x - \frac{n_0-1}{b}) \neq 0$ for *a.e.* $x \in [n_0 - 1, \frac{n_0-1}{b}]$ and $h(x+1) = 0$ for *a.e.* $x \in [n_0 - 1, \frac{n_0-1}{b}]$ by assumption. Considering (3.3) for $n = n_0 - 1$ leads to

$$h(x) = 0, \ a.e. \ x \in [n_0 - 1, \frac{n_0 - 1}{b}].$$

This completes our induction and so

$$h(x) = 0, \ a.e. \ x \in \bigcup_{k=1}^{N-1} [k, \frac{k}{b}].$$

By symmetry, considering (3.3) for $n = -1, -2, \cdots, -(N-1)$ leads to

$$h(x) = 0, \ a.e. \ x \in \bigcup_{k=1}^{N-1} [-\frac{k}{b}, -k].$$

□

Assuming that h is chosen continuously on $[-1,1]$, we now show that certain conditions on the interplay between h and the functions R_{n_+} and L_{n_-} imply that h is uniquely determined on the set $(-\tilde{J}) \cup \tilde{J}$. The result is formulated in terms of conditions on the zeros for g:

Lemma 3.3 *Let $N \in \mathbb{N} \setminus \{1\}$ and $b \in [\frac{N-1}{N}, \frac{N}{N+1}[$. Assume that $g \in V$, defined in* (2.1). *Assume that $h(x)$ is continuously chosen for $x \in [-1,1]$ so that the following five conditions hold:*

(1) $g(x)h(x) + g(x+1)h(x+1) = b, x \in [-1,0]$;

(2) $\lim_{x->1^-} \frac{b-g(x-1)h(x-1)}{g(x)} = 0$;

(3) $\lim_{x->(-1)^+} \frac{b-g(x+1)h(x+1)}{g(x)} = 0$.

(4) *If there exist $n_+ \in \{1,2,\cdots,N-1\}$ and $y_+ \in [0, n_+ - \frac{n_+}{b} + 1[$ such that $g(y_+) = 0$, then the limit*

$$\lim_{y \to y_+} \left\{ h(y + \frac{n_+}{b} - n_+) R_{n_+}(y) \right\} \tag{3.5}$$

exists; and if $g(n_+ - \frac{n_+}{b} + 1) = 0$, then

$$\lim_{y \to (n_+ - \frac{n_+}{b} + 1)^-} \left\{ h(y + \frac{n_+}{b} - n_+) R_{n_+}(y) \right\} = 0; \tag{3.6}$$

(5) *If there exist $n_- \in \{1,2,\cdots,N-1\}$ and $y_- \in]-n_- + \frac{n_-}{b} - 1, 0]$ such that $g(y_-) = 0$, then the limit*

$$\lim_{y \to y_-} \left\{ h(y - \frac{n_-}{b} + n_-) L_{n_-}(y) \right\}$$

exists; and if $g(-n_- + \frac{n_-}{b} - 1) = 0$, then

$$\lim_{y \to (-n_- + \frac{n_-}{b} - 1)^+} \left\{ h(y - \frac{n_-}{b} + n_-) L_{n_-}(y) \right\} = 0.$$

Then the equations, for $n = \pm 1, \pm 2, \cdots, \pm(N-1)$,

$$g(x - \frac{n}{b})h(x) + g(x - \frac{n}{b} + 1)h(x+1) = 0, \; x \in [\frac{n}{b} - 1, \frac{n}{b}] \tag{3.7}$$

determine $h(x)$ *continuously for* $x \in (-\tilde{J}) \cup \tilde{J}$. *Moreover,*

$$\lim_{x \to (n)^-} h(x) = \lim_{x \to (-n)^+} h(x) = 0, \; n = 1, 2, \cdots, N \tag{3.8}$$

and

$$\lim_{x \to (n/b)^+} h(x) = \lim_{x \to (-n/b)^-} h(x) = 0, \; n = 1, 2, \cdots, N-1. \tag{3.9}$$

Proof. We use induction to show that the equations (3.7) determine $h(x)$ continuously for $x \in (-\tilde{J}) \cup \tilde{J}$ and satisfy (3.8). First, by assumption, $h(x)$ is continuously chosen for $x \in [0,1] = \bigcup_{n=1}^{n_0} [\frac{n-1}{b}, n]$ with $n_0 = 1$, and

$$\lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} \frac{b - g(x-1)h(x-1)}{g(x)} = 0$$

by the conditions (1) and (2). With the purpose to perform an induction argument we now assume that, for some $1 \le n_0 \le N-1$, the function h is known to be continuous on $\bigcup_{n=1}^{n_0} [\frac{n-1}{b}, n]$ and $\lim_{x \to (n_0)^-} h(x) = 0$. We consider (3.7) for $n = n_0$, i.e.,

$$g(x - \frac{n_0}{b})h(x) + g(x - \frac{n_0}{b} + 1)h(x+1) = 0, \; x \in [\frac{n_0}{b} - 1, \frac{n_0}{b}]. \tag{3.10}$$

We will use (3.10) for x_0 in the subinterval $[\frac{n_0}{b} - 1, n_0]$ We split the argument into two cases:

1) We first assume that $g(x_0 - \frac{n_0}{b} + 1) \neq 0$. Then (3.10) implies

$$h(x_0 + 1) = -\frac{g(x_0 - \frac{n_0}{b})h(x_0)}{g(x_0 - \frac{n_0}{b} + 1)}; \tag{3.11}$$

and if $g(n_0 - \frac{n_0}{b} + 1) \neq 0$, then

$$\lim_{x \to (n_0)^-} h(x+1) = -\lim_{x \to (n_0)^-} \frac{g(x - \frac{n_0}{b})h(x)}{g(x - \frac{n_0}{b} + 1)} = -\frac{g(n_0 - \frac{n_0}{b}) \cdot 0}{g(n_0 - \frac{n_0}{b} + 1)} = 0.$$

2) We now assume $g(x_0 - \frac{n_0}{b} + 1) = 0$. Take $y := x - \frac{n_0}{b} + 1$ in the condition (3.5). Note that, for $n = 1, \cdots, n_0 - 1$,

$$[\frac{n+1}{b} - 1, n+1] - 1 = [\frac{n+1}{b} - 2, n] \subset [\frac{n}{b} - 1, n]. \tag{3.12}$$

Combining with (3.7) for $n = n_0 - 1$ implies that

$$\frac{h(x)}{g(x - \frac{n_0}{b} + 1)} = -\frac{g(x - \frac{n_0-1}{b} - 1)h(x-1)}{g(x - \frac{n_0}{b} + 1)g(x - \frac{n_0-1}{b})}, \ x \in [\frac{n_0}{b} - 1, n_0],$$

which is well-defined except for a finite number of x-values. Applying (3.7) and (3.12) repeatedly for $n = 1, 2, \cdots, n_0 - 2$ in reverse order implies that

$$\begin{aligned} \frac{h(x)}{g(x - \frac{n_0}{b} + 1)} &= (-1)^{n_0-1} \frac{g(x - \frac{n_0-1}{b} - 1) \cdots g(x - \frac{1}{b} - n_0 + 1)h(x - n_0 + 1)}{g(y - \frac{n_0}{b} + 1) \cdots g(x - \frac{1}{b} - n_0 + 2)} \\ &= (-1)^{n_0-1} \left(h(x - n_0 + 1) R_{n_0}(x - \frac{n_0}{b} + 1) \right). \end{aligned}$$

If $x_0 \in [\frac{n_0}{b} - 1, n_0[$, i.e., $x_0 \neq n_0$, then the limit

$$\lim_{x \to x_0} \frac{h(x)}{g(x - \frac{n_0}{b} + 1)} = (-1)^{n_0-1} \lim_{x \to x_0} \left(h(x - n_0 + 1) R_{n_0}(x - \frac{n_0}{b} + 1) \right)$$

exists by (3.5). Thus we can define

$$h(x_0 + 1) = -\lim_{x \to x_0} \left(\frac{h(x)}{g(x - \frac{n_0}{b} + 1)} \right) g(x_0 - \frac{n_0}{b}); \tag{3.13}$$

and if $x_0 = n_0$, i.e., $g(n_0 - \frac{n_0}{b} + 1) = 0$, then

$$\begin{aligned} \lim_{x \to (n_0)^-} \frac{h(x)}{g(x - \frac{n_0}{b} + 1)} &= (-1)^{n_0-1} \lim_{x \to (n_0)^-} \left(h(x - n_0 + 1) R_{n_0}(x - \frac{n_0}{b} + 1) \right) \\ &= 0, \end{aligned}$$

by (3.6). So $\lim_{x \to (n_0)^-} h(x+1) = 0$.

Note that $g(x - \frac{n_0}{b}), g(x - \frac{n_0}{b} - 1)$ and $h(x-1)$ are continuous for $x \in [\frac{n_0}{b}, n_0 + 1] \subset [\frac{n_0-1}{b} + 1, n_0 + 1]$. Hence $h(x)$ is determined and continuous for $x \in [\frac{n_0}{b}, n_0 + 1]$ by (3.11) and (3.13). By induction, $h(x)$ is continuous for $x \in \tilde{J}$, and $h(n) = 0$ for $n = 1, \cdots, N$.

On the other hand, for $x \in [n, \frac{n}{b}],\ n = 1, 2, \cdots, N-1$, the equation

$$g(x - \frac{n}{b})h(x) + g(x - \frac{n}{b} + 1)h(x+1) = 0$$

only involves $x \in [n, \frac{n}{b}]$ and $x + 1 \in [n+1, \frac{n}{b} + 1]$ for h, and

$$\Big([n, \frac{n}{b}] \cup [n+1, \frac{n}{b} + 1]\Big) \cap \tilde{J} = \emptyset,\ n = 1, 2, \cdots, N-1.$$

By symmetry, considering (3.7) for $n = -1, -2. \cdots, -N+1$ determines $h(x)$ continuously for $x \in (-\tilde{J})$. This proves that $h(x)$ is continuously determined for $x \in (-\tilde{J}) \cup \tilde{J}$ and satisfies (3.8).

For (3.9), the condition (1) and $g(-1) = 0$ imply that $g(0) \neq 0$. So the condition (a) implies that

$$\lim_{x \to (n/b)^+} h(x) = -\lim_{x \to (n/b)^+} \frac{g(x - \frac{n}{b} - 1)h(x-1)}{g(x - \frac{n}{b})} = \frac{g(-1)h(n/b - 1)}{g(0)} = 0,$$

for $n = 1, \cdots, N-1$. By symmetry,

$$\lim_{x \to (-n/b)^-} h(x) = 0 \quad \text{for} \quad n = 1, \cdots, N-1.$$

$\square$

Proposition 3.4 *Under the assumptions in Lemma 3.3, there exists a unique extension of h to a function with* supp $h \subseteq [-N, N]$ *so that for*

$$n = \pm 1, \pm 2, \cdots, \pm(N-1),$$

$$g(x - \frac{n}{b})h(x) + g(x - \frac{n}{b} + 1)h(x+1) = 0,\ x \in [\frac{n}{b}, \frac{n}{b} + 1]. \tag{3.14}$$

This function h is continuous.

Proof. We define $h(x)$ for $x \in (-\tilde{J}) \cup [-1, 1] \cup \tilde{J}$ as in the proof in Lemma 3.3 and

$$h(x) = 0,\ x \notin (-\tilde{J}) \cup [-1, 1] \cup \tilde{J}. \tag{3.15}$$

From Lemma 3.3, $h(x)$ is a continuous function with supp $h \subseteq [-N, N]$ satisfying (3.14) for $n = \pm 1, \pm 2, \cdots, \pm(N-1)$. $\square$

Proof of Theorem 2.2: (1) $\Rightarrow$ (3) : Suppose g generates a Gabor frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$. By Theorem 2.1, there exists a dual window $h \in L^2(\mathbb{R})$ with $\text{supp } h \subseteq [-N, N]$. Note that such a function h is essentially bounded due to the frame assumption. By Corollary 1.2, for $n = \pm 1, \pm 2, \cdots, \pm(N-1)$ we have that

$$g(x - \frac{n}{b})h(x) + g(x - \frac{n}{b} + 1)h(x+1) = 0, \ a.e.\ x \in [\frac{n}{b} - 1, \frac{n}{b}]; \quad (3.16)$$

further, by a shift of the equation in (1.2) with $n = 0$,

$$g(x)h(x) + g(x+1)h(x+1) = b, \ a.e.\ x \in [-1, 0]. \quad (3.17)$$

We now verify that the conditions in Theorem 2.2(3)(i)-(iv) are satisfied:
(i): Since g is continuous and $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is a frame with lower bound A,

$$\sum_{m\in\mathbb{Z}} |g(x-m)| \geq bA$$

for all $x \in \mathbb{R}$; since $\text{supp}\, g = [-1, 1]$, this leads to (i).
(ii): Suppose n_+ and y_+ satisfy the assumption in (ii). Via (3.4),

$$y_+ \in [0, n_+ - \frac{n_+}{b} + 1] \subset [0, 1[.$$

Let

$$x_+ := y_+ + \frac{n_+}{b} - 1 \in [\frac{n_+}{b} - 1, n_+].$$

Consider (3.16) with $n = n_+$, i.e.,

$$g(x - \frac{n_+}{b})h(x) + g(x - \frac{n_+}{b} + 1)h(x+1) = 0, \ a.e.\ x \in [\frac{n_+}{b} - 1, n_+].$$

Since g has a finite number of zeros in $[-1, 1]$, it follows that

$$\frac{h(x)}{g(x - \frac{n_+}{b} + 1)} = -\frac{h(x+1)}{g(x - \frac{n_+}{b})}, \ a.e.\ x \in [\frac{n_+}{b} - 1, n_+];$$

since $g(x_+ - \frac{n_+}{b}) = g(y_+ - 1) \neq 0$ by (i) and h is essentially bounded, it follows that

$$\limsup_{L_h \ni x \to x_+} \left| \frac{h(x)}{g(x - \frac{n_+}{b} + 1)} \right| =: M < \infty,$$

where L_h is the set of Lebesgue points of h. As in the proof of Lemma 3.3, we have

$$\limsup_{L_h \ni x \to x_+} \left| \frac{h(x)}{g(x - \frac{n_+}{b} + 1)} \right| = \limsup_{L_h \ni x \to x_+} \left| h(x - n_+ + 1) R_{n_+}(x - \frac{n_+}{b} + 1) \right|.$$

Since $\lim_{x \to x_+} \left| R_{n_+}(x - \frac{n_+}{b} + 1) \right| = \infty$, we conclude that

$$\lim_{L_h \ni x \to x_+} h(x - n_+ + 1) = 0,$$

i.e.,

$$\lim_{L_h \ni y \to y_+} h(y + \frac{n_+}{b} - n_+) = 0. \tag{3.18}$$

By (3.17) and (3.18),

$$\begin{aligned} b &= \lim_{L_h \ni x \to x_+} \{ g(x - n_+) h(x - n_+) + g(x - n_+ + 1) h(x - n_+ + 1) \} \\ &= \lim_{L_h \ni x \to x_+} g(x - n_+) h(x - n_+). \end{aligned}$$

Since $h(x)$ is essentially bounded and $g(x)$ is continuous, we have

$$g(x_+ - n_+) \neq 0,$$

i.e.,

$$g(y_+ + \frac{n_+}{b} - n_+ - 1) \neq 0.$$

This proves that (ii) holds.

(iii): This is similar to the proof of (ii) by symmetry, so we skip it. But we note for use in the proof of (iv) that the result corresponding to (3.18) is

$$\lim_{L_h \ni y \to y_-} h(y - \frac{n_-}{b} + n_-) = 0. \tag{3.19}$$

(iv): Suppose that y_+, n_+ and y_-, n_- are as in (ii) and (iii), respectively. Then the results in (3.18) and (3.19) holds, i.e,

$$\lim_{L_h \ni y \to y_+} h(y + \frac{n_+}{b} - n_+) = 0 \tag{3.20}$$

and

$$\lim_{L_h \ni y \to y_-} h(y - \frac{n_-}{b} + n_-) = 0. \tag{3.21}$$

Note that $y_+ + \frac{n_+}{b} - n_+,\ y_- - \frac{n_-}{b} + n_- + 1 \in [0,1]$. If

$$y_+ + \frac{n_+}{b} - n_+ = y_- - \frac{n_-}{b} + n_- + 1,$$

then by (3.17),

$$b = \lim_{L_h \ni y \to y_+} \left\{ g(y + \frac{n_+}{b} - n_+ - 1)h(y + \frac{n_+}{b} - n_+ - 1) + g(y + \frac{n_+}{b} - n_+)h(y + \frac{n_+}{b} - n_+) \right\};$$

however, this contradicts (3.20) and (3.21). Hence

$$y_+ + \frac{n_+}{b} - n_+ \neq y_- - \frac{n_-}{b} + n_- + 1,$$

i.e., (iv) holds.

(3) $\Rightarrow$ (2) : Assume that (i)-(iv) in Theorem 2.2(3) hold. We construct $h(x)$ on $[-1,1]$ satisfying the hypotheses described in Lemma 3.3. For $m, n = 1, 2, \cdots, N-1$ we define the sets Y_n and W_m by

$$Y_n = \{y_{n,i} \in]0, n - \frac{n}{b} + 1[\ :\ g(y_{n,i}) = 0 \text{ and } \lim_{y \to y_{n,i}} |R_n(y)| = \infty\}_{i=1,2,\cdots,r_n}$$

and

$$W_m = \{w_{m,j} \in]-m + \frac{m}{b} - 1, 0[\ :\ g(w_{m,j}) = 0 \text{ and } \lim_{y \to w_{m,j}} |L_m(y)| = \infty\ \}_{j=1,2,\cdots,l_m}$$

where r_n and l_m are the cardinalities of Y_n and W_m, respectively. We denote the open interval of radius $\epsilon > 0$ centered at x by

$$B(x;\epsilon) =]x - \epsilon, x + \epsilon[.$$

Let $y_{n,i} \in Y_n$, $w_{m,j} \in W_m$ for $n, m = 1, 2, \cdots, N-1$ and

$$\tilde{y}_{n,i} := y_{n,i} - n + \frac{n}{b},\ \hat{w}_{m,j} := w_{m,j} - \frac{m}{b} + m.$$

By the conditions (ii), (iii) and (iv),

$$g(\tilde{y}_{n,i} - 1) \neq 0 \neq g(\hat{w}_{m,j} + 1), \tag{3.22}$$

and

$$\tilde{y}_{n,i} \neq \hat{w}_{m,j} + 1. \tag{3.23}$$

Since

$$0 < \frac{n}{b} - n < \tilde{y}_{n,i} < 1 \tag{3.24}$$

and

$$-1 < \hat{w}_{m,j} < m - \frac{m}{b} < 0, \tag{3.25}$$

we can by (3.22) choose $\epsilon_0 > 0$ so that $g(x) \neq 0$ for

$$x \in]-1, -1+\epsilon_0[\cup B(\tilde{y}_{n,i} - 1; \epsilon_0) \cup B(\hat{w}_{m,j} + 1; \epsilon_0) \cup]1-\epsilon_0, 1[\tag{3.26}$$

and

$$\begin{array}{ll} B(\tilde{y}_{n,i}; \epsilon_0) \cap B(\hat{w}_{m,j} + 1; \epsilon_0) = \emptyset & \text{(by (3.23))} \\ B(\tilde{y}_{n,i}; \epsilon_0) \cap \{]0, \epsilon_0[\cup]1-\epsilon_0, 1[\} = \emptyset & \text{(by (3.24))} \\ B(\hat{w}_{m,j}; \epsilon_0) \cap \{]-1, -1+\epsilon_0[\cup]-\epsilon_0, 0[\} = \emptyset & \text{(by (3.25))} \end{array} \tag{3.27}$$

for $m, n = 1, 2, \cdots, N-1$, and $i = 1, 2, \cdots, r_n$ and $j = 1, 2, \cdots, l_m$.

First, we define $h(x)$ on $]-\epsilon_0, 0[\cup]1-\epsilon_0, 1[$: By the condition (i), $g(1) = 0$ implies $g(0) \neq 0$. Define $h(1) = 0$ and $h(0) = b/g(0)$. We split into two cases:

1) If $g(n - \frac{n}{b} + 1) \neq 0$ for $n = 1, 2, \cdots, N-1$, choose $h(x)$ on $]-\epsilon_0, 0[$ so that

$$\lim_{x->1^-} \frac{b - g(x-1)h(x-1)}{g(x)} = 0. \tag{3.28}$$

2) If the assumption in 1) does not hold, then there exists $\{n_k\}_{k=1}^{k_0} \subset \{1, \cdots, N-1\}$ such that

$$g(n_k - \frac{n_k}{b} + 1) = 0 \text{ for } k = 1, \cdots, k_0,$$

and

$$g(n - \frac{n}{b} + 1) \neq 0 \quad \text{for } n \in \{1, \cdots, N-1\} \setminus \{n_k\}_{k=1}^{k_0}.$$

Choose $h(x)$ on $]-\epsilon_0, 0[$ so that

$$\lim_{x->1^-} \frac{b - g(x-1)h(x-1)}{g(x)} = 0 \tag{3.29}$$

and

$$\lim_{x->1^-} \left\{ \frac{b - g(x-1)h(x-1)}{g(x)} R_{n_k}(x - \frac{n_k}{b} + n_k) \right\} = 0 \tag{3.30}$$

for $k = 1, \cdots, k_0$. We remark that there is a certain freedom in the choice of $h(x)$ on $]-\epsilon_0, 0[$.

Now, we define $h(x)$ on $]1-\epsilon_0, 1[$ by

$$h(x) := \frac{b - g(x-1)h(x-1)}{g(x)};$$

h is well-defined since $g(x) \neq 0$ for $x \in]1-\epsilon_0, 1[$. Then

$$g(x)h(x) + g(x+1)h(x+1) = b, \ x \in]-\epsilon_0, 0[.$$

Secondly, we define $h(x)$ on $B(\tilde{y}_{n,i} - 1; \epsilon_0) \cup B(\tilde{y}_{n,i}; \epsilon_0)$. We can choose $h(x)$ continuously on $B(\tilde{y}_{n,i}; \epsilon_0)$ so that

$$\lim_{y \to y_{n,i}} h(y + \frac{n}{b} - n) = 0 =: h(\tilde{y}_{n,i})$$

and the limit

$$\lim_{y \to y_{n,i}} \left\{ h(y + \frac{n}{b} - n) R_n(y) \right\}$$

do exist. Now, define $h(x)$ on $B(\tilde{y}_{n,i} - 1; \epsilon_0)$ by

$$h(x) = \frac{b - g(x+1)h(x+1)}{g(x)},$$

which is well-defined by (3.26). Then

$$g(x)h(x) + g(x+1)h(x+1) = b, \ x \in B(\tilde{y}_{n,i} - 1; \epsilon_0).$$

Thirdly, we define $h(x)$ on $]-1, \epsilon_0 - 1[\cup]0, \epsilon_0[$: Define $h(-1) = 0$. We split into two cases:

1) If $g(-n + \frac{n}{b} - 1) \neq 0$ for $n = 1, 2, \cdots, N-1$, choose $h(x)$ on $]0, \epsilon_0[$ so that

$$\lim_{x -> (-1)^+} \frac{b - g(x+1)h(x+1)}{g(x)} = 0.$$

2) If the assumption in 1) does not hold, then there exists $\{n_k\}_{k=1}^{k_1} \subset \{1, \cdots, N-1\}$ such that $g(-n_k + \frac{n_k}{b} - 1) = 0$ for $k = 1, \cdots, k_1$ and $g(-n + \frac{n}{b} - 1) \neq 0$ for $n \in \{1, \cdots, N-1\} \setminus \{n_k\}_{k=1}^{k_1}$. Choose $h(x)$ on $]0, \epsilon_0[$ so that

$$\lim_{x -> (-1)^+} \frac{b - g(x+1)h(x+1)}{g(x)} = 0$$

and

$$\lim_{x->(-1)^+}\left\{\frac{b-g(x+1)h(x+1)}{g(x)}\ L_{n_k}(x+\frac{n_k}{b}-n_k)\right\}=0$$

for $k=1,\cdots,k_0$.

We now define $h(x)$ on $]-1,\epsilon_0-1[$ by

$$h(x):=\frac{b-g(x+1)h(x+1)}{g(x)},$$

which is well-defined since $g(x)\neq 0$ for $x\in]-1,-1+\epsilon[$. Then

$$g(x)h(x)+g(x+1)h(x+1)=b,\ x\in]-1,-1+\epsilon_0[.$$

Fourthly, we define $h(x)$ on $B(\hat{w}_{m,j};\epsilon_0)\cup B(\hat{w}_{m,j}+1;\epsilon_0)$. Choose $h(x)$ continuously on $B(\hat{w}_{m,j};\epsilon_0)$ so that

$$\lim_{y\to w_{m,j}} h(y-\frac{m}{b}+m)=0=:h(\hat{w}_{m,j})$$

and the limit

$$\lim_{y\to w_{m,j}}\left\{h(y-\frac{m}{b}+m)L_m(y)\right\}$$

do exist. Now, define $h(x)$ on $B(\hat{w}_{m,j}+1;\epsilon_0)$ by

$$h(x)=\frac{b-g(x-1)h(x-1)}{g(x)},$$

which is well-defined by (3.26). Then

$$g(x)h(x)+g(x+1)h(x+1)=b,\ x\in B(\hat{w}_{m,j};\epsilon_0).$$

To summarize all these, let

$$A:=(-\epsilon_0,0]\cup[-1,-1+\epsilon_0)\cup\left(\cup_{n=1}^{N-1}\cup_{i=1}^{r_n}B(\tilde{y}_{n,i}-1;\epsilon_0)\right)\cup\left(\cup_{m=1}^{N-1}\cup_{j=1}^{l_m}B(\hat{w}_{m,j};\epsilon_0)\right).$$

Note that $A\subset[-1,0]$. We have defined $h(x)$ on $A\cup(A+1)$ such that

$$g(x)h(x)+g(x+1)h(x+1)=b,\ x\in A.$$

Finally, we choose $h(x)$ on $[-1,1]\backslash(A\cup(A+1))$ so that $h(x)$ be continuous on $[-1,1]$ and

$$g(x)h(x)+g(x+1)h(x+1)=b,\ x\in[-1,0]\setminus A,$$

by the condition (i).

By Proposition 3.4, the function h can be extended to a continuous function supported on $[-N,N]$ that is a dual window.

(2) $\Rightarrow$ (1) : This is well known. See [2] for example. □

3.3 Proof of Theorem 2.4

For each cases, we check the conditions (ii)-(iv) of (3) in Theorem 2.2.

(a) : The conditions (ii)-(iv) are trivially satisfied.

(b) : The conditions (iii) and (iv) are trivially satisfied. For (ii), if $k > k_0$, then

$$y_{\ell+k} \notin [0, 2 - \frac{1}{b}].$$

Thus

$$y_{\ell+k} \notin [0, n - \frac{n}{b} + 1],\ n = 1, 2, \cdots, N-1,$$

since $n - \frac{n}{b} + 1$ is decreasing as a function of n.

Now fix $k \in \{1, \cdots, k_0\}$. By Definition 2.3,

$$y_{\ell+k} \in [0, n - \frac{n}{b} + 1] \text{ iff } n = 1, 2, \cdots, n_k.$$

By (2.4), we have

$$g(y_{\ell+k} + \frac{n}{b} - n - 1) \neq 0,\ n = 1, 2, \cdots, n_k.$$

This proves (ii) and (iv).

(c): Similar to the proof of (b).

(d): As in the proof of (b) and (c), (2.4) and (2.5) imply (ii) and (iii). The condition (iv) follows from (2.6).

□

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