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Abstract

We introduce the pseudo-Butterworth refinable functions with order $(n; m, \ell)$ which is defined by the pseudo-Butterworth masks

$$M_{n;m,\ell}(w) := \frac{\cos^{2nm}(w/2)\left(\sum_{j=0}^{\ell}\binom{m+\ell}{j}\sin^{2nj}(w/2)\cos^{2n(\ell-j)}(w/2)\right)}{(\cos^{2n}(w/2)+\sin^{2n}(w/2))^{m+\ell}}$$

with positive integers n, m and nonnegative integer $\ell \leq m-1$. This family contains pseudo-splines (when $n = 1$) and provide a rich family of refinable functions. The pseudo-Butterworth refinable functions are not compactly supported (if $n > 1$) but have exponential decay to compensate for the lack of compact support. This paper gives a comprehensive analysis of the pseudo-Butterworth refinable functions, such as regularity, asymptotic analysis, approximation order, asymptotic behavior as a parameter grows to the infinity and wavelet constructions, etc, comparable to the analysis of pseudo-splines of Dong and Shen [8] and to the asymptotic behavior of Battle-Lemarie refinable function of Kim, Kim and Ku [12].

1 Introduction

Pseudo-Butterworth refinable functions will be defined as an extension of pseudo-splines. Starting from the simple identity,

$$1 = \left(\frac{\cos^{2n}(w/2)+\sin^{2n}(w/2)}{\cos^{2n}(w/2)+\sin^{2n}(w/2)}\right)^{m+\ell}, \tag{1.1}$$

the pseudo-Butterworth masks with order $(n; m, \ell)$ for given positive integers n, m and nonnegative integer $\ell \leq m-1$, are defined by the sum of the first $\ell+1$ terms of the binomial expansion of numerator of (1.1) for the numerator of the mask as follows:

$$M_{n;m,\ell}(w) := \frac{\cos^{2nm}(w/2)\left(\sum_{j=0}^{\ell}\binom{m+\ell}{j}\sin^{2nj}(w/2)\cos^{2n(\ell-j)}(w/2)\right)}{(\cos^{2n}(w/2)+\sin^{2n}(w/2))^{m+\ell}}. \tag{1.2}$$

It reduces to the mask of pseudo-spline of type II [8] when $n = 1$ and to the Butterworth mask when $m = 1$. The case $\ell = 0$ was used in [13] to construct tight wavelet frames by the unitary extension principle. The pseudo-Butterworth refinable function $\Phi_{n;m,\ell}$ with order $(n; m, \ell)$ is defined, via Fourier transform, as

$$\hat{\Phi}_{n;m,\ell}(w) := \prod_{j=1}^{\infty} M_{n;m,\ell}(w/2^j),$$

and can be regarded as an extension of pseudo-spline of type II ($n = 1$) with order (m, l). Pseudo-splines are first introduced in [5, 8] in order to construct tight framelets with required approximation order of the truncated frame series. They are refinable and compactly supported and provide large flexibilities in wavelet and framelet constructions and filter designs. The systematic analysis of pseudo-splines is given in [8]. Pseudo-Butterworth refinable functions provide more variety by introducing one more parameter in wavelet constructions, framelet constructions and filter designs. The lack of compact support of pseudo-Butterworth refinable functions can be compensated by the exponential decay. We remark that $\Phi_{n;1,0}$ is the Butterworth refinable function corresponding to the Butterworth mask $M_{n;1,0}$ and $\Phi_{n;m,0}$ is the m^{th} convolution of the Butterworth refinable function $\Phi_{n;1,0}$, which becomes to the B-spline B_m of order m when $n = 1$. We also remark that $\Phi_{n;m,m-1}$ is an interpolatory refinable function which reduces to the D-D(Deslauriers-Dubuc) interpolatory refinable function of order m when $n = 1$ [6, 14].

In this paper, we make a comprehensive analysis of the pseudo-Butterworth refinable functions, such as regularity, asymptotic analysis, approximation order, asymptotic behavior as a parameter grows to the infinity and wavelet constructions, etc, comparable to the analysis of pseudo-splines of Dong and Shen [8] and to the asymptotic behavior of Battle-Lemarie refinable functions of Kim, Kim and Ku [12]. In section 2, the technical lemmas are collected for the subsequent computations. In section 3, the Sobolev exponent $s_\infty(\Phi_{n;m,\ell})$ is computed in terms of parameters and its dependance on the parameters are analyzed (Theorem 3.3 and Proposition 3.4). The asymptotic analysis of the Sobolev exponent $s_\infty(\Phi_{n;m,\ell})$ as $n \to \infty$ or as $m \to \infty$ are given in Section 4 (Theorem 4.1 $\sim$ 4.3). In section 5, we show that the pseudo-Butterworth refinable function $\Phi_{n;m,\ell}$ can generate a Riesz wavelet $\Psi_{n;m,\ell}$ whose dilation and translation forms a Riesz basis for $L^2(\mathbb{R})$ (Theorem 5.3) and compute its approximation order for the proper projection (Theorem 5.5). In section 6, we show that $\Phi_{n;m,\ell}$ converges to Shannon refinable function in $L^q(\mathbb{R})(2 \le q \le \infty)$ as $n \to \infty$ or as $m \to \infty$ with $\ell = m - 1$ (Theorem 6.5). Finally, in section 7, we consider another class of pseudo-Butterworth refinable function coming from the Riesz factorization of pseudo-Butterworth masks and give some relevant examples of pseudo-Butterworth refinable functions and their graphs.

2 Technical Lemmas

For the notational simplicity we will use the following notations throughout the paper:

$$P_{n;m,\ell}(y) := \left(\sum_{j=0}^{\ell} \binom{m+\ell}{j} y^{nj}(1-y)^{n(\ell-j)} \right) \Big/ \left((1-y)^n + y^n\right)^\ell,$$

$$\tilde{M}_{n;m,\ell}(y) := (1-y)^{nm} P_{n;m,\ell}(y) / \left((1-y)^n + y^n\right)^m = M_{n;m,\ell}(w),$$

with $y = \sin^2(w/2)$. We also introduce an auxiliary function

$$\Lambda(y) := \Lambda_n(y) := \frac{y^n}{(1-y)^n + y^n}.$$

and note that

$$P_{n;m,\ell}(y) = \sum_{j=0}^{\ell} \binom{m+\ell}{j} \Lambda^j(y) \Lambda^{\ell-j}(1-y), \tag{2.1}$$

and

$$\tilde{M}_{n;m,\ell}(y) = \Lambda^m(1-y)P_{n;m,\ell}(y) \tag{2.2}$$

The computations here are more complicated by the additional parameter n but are analoguous to those in [8] with $\Lambda(y)$ in place of y and $\Lambda(1-y)$ in place of $1-y$.

Lemma 2.1 *For given positive integer m and nonnegative integers j, ℓ, we have:*

(1) $\binom{m+1}{j} = \binom{m}{j} + \binom{m}{j-1}$ *for* $j \geq 1$ *and* $(j+1)\binom{m+j}{j+1} = (m+j)\binom{m-1+j}{j}$.

(2) $2(m+1)\sum_{j=0}^{\ell-1}\binom{m+\ell}{j} - \ell\sum_{j=0}^{\ell}\binom{m+\ell}{j} \geq 0$ *for* $m \geq 1$ *and* $1 \leq \ell \leq m-1$.

(3) $2^{2\ell}\binom{m+\ell}{\ell} \leq \left(\sum_{j=0}^{\ell}\binom{m+\ell}{j}\right)^2$ *for all* $m \geq 1$ *and* $0 \leq \ell \leq m-1$.

Proof. [8, Lemma 2.1]. □

Lemma 2.2 *For positive integers n, m and nonnegative integer $\ell \leq m-1$, let $P_{n;m,\ell}$ be the rational function defined in* (2.1). *Then*

(1) $P_{n;m,\ell+1}(y) = P_{n;m,\ell}(y) + \binom{m+\ell}{\ell+1}\Lambda^{\ell+1}(y)$ *for* $0 \leq \ell \leq m-2$.

(2) $P_{n;m,\ell}(y) = \sum_{j=0}^{\ell}\binom{m-1+j}{j}\Lambda^j(y)$.

(3) $\tilde{M}'_{n;m,\ell}(y) = -(m+\ell)\binom{m+\ell-1}{\ell}\Lambda^\ell(y)\Lambda^{m-1}(1-y)\Lambda'(y)$.

(4) *For* $T(y) := (1-y)^n + y^n$,

$$\min_{y\in[0,1]} T(y) = T\left(\frac{1}{2}\right) = 2^{1-n}.$$

(5) *For* $S(y) := \tilde{M}^2_{n;m,\ell}(y) + \tilde{M}^2_{n;m,\ell}(1-y)$,

$$\min_{y\in[0,1]} S(y) = S\left(\frac{1}{2}\right) = 2^{1-2m-2\ell}\left(\sum_{j=0}^{\ell}\binom{m+\ell}{j}\right)^2.$$

Proof. (1): By use of Lemma 2.1 (1), we note that

$$\begin{aligned}
P_{n;m,\ell+1}(y) =& \sum_{j=0}^{\ell+1} \binom{m+\ell+1}{j} \Lambda^j(y)\Lambda^{\ell+1-j}(1-y) \\
=& \Lambda^{\ell+1}(1-y) + \sum_{j=1}^{\ell+1} \binom{m+\ell+1}{j} \Lambda^j(y)\Lambda^{\ell+1-j}(1-y) \\
=& \Lambda^{\ell+1}(1-y) \\
&+ \sum_{j=1}^{\ell+1} \binom{m+\ell}{j} \Lambda^j(y)\Lambda^{\ell+1-j}(1-y) + \sum_{j=0}^{\ell} \binom{m+\ell}{j} \Lambda^{j+1}(y)\Lambda^{\ell-j}(1-y) \\
=& \Lambda(1-y)P_{n;m,\ell}(y) + \binom{m+\ell}{\ell+1}\Lambda^{\ell+1}(y) + \Lambda(y)P_{n;m,\ell}(y) \\
=& P_{n;m,\ell}(y) + \binom{m+\ell}{\ell+1}\Lambda^{\ell+1}(y).
\end{aligned}$$

(2): We induct on ℓ. It is obviously true for $\ell = 0$. Assume that (2) holds for $\ell = \ell_0$. Then (1) with induction hypothesis for $\ell = \ell_0$ implies

$$\begin{aligned}
P_{n;m,\ell_0+1} &= \sum_{j=0}^{\ell_0} \binom{m-1+j}{j} \Lambda^j(y) + \binom{m+\ell_0}{\ell_0+1}\Lambda^{\ell_0+1}(y) \\
&= \sum_{j=0}^{\ell_0+1} \binom{m-1+j}{j} \Lambda^j(y).
\end{aligned}$$

(3): We induct on ℓ again. It is obviously true for $\ell = 0$. Assume that (3) holds for $\ell = \ell_0$, i.e.,

$$\tilde{M}'_{n;m,\ell_0}(y) = -(m+\ell_0)\binom{m+\ell_0-1}{\ell_0}\Lambda^{\ell_0}(y)\Lambda^{m-1}(1-y)\Lambda'(y),$$

and consider the case $\ell = \ell_0 + 1 \le m-1$. Using (1) and the definition of $\tilde{M}_{n;m,\ell}(y)$ in (2.2), we have

$$\begin{aligned}
\tilde{M}_{n;m,\ell_0+1}(y) &= \Lambda^m(1-y)P_{n;m,\ell_0+1}(y) \\
&= \Lambda^m(1-y)\left(P_{n;m,\ell_0}(y) + \binom{m+\ell_0}{\ell_0+1}\Lambda^{\ell_0+1}(y)\right) \\
&= \tilde{M}_{n;m,\ell_0}(y) + \binom{m+\ell_0}{\ell_0+1}\Lambda^{\ell_0+1}(y)\Lambda^m(1-y).
\end{aligned}$$

Since $\Lambda'(1-y) = \Lambda'(y)$, we have

$$\begin{aligned}\tilde{M}'_{n;m,\ell_0+1}(y) =& -(m+\ell_0)\binom{m+\ell_0-1}{\ell_0}\Lambda^{\ell_0}(y)\Lambda^{m-1}(1-y)\Lambda'(y)\\ &+\binom{m+\ell_0}{\ell_0+1}(\ell_0+1)\Lambda^{\ell_0}(y)\Lambda^{m}(1-y)\Lambda'(y)\\ &-\binom{m+\ell_0}{\ell_0+1}m\Lambda^{\ell_0+1}(y)\Lambda^{m-1}(1-y)\Lambda'(y)\\ =&\Lambda^{\ell_0}(y)\Lambda^{m-1}(1-y)\Lambda'(y)\left(-(m+\ell_0)\binom{m+\ell_0-1}{\ell_0}\right.\\ &\left.+(\ell_0+1)\binom{m+\ell_0}{\ell_0+1}\Lambda(1-y)-m\binom{m+\ell_0}{\ell_0+1}\Lambda(y)\right).\end{aligned}$$

Since $\Lambda(1-y) = 1-\Lambda(y)$, by using Lemma 2.1 (1), we obtain

$$\tilde{M}'_{n;m,\ell_0+1}(y) = -(m+\ell_0+1)\binom{m+\ell_0}{\ell_0+1}\Lambda^{\ell_0+1}(y)\Lambda^{m-1}(1-y)\Lambda'(y).$$

(4): Since $T'(y) = n(y^{n-1}-(1-y)^{n-1})$, $T'(y) \le 0$ on $[0, \frac{1}{2}]$ and $T'(y) \ge 0$ on $[\frac{1}{2}, 1]$. Therefore, $T(y)$ attains its minimum value at the point $y = \frac{1}{2}$. That is,

$$\min_{y\in[0,1]} T(y) = T\left(\frac{1}{2}\right) = 2^{1-n}.$$

(5): We compute $S'(y)$:

$$S'(y) = 2\tilde{M}_{n;m,\ell}(y)\tilde{M}'_{n;m,\ell}(y) + 2\tilde{M}_{n;m,\ell}(1-y)(\tilde{M}_{n;m,\ell}(1-y))'.$$

Using the identities

$$\begin{aligned}\tilde{M}_{n;m,\ell}(y) &= \Lambda^m(1-y)P_{n;m,\ell}(y),\\ \tilde{M}'_{n;m,\ell}(y) &= -(m+\ell)\binom{m+\ell-1}{\ell}\Lambda^{\ell}(y)\Lambda^{m-1}(1-y)\Lambda'(y)\end{aligned}$$

and

$$(\tilde{M}_{n;m,\ell}(1-y))' = (m+\ell)\binom{m+\ell-1}{\ell}\Lambda^{m-1}(y)\Lambda^{\ell}(1-y)\Lambda'(y),$$

we obtain

$$\begin{aligned}&\frac{S'(y)}{2(m+\ell)\binom{m+\ell-1}{\ell}}\\ &=\sum_{j=0}^{\ell}\binom{m-1+j}{j}\left(\Lambda^{2m-1-\ell-j}(y)-\Lambda^{2m-1-\ell-j}(1-y)\right)\Lambda^{\ell+j}(y)\Lambda^{\ell+j}(1-y)\Lambda'(y).\end{aligned}$$

For each $0 \le j \le \ell \le m-1$, we note

$$\begin{aligned}(\Lambda(y))^{2m-1-\ell-j} &\le (\Lambda(1-y))^{2m-1-\ell-j} \qquad \text{for} \quad y \in \left[0, \frac{1}{2}\right],\\ (\Lambda(y))^{2m-1-\ell-j} &\ge (\Lambda(1-y))^{2m-1-\ell-j} \qquad \text{for} \quad y \in \left[\frac{1}{2}, 1\right].\end{aligned}$$

Therefore, $S'(y) \le 0$ on $[0, \frac{1}{2}]$ and $S'(y) \ge 0$ on $[\frac{1}{2}, 1]$, and so $S(y)$ attains its minimum value at the point $y = \frac{1}{2}$. That is,

$$\min_{y\in[0,1]} S(y) = S\left(\frac{1}{2}\right) = 2\tilde{M}^2_{n;m,\ell}\left(\frac{1}{2}\right) = 2^{1-2m-2\ell}\left(\sum_{j=0}^{\ell}\binom{m+\ell}{j}\right)^2.$$

□

3 Regularity

The regularity of a function $\varphi \in L^2(\mathbb{R})$ can be measured by the Sobolev exponent defined by

$$s_\infty(\varphi) := \sup\{s : \sup_w |\hat{\varphi}(w)|(1+|w|)^s < \infty\}.$$

For example, it gives the regularity of $\varphi \in C^s$ for any $s < s_\infty(\varphi) - 1$. The following proposition is [4, Lemma 7.1.7] adapted for our purpose and is used to compute the Sobolev exponents.

Proposition 3.1 *Let φ be the refinable function with the refinement mask $m(w)$ of the form*

$$|m(w)| := \cos^{2N}(w/2)|P(\sin^2(w/2))|, \quad w \in \mathbb{T} := [-\pi, \pi],$$

for $P \in C^1(\mathbb{R})$. Suppose that

(1) $|P(y)| \le |P(3/4)|$ *for* $0 \le y \le 3/4$*;*

(2) $|P(y)P(4y(1-y))| \le |P(3/4)|^2$ *for* $3/4 \le y \le 1$.

Then $s_\infty(\varphi) = 2N - \kappa$ with $\kappa = \log_2(|P(3/4)|)$. Consequently, $\varphi \in C^{2N-\kappa-1-\epsilon}$ for any $\varepsilon > 0$.

In this section, we compute the Sobolev exponent $s_\infty(\Phi_{n;m,\ell})$ of the pseudo-Butterworth refinable function $\Phi_{n;m,\ell}$ and investigate its dependance on the parameters n, m and ℓ. The computational technique is largely the same as in [8] but more complicated because of three parameters instead of two.

Lemma 3.2 *Let $P_{n;m,\ell}$ be defined as in (2.2), where n, m are positive integers and ℓ is nonnegative integer $\le m-1$. Then*

(1) $P_{n;m,\ell}(y) \le P_{n;m,\ell}(3/4)$ *for* $y \in [0, 3/4]$*;*

(2) $P_{n;m,\ell}(y)P_{n;m,\ell}(4y(1-y)) \le (P_{n;m,\ell}(3/4))^2$ *for* $y \in [3/4, 1]$.

Proof. From Lemma 2.2 (2), $P_{n;m,\ell}(y)$ is positive and monotonically increasing on $[0,1]$. Hence (1) is satisfied. Next, fix $n, m \in \mathbb{N}$. For the proof of (2), we will show that, for any $\ell = 0, \ldots, m-2$,

$$W_{n;m,\ell+1}(y) - W_{n;m,\ell}(y) \leq 0, \quad y \in [3/4, 1], \tag{3.1}$$

where

$$W_{n;m,\ell}(y) := P_{n;m,\ell}(y) P_{n;m,\ell}(4y(1-y)) - \left(P_{n;m,\ell}(3/4)\right)^2. \tag{3.2}$$

Note that $P_{n;m,0}(y) \equiv 1$, and hence $W_{n;m,0}(y) \equiv 0$. Then (2) follows immediately from (3.1). Let us observe from (3.2) and Lemma 2.2 (1) that

$$\begin{aligned}
& W_{n;m,\ell+1}(y) - W_{n;m,\ell}(y) \\
&= \left(P_{n;m,\ell}(y) + \binom{m+\ell}{\ell+1} \Lambda^{\ell+1}(y) \right) P_{n;m,\ell+1}(4y(1-y)) \\
&\quad - P_{n;m,\ell}(y) P_{n;m,\ell}(4y(1-y)) - (P_{n;m,\ell+1}(3/4))^2 + (P_{n;m,\ell}(3/4))^2 \\
&= P_{n;m,\ell}(y) \left(P_{n;m,\ell+1}(4y(1-y)) - P_{n;m,\ell}(4y(1-y)) \right) \\
&\quad + \binom{m+\ell}{\ell+1} \Lambda^{\ell+1}(y) P_{n;m,\ell+1}(4y(1-y)) \\
&\quad - (P_{n;m,\ell+1}(3/4))^2 + (P_{n;m,\ell}(3/4))^2 \\
&= \binom{m+\ell}{\ell+1} \Big(\Lambda^{\ell+1}(4y(1-y)) P_{n;m,\ell}(y) + \Lambda^{\ell+1}(y) P_{n;m,\ell+1}(4y(1-y)) \Big) \\
&\quad - (P_{n;m,\ell+1}(3/4))^2 + (P_{n;m,\ell}(3/4))^2.
\end{aligned}$$

Since $W_{n;m,\ell+1}(3/4) - W_{n;m,\ell}(3/4) = 0$, in order to show the relation (3.1), it suffices to show that $W_{n;m,\ell+1}(y) - W_{n;m,\ell}(y)$ decreases monotonically on $[3/4, 1]$. Since $P_{n;m,\ell+1}(y) \geq P_{n;m,\ell}(y)$ for any $y \in [0,1]$, it suffices to show that

$$G(y) := \Lambda^{\ell+1}(4y(1-y)) P_{n;m,\ell}(y) + \Lambda^{\ell+1}(y) P_{n;m,\ell+1}(4y(1-y))$$

decreases monotonically on $[3/4, 1]$, i.e., $G'(y) \leq 0$, $y \in [3/4, 1]$. We compute G':

$$\begin{aligned}
G'(y) =& (\ell+1)(4-8y) \Lambda^{\ell}(4y(1-y)) \Lambda'(4y(1-y)) P_{n;m,\ell}(y) \\
&+ \Lambda^{\ell+1}(4y(1-y)) P'_{n;m,\ell}(y) \\
&+ (\ell+1) \Lambda^{\ell}(y) \Lambda'(y) P_{n;m,\ell+1}(4y(1-y)) \\
&+ \Lambda^{\ell+1}(y)(4-8y) P'_{n;m,\ell+1}(4y(1-y)).
\end{aligned}$$

From Lemma 2.2 (1), we have the identity

$$P'_{n;m,\ell}(y) = P'_{n;m,\ell+1}(y) - \binom{m+\ell}{\ell+1} (\ell+1) \Lambda^{\ell}(y) \Lambda'(y).$$

This together with Lemma 2.2 (1) implies that

$$\begin{aligned}
G'(y) =& (\ell+1)(4-8y) \Lambda^{\ell}(4y(1-y)) \Lambda'(4y(1-y)) P_{n;m,\ell}(y) \\
&+ \Lambda^{\ell+1}(4y(1-y)) P'_{n;m,\ell+1}(y) \\
&+ (\ell+1) \Lambda^{\ell}(y) \Lambda'(y) P_{n;m,\ell}(4y(1-y)) \\
&+ \Lambda^{\ell+1}(y)(4-8y) P'_{n;m,\ell+1}(4y(1-y)).
\end{aligned}$$

Using (2.1) and Lemma 2.1 (1), we have

$$
\begin{aligned}
G'(y) = &\sum_{j=0}^{\ell} \binom{m-1+j}{j} \Lambda^j(y)\Lambda^j(4y(1-y)) \\
&\Big((\ell+1)(4-8y)\Lambda^{\ell-j}(4y(1-y))\Lambda'(4y(1-y)) \\
&+ (m+j)\Lambda^{\ell+1-j}(4y(1-y))\Lambda'(y) + (\ell+1)\Lambda^{\ell-j}(y)\Lambda'(y) \\
&+(m+j)(4-8y)\Lambda^{\ell+1-j}(y)\Lambda'(4y(1-y))\Big).
\end{aligned}
$$

For $0 \le j \le \ell \le m-2$, consider

$$
\begin{aligned}
g_{\ell,j}(y) :=&(\ell+1)(4-8y)\Lambda^{\ell-j}(4y(1-y))\Lambda'(4y(1-y)) \\
&+ (m+j)\Lambda^{\ell+1-j}(4y(1-y))\Lambda'(y) + (\ell+1)\Lambda^{\ell-j}(y)\Lambda'(y) \\
&+ (m+j)(4-8y)\Lambda^{\ell+1-j}(y)\Lambda'(4y(1-y)) \\
=&(\ell+1)\Lambda^{\ell-j}(4y(1-y))h_1(y) + (\ell+1)\Lambda^{\ell-j}(y)h_2(y) + (m+j-(\ell+1))h_3(y),
\end{aligned}
$$

where h_1, h_2 and h_3 are given by

$$
\begin{aligned}
h_1(y) &:= -(8y-4)\Lambda'(4y(1-y)) + \Lambda(4y(1-y))\Lambda'(y); \\
h_2(y) &:= \Lambda'(y) - (8y-4)\Lambda(y)\Lambda'(4y(1-y)); \\
h_3(y) &:= \Lambda^{\ell+1-j}(4y(1-y))\Lambda'(y) - (8y-4)\Lambda^{\ell+1-j}(y)\Lambda'(4y(1-y)).
\end{aligned}
$$

Since $3/4 \le \Lambda(y) \le 1,\ 2 \le 8y-4$ for $y \in [3/4, 1]$ and $\Lambda(y) \ge 0,\ \Lambda'(y) \ge 0$ for $y \in [0,1]$, we obtain

$$
\begin{aligned}
h_1(y) &\le -2\Lambda'(4y(1-y)) + \Lambda'(y); \\
h_2(y) &\le \Lambda'(y) - 2\Lambda(y)\Lambda'(4y(1-y)) \le \Lambda'(y) - \frac{3}{2}\Lambda'(4y(1-y)); \\
h_3(y) &\le \Lambda^{\ell+1-j}(y)\Lambda'(y) - 2\Lambda^{\ell+1-j}(y)\Lambda'(4y(1-y)) = \Lambda^{\ell+1-j}(y)\left(\Lambda'(y) - 2\Lambda'(4y(1-y))\right).
\end{aligned}
$$

Hence, in order to show $G'(y) \le 0$, we prove

$$
\Lambda'(4y(1-y)) \ge \Lambda'(y),\ \ y \in [3/4, 1].
$$

A direct calculation shows that

$$
\Lambda'(y) = \frac{ny^{n-1}(1-y)^{n-1}}{(y^n + (1-y)^n)^2} \tag{3.3}
$$

increases on $[0, 1/2]$ and decreases on $[1/2, 1]$. If $y \ge 3/4$ and $4y(1-y) \ge 1/2$, *i.e.*, $3/4 \le y \le (2+\sqrt{2})/4$, then $\Lambda'(4y(1-y)) \ge \Lambda'(y)$ since $1/2 \le 4y(1-y) \le y$. If $y \ge 3/4$ and $4y(1-y) \le 1/2$, *i.e.*, $(2+\sqrt{2})/4 \le y \le 1$, then $\Lambda'(y) \le \Lambda'(4y(1-y))$ since $1-y \le 4y(1-y) \le 1/2$ and $\Lambda'(y) = \Lambda'(1-y)$ by (3.3). This completes the proof. □

The following main theorem in this section now follows from Proposition 3.1 , Lemma 3.2 and [11, Theorem 3]. The case $n = 1$ was given in [8, Theorem 3.4] and the case $m = 1$ was given in [11, Thoerem 3].

Theorem 3.3 *Let $\Phi_{n;m,\ell}$ be the pseudo-Butterworth refinable function with order $(n;m,\ell)$. Then $s_\infty(\Phi_{n;m,\ell}) = m\log_2(1+3^n) - \log_2 P_{n;m,\ell}(3/4)$.*

Proof. The pseudo-Butterworth mask with order $(n;m,\ell)$ is

$$M_{n;m,\ell}(w) = \cos^{2nm}(w/2)\left(L_n(\sin^2(w/2))\right)^m P_{n;m,\ell}(\sin^2(w/2)),$$

where $L_n(y) := 1/\left((1-y)^n + y^n\right)$. By Proposition 3.1, Lemma 3.2 and [11, Theorem 3], the Sobolev exponent $s_\infty(\Phi_{n;m,\ell})$ is

$$\begin{aligned} s_\infty(\Phi_{n;m,\ell}) &= 2nm - \log_2|L_n(3/4)|^m - \log_2|P_{n;m,\ell}(3/4)| \\ &= \log_2(1+3^n)^m - \log_2 P_{n;m,\ell}(3/4). \end{aligned}$$

□

We now analyze the dependance of the Sobolev exponent $s_\infty(\Phi_{n;m,\ell})$ of pseudo-Butterworth refinable functions $\Phi_{n;m,\ell}$ on parameters n, m and ℓ in the following proposition. (1)∼(3) in the following for the case $n = 1$ was given in [8, Proposition 3.5].

Proposition 3.4 *Let $\beta_{n;m,\ell} := s_\infty(\Phi_{n;m,\ell})$. Then*

(1) *For fixed n, m, $\beta_{n;m,\ell}$ decreases as ℓ increases.*

(2) *For fixed n, ℓ, $\beta_{n;m,\ell}$ increases as m increases.*

(3) *For fixed n, when $\ell = m-1$, $\beta_{n;m,\ell}$ increases as m increases.*

(4) *For fixed m, ℓ, $\beta_{n;m,\ell}$ increases as n increases.*

Proof. (1) follows directly from Lemma 2.2 (2), which shows that $P_{n;m,\ell}(3/4)$ increases as ℓ increases for fixed n, m.
(2): Note that

$$\begin{aligned} \beta_{n;m,\ell} &= \log_2(1+3^n)^m - \log_2(P_{n;m,\ell}(3/4)), \; \textit{i.e.}, \\ 2^{\beta_{n;m,\ell}} &= \frac{(1+3^n)^m}{P_{n;m,\ell}(\frac{3}{4})} = \frac{1}{(1+3^n)^{-m}P_{n;m,\ell}(\frac{3}{4})}. \end{aligned}$$

(2) is equivalent to the claim that

$$I_m := (1+3^n)^{-m} P_{n;m,\ell}\left(\frac{3}{4}\right)$$

decreases as m increases for fixed n, ℓ, which is again equivalent to the claim that for fixed $0 \le \ell \le m-1$,

$$I_{m+1} - I_m < 0. \tag{3.4}$$

Note that

$$\begin{aligned} I_{m+1} - I_m &= (1+3^n)^{-m-1} P_{n;m+1,\ell}\left(\frac{3}{4}\right) - (1+3^n)^{-m} P_{n;m,\ell}\left(\frac{3}{4}\right) \\ &= (1+3^n)^{-m-1} \sum_{j=0}^{\ell} \left(\binom{m+j}{j} - (1+3^n)\binom{m-1+j}{j} \right) \left(\frac{3^n}{1+3^n}\right)^j. \end{aligned}$$

Inequality (3.4) follows from the fact that for $0 \le j \le m-1, n \ge 1$,

$$\begin{aligned}\binom{m+j}{j} &= \frac{m+j}{m}\binom{m-1+j}{j} = \left(1+\frac{j}{m}\right)\binom{m-1+j}{j} \\ &< 2\binom{m-1+j}{j} < (1+3^n)\binom{m-1+j}{j}.\end{aligned}$$

This completes the proof of (2).
(3): Using a similar argument as in the proof of (2), (3) follows if we show that

$$J_m := (1+3^n)^{-m} P_{n;m,m-1}\left(\frac{3}{4}\right)$$

decreases as m increases for fixed n, which is equivalent to showing that for fixed $n \ge 1$,

$$J_{m+1} - J_m < 0. \tag{3.5}$$

Note that, similarly to the proof of (2), we have

$$J_{m+1} - J_m = (1+3^n)^{-m-1} M,$$

where

$$\begin{aligned} M &:= \sum_{j=0}^{m}\binom{m+j}{j}\left(\frac{3^n}{1+3^n}\right)^j - (1+3^n)\sum_{j=0}^{m-1}\binom{m-1+j}{j}\left(\frac{3^n}{1+3^n}\right)^j \qquad (3.6)\\ &= \sum_{j=0}^{m-1}\binom{m+j}{j}\left(\frac{3^n}{1+3^n}\right)^j + \binom{2m}{m}\left(\frac{3^n}{1+3^n}\right)^m - (1+3^n)\sum_{j=0}^{m-1}\binom{m-1+j}{j}\left(\frac{3^n}{1+3^n}\right)^j \\ &= \sum_{j=1}^{m-1}\binom{m-1+j}{j-1}\left(\frac{3^n}{1+3^n}\right)^j - 3^n\sum_{j=0}^{m-1}\binom{m-1+j}{j}\left(\frac{3^n}{1+3^n}\right)^j + \binom{2m}{m}\left(\frac{3^n}{1+3^n}\right)^m, \end{aligned}$$

where the last identity follows from Lemma 2.1 (1). Now, (3.5) is equivalent to $M < 0$ for $m \ge 1$. It is easy to check that $M < 0$ for $m = 1$. We now consider the case $m \ge 2$. Substituting j for $j-1$ in the first term of the last expression in (3.6), one obtains that

$$\begin{aligned} M = &\frac{3^n}{1+3^n}\sum_{j=0}^{m-2}\binom{m+j}{j}\left(\frac{3^n}{1+3^n}\right)^j - 3^n\sum_{j=0}^{m-1}\binom{m-1+j}{j}\left(\frac{3^n}{1+3^n}\right)^j \\ &+ \binom{2m}{m}\left(\frac{3^n}{1+3^n}\right)^m. \end{aligned} \tag{3.7}$$

Splitting the second term in (3.7), one obtains

$$\begin{aligned} M = &\frac{3^n}{1+3^n}\sum_{j=0}^{m-2}\binom{m+j}{j}\left(\frac{3^n}{1+3^n}\right)^j - 3^n\sum_{j=0}^{m-2}\binom{m-1+j}{j}\left(\frac{3^n}{1+3^n}\right)^j \\ &+ \binom{2m}{m}\left(\frac{3^n}{1+3^n}\right)^m - 3^n\binom{2m-2}{m-1}\left(\frac{3^n}{1+3^n}\right)^{m-1}. \end{aligned} \tag{3.8}$$

For the last two terms of (3.8), we have

$$
\begin{aligned}
\binom{2m}{m}\left(\frac{3^n}{1+3^n}\right)^m &- 3^n\binom{2m-2}{m-1}\left(\frac{3^n}{1+3^n}\right)^{m-1} \\
&= \left(\frac{3^n}{1+3^n}\right)^m\left(\binom{2m}{m} - (1+3^n)\binom{2m-2}{m-1}\right) \\
&= \left(\frac{3^n}{1+3^n}\right)^m\left(\left(4-\frac{2}{m}\right)\binom{2m-2}{m-1} - (1+3^n)\binom{2m-2}{m-1}\right) \\
&< 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
M &< \frac{3^n}{1+3^n}\sum_{j=0}^{m-2}\binom{m+j}{j}\left(\frac{3^n}{1+3^n}\right)^j - 3^n\sum_{j=0}^{m-2}\binom{m-1+j}{j}\left(\frac{3^n}{1+3^n}\right)^j \\
&< \sum_{j=0}^{m-2}\binom{m+j}{j}\left(\frac{3^n}{1+3^n}\right)^j - 3^n\sum_{j=0}^{m-2}\binom{m-1+j}{j}\left(\frac{3^n}{1+3^n}\right)^j \\
&< \sum_{j=0}^{m-2}\left(\binom{m+j}{j} - 3^n\binom{m-1+j}{j}\right)\left(\frac{3^n}{1+3^n}\right)^j.
\end{aligned}
$$

Inequality (3.5) now follows from the fact that, for $0 \le j \le m-2$ and $n \ge 1$,

$$
\begin{aligned}
\binom{m+j}{j} &= \frac{m+j}{m}\binom{m-1+j}{j} = \left(1+\frac{j}{m}\right)\binom{m-1+j}{j} \\
&< 2\binom{m-1+j}{j} < 3^n\binom{m-1+j}{j}
\end{aligned}
$$

This completes the proof of (3).
(4) follows if we show that

$$
K_n := (1+3^n)^{-m}P_{n;m,\ell}\left(\frac{3}{4}\right) = (1+3^n)^{-m}\sum_{j=0}^{\ell}\binom{m-1+j}{j}\left(\frac{3^n}{1+3^n}\right)^j
$$

decreases as n increases for fixed m, ℓ. Define

$$
f(x) := (1+3^x)^{-m}\sum_{j=0}^{\ell}\binom{m-1+j}{j}\left(\frac{3^x}{1+3^x}\right)^j,
$$

for $x > 0$. Note that

$$
f'(x) = (1+3^x)^{-m-1}\log 3\left(\sum_{j=1}^{\ell}\binom{m-1+j}{j}(j-m3^x)\left(\frac{3^x}{1+3^x}\right)^j - m3^x\right).
$$

Since $j - m3^x < 0$ for $x > 0$, $f'(x) < 0$ for $x > 0$. Hence, $f(x)$ is decreasing for $x > 0$, which shows that K_n decreases as n increases for fixed m, ℓ. This completes the proof of (4). □

4 Asymptotic analysis of Sobolev exponents

In this section, we give an asymptotic analysis of the Sobolev exponent $s_\infty(\Phi_{n;m,\ell})$ of pseudo-Butterworth refinable function $\Phi_{n;m,\ell}$ as $m\to\infty$ or as $n\to\infty$. The case $n=1$ of the following theorem is given in [8, theorem 3.6]. We follow the technique in [8] but we include the proof for the sake of completeness.

Theorem 4.1 *Let $\Phi_{n;m,\ell}$ be the pseudo-Butterworth refinable function with order $(n;m,\ell)$. For $0\le\lambda\le 1$, let $\ell=\lfloor\lambda m\rfloor$, the largest integer smaller than or equal to λm. Then*

$$\lim_{m\to\infty}\frac{s_\infty(\Phi_{n;m,\ell})}{m}=\mu_{n,\lambda}:=\log_2\left(\frac{1+3^n}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3^n}\right)^{\lambda}.$$

This implies that $\hat{\Phi}_{n;m,\ell}$ has the optimal decay $|\hat{\Phi}_{n;m,\ell}(w)|\le C(1+|w|)^{-\mu_{n,\lambda}m}$.

Proof. We first note that, for $0<x\le y\le 1$,

$$\Lambda^{-\ell}(x)P_{n;m,\ell}(x)\ge\Lambda^{-\ell}(y)P_{n;m,\ell}(y). \tag{4.1}$$

In fact, the fact $\Lambda(x)\le\Lambda(y)$ for $0<x\le y\le 1$ and Lemma 2.2 (2) imply that, for $0<x\le y\le 1$,

$$\begin{aligned}\Lambda^{-\ell}(x)P_{n;m,\ell}(x)&=\sum_{j=0}^{\ell}\binom{m-1+j}{j}\Lambda^{j-\ell}(x)\\&\ge\sum_{j=0}^{\ell}\binom{m-1+j}{j}\Lambda^{j-\ell}(y)=\Lambda^{-\ell}(y)P_{n;m,\ell}(y).\end{aligned}$$

The key step to compute the asymptotic rate of $s_\infty(\Phi_{n;m,\ell})$ is to estimate the upper and lower bounds of $P_{n;m,\ell}\left(\frac{3}{4}\right)$ in terms of n,m and ℓ. Take $x=\frac{3}{4}$ and $y=1$ in (4.1) and note that $\Lambda(1)=1$. We have

$$P_{n;m,\ell}\left(\frac{3}{4}\right)\ge\Lambda^{\ell}\left(\frac{3}{4}\right)P_{n;m,\ell}(1)=\Lambda^{\ell}\left(\frac{3}{4}\right)\binom{m+\ell}{\ell}. \tag{4.2}$$

With $x=\frac{1}{2}$ and $y=\frac{3}{4}$ in (4.1), we have

$$P_{n;m,\ell}\left(\frac{3}{4}\right)\le\Lambda^{-\ell}\left(\frac{1}{2}\right)\Lambda^{\ell}\left(\frac{3}{4}\right)P_{n;m,\ell}\left(\frac{1}{2}\right).$$

Since $\Lambda\left(\frac{1}{2}\right)=\frac{1}{2}$ and

$$P_{n;m,\ell}\left(\frac{1}{2}\right)=\sum_{j=0}^{\ell}\binom{m+\ell}{j}\Lambda^{j}\left(\frac{1}{2}\right)\Lambda^{\ell-j}\left(\frac{1}{2}\right)=2^{-\ell}\sum_{j=0}^{\ell}\binom{m+\ell}{j},$$

we obtain

$$P_{n;m,\ell}\left(\frac{3}{4}\right)\le\Lambda^{\ell}\left(\frac{3}{4}\right)\sum_{j=0}^{\ell}\binom{m+\ell}{j}\le m\Lambda^{\ell}\left(\frac{3}{4}\right)\binom{m+\ell}{\ell}, \tag{4.3}$$

since for $\ell \le m-1$

$$\sum_{j=0}^{\ell} \binom{m+\ell}{j} \le m \binom{m+\ell}{\ell}.$$

Putting (4.2) and (4.3) together, we obtain the following estimates of $P_{n;m,\ell}\left(\frac{3}{4}\right)$:

$$\Lambda^{\ell}\left(\frac{3}{4}\right)\binom{m+\ell}{\ell} \le P_{n;m,\ell}\left(\frac{3}{4}\right) \le m\Lambda^{\ell}\left(\frac{3}{4}\right)\binom{m+\ell}{\ell}. \tag{4.4}$$

Next, we will use this estimate to compute the limit in the theorem. The upper bound of $P_{n;m,\ell}\left(\frac{3}{4}\right)$ in (4.4) implies

$$\log_2(1+3^n)^m - \log_2 P_{n;m,\ell}\left(\frac{3}{4}\right) \ge \log_2(1+3^n)^m - \log_2\left(m\Lambda^{\ell}\left(\frac{3}{4}\right)\binom{m+\ell}{\ell}\right).$$

Recall the Stirling's formula, i.e., $m! \sim \sqrt{2\pi}e^{(m+\frac{1}{2})\log m - m}$ [9], where $a_m \sim b_m$ meaning $\frac{a_m}{b_m} \to 1$ as $m \to \infty$, or

$$\log m! \sim \log \sqrt{2\pi}e^{(m+\frac{1}{2})\log m - m} \sim m\log m - m. \tag{4.5}$$

Applying (4.5), we obtain

$$\begin{aligned}\log\binom{m+\ell}{\ell} &= \log(m+\ell)! - \log m! - \log \ell! \\ &\sim (m+\ell)\log(m+\ell) - (m+\ell) - (m\log m - m) - (\ell\log\ell - \ell) \\ &\sim (m+\ell)\log(m+\ell) - m\log m - \ell\log\ell,\end{aligned}$$

or

$$\log_2\binom{m+\ell}{\ell} \sim (m+\ell)\log_2(m+\ell) - m\log_2 m - \ell\log_2\ell.$$

Therefore, we have

$$\begin{aligned}&\log_2(1+3^n)^m - \log_2\left(m\Lambda^{\ell}\left(\frac{3}{4}\right)\binom{m+\ell}{\ell}\right) \qquad (4.6)\\ &= m\log_2(1+3^n) - \log_2 m - \ell\log_2\left(\frac{3^n}{1+3^n}\right) - \log_2\binom{m+\ell}{\ell} \\ &\sim m\left(\log_2(1+3^n) - \frac{\ell}{m}\log_2\left(\frac{3^n}{1+3^n}\right) - \left(1+\frac{\ell}{m}\right)\log_2(m+\ell) + \log_2 m + \frac{\ell}{m}\log_2\ell\right) \\ &\sim m\left(\log_2(1+3^n) - \lambda\log_2\left(\frac{3^n}{1+3^n}\right) - \log_2(1+\lambda) - \lambda\log_2\left(\frac{1}{\lambda}+1\right)\right) \\ &= m\log_2\left(\frac{1+3^n}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3^n}\right)^{\lambda},\end{aligned}$$

since $\frac{\ell}{m} \sim \lambda$ as $m \to \infty$. From (4.4) and (4.6), we have

$$\begin{aligned}\liminf_{m\to\infty}\frac{\log_2(1+3^n)^m - \log_2 P_{n;m,\ell}\left(\frac{3}{4}\right)}{m} &\ge \lim_{m\to\infty}\frac{\log_2(1+3^n)^m - \log_2\left(m\Lambda^{\ell}\left(\frac{3}{4}\right)\binom{m+\ell}{\ell}\right)}{m} \qquad (4.7)\\ &= \log_2\left(\frac{1+3^n}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3^n}\right)^{\lambda}.\end{aligned}$$

By a similar argument as above we note

$$\begin{aligned}
&\log_2(1+3^n)^m - \log_2\left(\Lambda^\ell\left(\frac{3}{4}\right)\binom{m+\ell}{\ell}\right) \qquad (4.8)\\
&\sim m\left(\log_2(1+3^n) - \frac{\ell}{m}\log_2\left(\frac{3^n}{1+3^n}\right) - \left(1+\frac{\ell}{m}\right)\log_2(m+\ell) + \log_2 m + \frac{\ell}{m}\log_2 \ell\right)\\
&\sim m\log_2\left(\frac{1+3^n}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3^n}\right)^{\lambda}.
\end{aligned}$$

From (4.4) and (4.8), we have

$$\begin{aligned}
\limsup_{m\to\infty}\frac{\log_2(1+3^n)^m - \log_2 P_{n;m,\ell}\left(\frac{3}{4}\right)}{m} &\le \lim_{m\to\infty}\frac{\log_2(1+3^n)^m - \log_2\left(\Lambda^\ell\left(\frac{3}{4}\right)\binom{m+\ell}{\ell}\right)}{m} \qquad (4.9)\\
&= \log_2\left(\frac{1+3^n}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3^n}\right)^{\lambda}.
\end{aligned}$$

Combining (4.7) and (4.9), we conclude that

$$\lim_{m\to\infty}\frac{\log_2(1+3^n)^m - \log_2 P_{n;m,\ell}\left(\frac{3}{4}\right)}{m} = \log_2\left(\frac{1+3^n}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3^n}\right)^{\lambda}.$$

This complete the proof. □

We can expect that the asymptotic behavior of the Sobolev exponent $s_\infty(\Phi_{n;m,\ell})$ as $m\to\infty$ for fixed n and ℓ must be the same as in Theorem 4.1 with $\lambda = 0$ and can prove that it is true in the next theorem. The proof is a modification of that of Theorem 4.1 and is omitted.

Theorem 4.2 *Let $\Phi_{n;m,\ell}$ be the pseudo-Butterworth refinable function with order $(n;m,\ell)$. Fix $\ell\in\mathbb{N}$. Then*

$$\lim_{m\to\infty}\frac{s_\infty(\Phi_{n;m,\ell})}{m} = \mu_n := \log_2(1+3^n).$$

This implies that $\hat{\Phi}_{n;m,\ell}$ has the optimal decay $|\hat{\Phi}_{n;m,\ell}(w)| \le C(1+|w|)^{-\mu_n m}$.

The following theorem gives the asymptotic analysis of the Sobolev exponent $s_\infty(\Phi_{n;m,\ell})$ as $n\to\infty$ for fixed m and ℓ. We remark that the limit does not depend on ℓ.

Theorem 4.3 *Let $\Phi_{n;m,\ell}$ be the pseudo-Butterworth refinable function with order $(n;m,\ell)$. Then for fixed m and ℓ,*

$$\lim_{n\to\infty}\frac{s_\infty(\Phi_{n;m,\ell})}{n} = \mu_m := m\log_2 3.$$

This implies that $\hat{\Phi}_{n;m,\ell}$ has the optimal decay $|\hat{\Phi}_{n;m,\ell}(w)| \le C(1+|w|)^{-\mu_m n}$.

Proof. From the proof of Theorem 4.1, we have

$$\log_2(1+3^n)^m - \log_2\left(m\Lambda^\ell\left(\frac{3}{4}\right)\binom{m+\ell}{\ell}\right) \leq s_\infty(\Phi_{n;m,\ell}) \leq \log_2(1+3^n)^m - \log_2\left(\Lambda^\ell\left(\frac{3}{4}\right)\binom{m+\ell}{\ell}\right).$$

Since

$$\frac{1}{n}\log_2(1+3^n)^m = \frac{m}{n}\log_2(1+3^n) \to m\log_2 3 \quad \text{as} \quad n\to\infty$$

and

$$\frac{1}{n}\log_2\left(\Lambda^\ell\left(\frac{3}{4}\right)\binom{m+\ell}{\ell}\right) = \frac{\ell}{n}\log_2\left(\frac{3^n}{1+3^n}\right) + \frac{1}{n}\log_2\binom{m+\ell}{\ell}$$
$$= \ell\log_2 3 - \frac{\ell}{n}\log_2(1+3^n) + \frac{1}{n}\log_2\binom{m+\ell}{\ell} \to 0 \quad \text{as} \quad n\to\infty,$$

we obtain

$$\limsup_{n\to\infty}\frac{s_\infty(\Phi_{n;m,\ell})}{n} \leq m\log_2 3.$$

We also obtain

$$\liminf_{n\to\infty}\frac{s_\infty(\Phi_{n;m,\ell})}{n} \geq m\log_2 3,$$

since

$$\frac{1}{n}\log_2\left(m\Lambda^\ell\left(\frac{3}{4}\right)\binom{m+\ell}{\ell}\right) = \frac{1}{n}\log_2 m + \frac{1}{n}\log_2\left(\Lambda^\ell\left(\frac{3}{4}\right)\binom{m+\ell}{\ell}\right)$$
$$\to 0 \quad \text{as} \quad n\to\infty.$$

This completes the proof. □

5 Wavelet construction and approximation order

For a large class of refinable functions φ with mask $m(w)$, we can associate a wavelet ψ, via

$$\hat{\psi}(w) = e^{-iw/2}\overline{m(w/2+\pi)}\hat{\phi}(w/2),$$

for which $\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k), j,k \in \mathbb{Z}$ form a Riesz basis for $L^2(\mathbb{R})$. In this section, we can also associate the pseudo-Butterworth wavelet to the pseudo-Butterworth refinable function (Theorem 5.3) and compute their approximation order (Theorem 5.5).

We first quote the following proposition which gives a sufficient condition for the associated wavelet ψ to generate a Riesz basis for $L^2(\mathbb{R})$.

Proposition 5.1 [10, Corollarty 3.3] *Let m be a refinement mask of a refinable function $\varphi \in L^2(\mathbb{R})$ with $m(0) = 1$ and $m(\pi) = 0$, so that m can be factorized into the form*

$$|m(w)| = \left| \left(\frac{1 + e^{-iw}}{2} \right)^n \mathcal{L}(w) \right| = \cos^n(w/2) |\mathcal{L}(w)|, \qquad w \in [-\pi, \pi], \tag{5.1}$$

where $\mathcal{L}$ has the Fourier series expansion of polynomial decay satisfying $\mathcal{L}(\pi) \neq 0$. Suppose that

$$|m(w)|^2 + |m(w + \pi)|^2 \neq 0, \qquad w \in [-\pi, \pi].$$

Define $\psi \in L^2(\mathbb{R})$ via

$$\hat{\psi}(2w) := e^{-iw} \overline{m(w + \pi)} \hat{\varphi}(w)$$

and let

$$\tilde{\mathcal{L}}(w) := \frac{\mathcal{L}(w)}{|m(w)|^2 + |m(w + \pi)|^2}.$$

Assume that

$$\|\mathcal{L}(w)\|_{L^\infty(\mathbb{R})} < 2^{n - \frac{1}{2}} \quad \text{and} \quad \|\tilde{\mathcal{L}}(w)\|_{L^\infty(\mathbb{R})} < 2^{n - \frac{1}{2}}.$$

Then ψ is a Riesz wavelet for $L^2(\mathbb{R})$, i.e., $\psi_{j,k}, j, k \in \mathbb{Z}$ generate a Riesz basis for $L^2(\mathbb{R})$.

Proof. A special case of [10, Corollary 3.3]. □

As we will show, the key step in the application of the above proposition is to estimate the upper bounds of $|\mathcal{L}(w)|$ and $|\tilde{\mathcal{L}}(w)|$. Recall that the refinement mask of pseudo-Butterworth refinable function $\Phi_{n;m,\ell}$ is, for $w \in [-\pi, \pi]$,

$$M_{n;m,\ell}(w) := \frac{\cos^{2nm}(w/2) \left(\sum_{j=0}^{\ell} \binom{m+\ell}{j} \sin^{2nj}(w/2) \cos^{2n(\ell-j)}(w/2) \right)}{(\cos^{2n}(w/2) + \sin^{2n}(w/2))^{m+\ell}}.$$

Hence, the corresponding $\mathcal{L}$ function in (5.1) for the pseudo-Butterworth refinable function $\Phi_{n;m,\ell}$ is

$$\begin{aligned} \mathcal{L}(w) : &= \frac{\sum_{j=0}^{\ell} \binom{m+\ell}{j} \sin^{2nj}(w/2) \cos^{2n(\ell-j)}(w/2)}{(\cos^{2n}(w/2) + \sin^{2n}(w/2))^{m+\ell}} \\ &= (L_n(y))^m P_{n;m,\ell}(y), \end{aligned}$$

with $y = \sin^2(w/2)$ and $L_n(y) = 1/\left((1-y)^n + y^n\right)$. We note the corresponding $\tilde{\mathcal{L}}(w)$ has the form

$$\tilde{\mathcal{L}}(w) = \frac{(L_n(y))^m P_{n;m,\ell}(y)}{\tilde{M}_{n;m,\ell}^2(y) + \tilde{M}_{n;m,\ell}^2(1-y)}. \tag{5.2}$$

Now we have following estimate of $\|\tilde{\mathcal{L}}\|_{L^\infty(\mathbb{R})}$:

Proposition 5.2 *Let n, m be positive integers and let ℓ be a nonnegative integer $\leq m - 1$ and $\tilde{\mathcal{L}}$ be defined in* (5.2)*. Then*

$$\|\tilde{\mathcal{L}}\|_{L^\infty(\mathbb{R})} = \sup_{y \in [0,1]} \frac{(L_n(y))^m P_{n;m,\ell}(y)}{\tilde{M}_{n;m,\ell}^2(y) + \tilde{M}_{n;m,\ell}^2(1-y)} \leq 2^{nm+m-1}.$$

Proof. Note from Lemma 2.2 (2) that

$$P_{n;m,\ell}(y) = \sum_{j=0}^{\ell} \binom{m+\ell}{j} \Lambda^j(y)\Lambda^{\ell-j}(1-y) = \sum_{j=0}^{\ell} \binom{m-1+j}{j} \Lambda^j(y), \quad y \in [0,1];$$

hence $P_{n;m,\ell}(y)$ attains its maximum value $P_{n;m,\ell}(1) = \binom{m+\ell}{\ell}$ on $[0,1]$ at $y = 1$. By (4) and (5) of Lemma 2.2, we obtain

$$\begin{aligned} \|\tilde{\mathcal{L}}\|_{L^\infty(\mathbb{R})} &= \sup_{y\in[0,1]} (L_n(y))^m \frac{P_{n;m,\ell}(y)}{\tilde{M}^2_{n;m,\ell}(y) + \tilde{M}^2_{n;m,\ell}(1-y)} \\ &\le \binom{m+\ell}{\ell} \max_{y\in[0,1]} (L_n(y))^m \max_{y\in[0,1]} \frac{1}{\tilde{M}^2_{n;m,\ell}(y) + \tilde{M}^2_{n;m,\ell}(1-y)} \\ &= 2^{nm+m+2\ell-1} \binom{m+\ell}{\ell} \Big/ \left(\sum_{j=0}^{\ell} \binom{m+\ell}{j} \right)^2 \le 2^{nm+m-1}. \end{aligned}$$

The last inequality follows from Lemma 2.1 (3). This completes the proof. □

Theorem 5.3 *Let $\Phi_{n;m,\ell}$ be the pseudo-Butterworth refinable function with order $(n;m,\ell)$. The refinement mask $M_{n;m,\ell}$ is given in* (1.2)*. Define $\Psi_{n;m,\ell} \in L^2(\mathbb{R})$ via*

$$\hat{\Psi}_{n;m,\ell}(2w) := e^{-iw}\overline{M_{n;m,\ell}(w+\pi)}\hat{\Phi}(w).$$

Then Ψ is a Riesz wavelet for $L^2(\mathbb{R})$.

Proof. To apply Theorem 5.1, we first note that

$$|M_{n;m,\ell}(w)|^2 + |M_{n;m,\ell}(w+\pi)|^2 = \tilde{M}^2_{n;m,\ell}(y) + \tilde{M}^2_{n;m,\ell}(1-y) \neq 0$$

for all $w \in [\pi,\pi]$ with $y = \sin^2(w/2)$. Next, we need to check both

$$\|\mathcal{L}\|_{L^\infty(\mathbb{R})} < 2^{2nm-\frac{1}{2}} \quad \text{and} \quad \|\tilde{\mathcal{L}}\|_{L^\infty(\mathbb{R})} < 2^{2nm-\frac{1}{2}}. \tag{5.3}$$

The second inequality in (5.3) follows from Proposition 5.2. For the first inequality in (5.3), we note that

$$|M_{n;m,\ell}(w)|^2 + |M_{n;m,\ell}(w+\pi)|^2 \le 1,$$

for all $w \in \mathbb{R}$. Hence $|\mathcal{L}(w)| \le |\tilde{\mathcal{L}}(w)|$ for all $w \in \mathbb{R}$. This concludes the proof. □

Next, we consider two projection operators $\mathcal{P}_j$ and $\mathcal{Q}_j$ from $L^2(\mathbb{R})$ onto $V_j = \overline{\text{span}}\{\Phi_{j,k} : k \in \mathbb{Z}\} = \overline{\text{span}}\{\Psi_{\ell,k} : \ell < j, k \in \mathbb{Z}\}$ via

$$\mathcal{P}_j f = \sum_{k\in\mathbb{Z}} < f, \Phi_{j,k} > \Phi_{j,k}, \tag{5.4}$$

and

$$\mathcal{Q}_j f = \sum_{\ell<j,k\in\mathbb{Z}} < f, \Psi_{\ell,k} > \Psi_{\ell,k},$$

where $\Phi = \Phi_{n;m,\ell}$ and $\Psi = \Psi_{n;m,\ell}$. It is known that $\mathcal{P}_j = \mathcal{Q}_j$ (See [5, Lemma 2.4]).

We say that $\mathcal{P}_j$ provides approximation order μ, if for all f in the Sobolev space $W_2^{\mu}(\mathbb{R})$,

$$\|f - \mathcal{P}_j f\|_{L^2(\mathbb{R})} = O(2^{-j\mu}).$$

The following proposition is a special case of [5, Theorem 2.8] adapted for our purpose.

Proposition 5.4 *Let $\Phi_{n;m,\ell}$ be a pseudo-Butterworth refinable function with order $(n; m, \ell)$ and $M_{n;m,\ell}$ be its refinement mask. Let $\mathcal{P}_j$ be the operator as defined in (5.4) with $\Phi_{n;m,\ell}$ as the underlying refinable function. Then the approximation order of the operator $\mathcal{P}_j$ is $\min\{2nm, m_1\}$, with m_1 the order of the zero of $1 - |M_{n;m,\ell}|^2$ at the origin.*

The following theorem shows that the pseudo-Butterworth refinable function $\Phi_{n;m,\ell}$ provides aproximation order $2n(\ell + 1)$ which is independent of m.

Theorem 5.5 *Let n, m be positive integers and let ℓ be a nonnegative integer $\leq m - 1$. Let $\Phi_{n;m,\ell}$ be the pseudo-Butterworth refinable function with order $(n; m, \ell)$. Then the corresponding operator $\mathcal{P}_j$ provides approximation order $2n(\ell + 1)$.*

Proof. We compute the order of zeros of $1 - |M_{n;m,\ell}|^2$ at the origin. We rewrite $1 - |M_{n;m,\ell}|^2$ as

$$1 - |M_{n;m,\ell}|^2 = 1 - \tilde{M}^2_{n;m,\ell}(y),$$

where $\tilde{M}_{n;m,\ell}(y)$ was defined in (2.2). It is obvious that $1 - \tilde{M}^2_{n;m,\ell}(y) = 0$ for $y = 0$. Recall that the derivative of $\tilde{M}_{n;m,\ell}(y)$ is given by Lemma 2.2 (3):

$$\tilde{M}'_{n;m,\ell}(y) = -(m + \ell)\binom{m + \ell - 1}{\ell}\Lambda^{\ell}(y)\Lambda^{m-1}(1 - y)\Lambda'(y). \tag{5.5}$$

Applying (5.5) to take the first derivative $1 - \tilde{M}^2_{n;m,\ell}(y)$ with respect to y, we obtain

$$\begin{aligned}(1 - \tilde{M}^2_{n;m,\ell}(y))' &= -2\tilde{M}_{n;m,\ell}(y)\tilde{M}'_{n;m,\ell}(y)\\ &= 2\tilde{M}_{n;m,\ell}(y)\left((m + \ell)\binom{m + \ell - 1}{\ell}\Lambda^{\ell}(y)\Lambda^{m-1}(1 - y)\Lambda'(y)\right).\end{aligned}$$

Since both $\tilde{M}_{n;m,\ell}(y)$ and $\Lambda^{m-1}(1 - y)$ are equal to 1 when $y = 0$, and since $\Lambda^{\ell}(y)$ has zero of order $n\ell$ at $y = 0$, and $\Lambda'(y)$ has zero of order $n - 1$ at $y = 0$, we conclude that

$$1 - |M_{n;m,\ell}(w)|^2 = 1 - \tilde{M}^2_{n;m,\ell}(y) = O(|y|^{n(\ell+1)}) = O(|w|^{2n(\ell+1)}).$$

where $y = \sin^2(w/2)$. Theorem 5.4 shows that the approximation order of $\mathcal{P}_j$ with $\Phi_{n;m,\ell}$ as the underlying refinable function is $\min\{2nm, 2n(\ell + 1)\} = 2n(\ell + 1)$ for $0 \leq \ell \leq m - 1$. This completes the proof. $\square$

6 Asymptotic behavior of $\Phi_{n,m,\ell}$

For $\ell = m-1$, $\Phi_{n;m,m-1}$ is an interpolatory refinable function, being the D-D interpolatory refinable function of Deslauriers and Dubuc when $n = 1$ [6, 14]. In this section, we show that $\Phi_{n;m,m-1}$ converges in $L^q(\mathbb{R})(2 \le q \le \infty)$, uniformly in particular, to the Shannon refinable function φ_{SH} as m tends to the infinity, where

$$\hat{\varphi}_{SH}(w) := \chi_{[-\pi,\pi]}(w).$$

In particular, the case $n = 1$ shows that the D-D interpolatory refinable function of order m tends to the Shannon refinable function φ_{SH} in $L^q(\mathbb{R})(2 \le q \le \infty)$ as $m \to \infty$. We also show that for fixed m and ℓ the pseudo-Butterworth refinable function $\Phi_{n;m,\ell}$ converges in $L^q(\mathbb{R})(2 \le q \le \infty)$, uniformly in particular, to the Shannon refinable function φ_{SH} as n tends to the infinity. The idea of the proof also appears in [2, 12] for the asymptotic behaviors of the corresponding family of refinable functions.

Lemma 6.1 *For $m \in \mathbb{N}$,*

(1) $M_{n;m+1,m}(w) \ge M_{n;m,m-1}(w)$ *for* $w \in [-\pi/2, \pi/2]$.

(2) $M_{n;m+1,m}(w) \le M_{n;m,m-1}(w)$ *for* $w \in [-\pi, -\pi/2] \cup [\pi/2, \pi]$.

Proof. (1) follows from the following with $0 \le y \le 1/2$:

$$\begin{aligned}
\tilde{M}_{n;m+1,m}(y) &= \Lambda^{m+1}(1-y)P_{n;m+1,m}(y)\\
&= \Lambda^{m+1}(1-y)\left(\sum_{j=0}^{m}\binom{m+j}{j}\Lambda^j(y)\right)\\
&= \Lambda^{m}(1-y)\left(\sum_{j=0}^{m}\binom{m+j}{j}\Lambda^j(y) - \sum_{j=1}^{m+1}\binom{m-1+j}{j-1}\Lambda^j(y)\right)\\
&= \Lambda^{m}(1-y)\left(1+\sum_{j=1}^{m}\left(\binom{m+j}{j} - \binom{m-1+j}{j-1}\right)\Lambda^j(y) - \binom{2m}{m}\Lambda^{m+1}(y)\right)\\
&= \Lambda^{m}(1-y)\left(\sum_{j=0}^{m}\binom{m-1+j}{j}\Lambda^j(y) - \binom{2m}{m}\Lambda^{m+1}(y)\right)\\
&= \Lambda^{m}(1-y)\left(\sum_{j=0}^{m-1}\binom{m-1+j}{j}\Lambda^j(y) + \binom{2m-1}{m}\Lambda^{m}(y) - \binom{2m}{m}\Lambda^{m+1}(y)\right)\\
&\ge \Lambda^{m}(1-y)\left(\sum_{j=0}^{m-1}\binom{m-1+j}{j}\Lambda^j(y)\right)\\
&= \tilde{M}_{n;m,m-1}(y).
\end{aligned}$$

(2) follows from (1), since $\tilde{M}_{n;m,m-1}(y) = 1 - \tilde{M}_{n;m,m-1}(1-y)$. $\square$

For given positive integers n, m and nonnegative integer $\ell \le m-1$, we easily check that the mask $M_{n;m,\ell}(w)$ converges pointwise to the mask

$$m_{SH}(w) = \begin{cases} 1, & |w| < \pi/2, \\ 0, & \pi/2 < |w| < \pi \end{cases}$$

of the Shannon refinable function φ_{SH} as $n \to \infty$.

Lemma 6.2 *(1) $M_{n;m,m-1} \to m_{SH}$ as $m \to \infty$.*

(2) $M_{n;m,\ell}(w) \to m_{SH}$ as $n \to \infty$.

Proof. (1) follows from the following with $1/2 < y \le 1$:

$$\begin{aligned} \tilde{M}_{n;m,m-1}(y) &= \Lambda^m(1-y) \sum_{j=0}^{m-1} \binom{m-1+j}{j} \Lambda^j(y) \le \Lambda^m(1-y) \sum_{j=0}^{m-1} 2^{m-1+j} \Lambda^j(y) \\ &= \Lambda^m(1-y) 2^{m-1} \frac{(2\Lambda(y))^m - 1}{2\Lambda(y) - 1} \le \frac{\{\Lambda(1-y) 2 (2\Lambda(y))\}^m}{2(2\Lambda(y)-1)} \\ &= \frac{(1-(2\Lambda(y)-1)^2)^m}{2(2\Lambda(y)-1)} \to 0 \end{aligned}$$

as $m \to \infty$. For $0 \le y < 1/2$, $\tilde{M}_{n;m,m-1}(y) = 1 - \tilde{M}_{n;m,m-1}(1-y) \to 1$ as $m \to \infty$.
(2) is trivial. $\square$

Before the statement and proof of the main result, we define an auxiliary mask

$$m_{aux}(w) = \begin{cases} 1, & |w| \le \frac{\pi}{2}, \\ \cos^4(w/2)\left(2 + 4\sin^2(w/2)\right), & \frac{\pi}{2} \le |w| \le \pi, \end{cases}$$

for the domination of $M_{n;m,\ell}(w)$. We need some technical lemmas.

Lemma 6.3 *(1) $M_{n;m,m-1}(w) \le m_{aux}(w),\ m \ge 2$.*

(2) $M_{n;m,\ell}(w) \le m_{aux}(w),\ m \ge 2$.

(3) $\hat{\varphi}_{aux}(w) := \prod_{j\in\mathbb{N}} m_{aux}(w/2^j)$ has the decay $|\hat{\varphi}_{aux}(w)| \le C(1+|w|)^{-3+\log_2 3}$.

(4) $|M_{n;m,m-1}(w) - 1| \le \begin{cases} 1, & \text{all } w, \\ \frac{2}{\pi}|w|, & |w| \le \pi/2. \end{cases}$

(5) $|M_{m;n,\ell}(w) - 1| \le \begin{cases} 1, & \text{all } w, \\ \frac{2^{m+1}}{\pi}|w|, & |w| \le \pi/2. \end{cases}$

Proof. (1): For $|w| \leq \frac{\pi}{2}$, we have

$$M_{n;m,m-1}(w) \leq M_{n;m,m-1}(w) + M_{n;m,m-1}(w+\pi) = 1.$$

For $\pi/2 \leq |w| \leq \pi$ and $m \geq 2$, we have $M_{n;m,m-1}(w) \leq M_{n;2,1}(w)$ by Lemma 6.1 (2) and

$$M_{n;2,1}(w) = \Lambda^2(1-y)(1+2\Lambda(y)).$$

Since $\Lambda(y) \leq 1$ and for $\pi/2 \leq |w| \leq \pi$

$$\begin{aligned} \Lambda(1-y) &= \frac{1}{(\tan^2(w/2))^n + 1} \\ &\leq \frac{1}{(\tan^2(w/2)) + 1} = \cos^2(w/2), \end{aligned}$$

we have

$$\begin{aligned} M_{n;2,1}(w) &\leq 3\cos^4(w/2) \\ &\leq \cos^4(w/2)\left(2 + 4\sin^2(w/2)\right) \\ &= m_{aux}(w). \end{aligned}$$

(2) follows from (1) and Lemma 2.2 (2).
(3): We note that $m_{aux}(w) = \cos^4(w/2)\mathcal{L}_{aux}(w)$, where

$$\mathcal{L}_{aux}(w) = \begin{cases} \dfrac{1}{\cos^4(w/2)}, & |w| \leq \frac{\pi}{2}, \\ 2 + 4\sin^2(w/2), & \frac{\pi}{2} \leq |w| \leq \pi, \end{cases}$$

and note that $\sup_w |\mathcal{L}_{aux}(w)| = 6$. Therefore, the decay of $\hat{\varphi}_{aux}(w)$ follows, for example, from [1, Theorem 5.5].

(4): For $w \in \mathbb{R}$, we have $|M_{n;m,m-1}(w) - 1| = |M_{n;m,m-1}(w+\pi)| \le 1$. For $|w| \le \pi/2$, we have

$$\begin{aligned}
|M_{n;m,m-1}(w) - 1| = |M_{n;m,m-1}(w+\pi)| &= \Lambda(y)\left(\sum_{j=0}^{m-1}\binom{2m-1}{j}\Lambda^j(1-y)\Lambda^{2m-2-j}(y)\right)\\
&= \Lambda(y)\left(\sum_{j=m}^{2m-1}\binom{2m-1}{j}\Lambda^{2m-1-j}(1-y)\Lambda^{j-1}(y)\right)\\
&= \Lambda(y)\left(\sum_{j=m-1}^{2m-2}\binom{2m-1}{j+1}\Lambda^{2m-2-j}(1-y)\Lambda^{j}(y)\right)\\
&\le \Lambda(y)\left(\sum_{j=m-1}^{2m-2}\binom{2m-1}{j}\Lambda^{2m-2-j}(1-y)\Lambda^{j}(y)\right)\\
&= \left(\frac{y}{1-y}\right)^n\Lambda(1-y)\left(\sum_{j=m-1}^{2m-2}\binom{2m-1}{j}\Lambda^{2m-2-j}(1-y)\Lambda^{j}(y)\right)\\
&= \tan^{2n}(w/2)\left(\sum_{j=m-1}^{2m-2}\binom{2m-1}{j}\Lambda^{2m-1-j}(1-y)\Lambda^{j}(y)\right)\\
&\le \tan^{2n}(w/2)\left(\Lambda(1-y)+\Lambda(y)\right)^{2m-1}\\
&= \tan^{2n}(w/2) \le |\tan(w/2)| \le \frac{2}{\pi}|w|,
\end{aligned}$$

where we used the inequality $\binom{2m-1}{j+1} \le \binom{2m-1}{j}$ for $m-1 \le j \le 2m-2$.
(5): Note that $M_{n;m,0}(w) \le M_{n;m,\ell}(w) \le 1$ by Lemma 2.2 (2). For $w \in \mathbb{R}$, we have

$$|M_{n;m,\ell}(w) - 1| \le |M_{n;m,0}(w) - 1| \le 1,$$

furthermore for $|w| \le \pi/2$, we have

$$\begin{aligned}
|M_{n;m,\ell}(w) - 1| \le |M_{n;m,0}(w) - 1| &= 1 - \Lambda^m(1-y)\\
&= \left(((1-y)^n + y^n)^m - (1-y)^{nm}\right)/((1-y)^n + y^n)^m\\
&= \left(\sum_{j=1}^{m}\binom{m}{j}y^{nj}(1-y)^{n(m-j)}\right)\Big/((1-y)^n + y^n)^m\\
&\le \left(\sum_{j=1}^{m}\binom{m}{j}y^{nj}(1-y)^{n(m-j)}\right)\Big/(1-y)^{nm}\\
&= \sum_{j=1}^{m}\binom{m}{j}\left(\frac{y}{1-y}\right)^{nj} \le \left(\frac{y}{1-y}\right)^n\sum_{j=1}^{m}\binom{m}{j}\\
&\le |\tan(w/2)|\, 2^m \le \frac{2^{m+1}}{\pi}|w|.
\end{aligned}$$

□

Lemma 6.4 (1) *For each fixed w, $\hat{\Phi}_{n;m,m-1}(w) = \prod_{j=1}^{\infty} M_{n;m,m-1}(w/2^j)$ converges uniformly on m.*

(2) *$\hat{\Phi}_{n;m,m-1}(w) \to \hat{\varphi}_{SH}(w)$ pointwise a.e. as $m \to \infty$.*

(3) *For each fixed w, $\hat{\Phi}_{n;m,\ell}(w) = \prod_{j=1}^{\infty} M_{n;m,\ell}(w/2^j)$ converges uniformly on n.*

(4) *$\hat{\Phi}_{n;m,\ell}(w) \to \hat{\varphi}_{SH}(w)$ pointwise a.e. as $n \to \infty$.*

Proof. (1): Fix w and choose j_0 so that $|w/2^{j_0}| \leq \pi/2$. By Lemma 6.3 (4),

$$\sum_{j=1}^{\infty} \left| M_{n;m,m-1}(\frac{w}{2^j}) - 1 \right| = \sum_{j=1}^{j_0} \left| M_{n;m,m-1}(\frac{w}{2^j}) - 1 \right| + \sum_{j=j_0+1}^{\infty} \left| M_{n;m,m-1}(\frac{w}{2^j}) - 1 \right|$$
$$\leq j_0 + \sum_{j=j_0+1}^{\infty} \frac{2}{\pi} \frac{|w|}{2^j} = j_0 + \frac{2}{\pi} \frac{|w|}{2^{j_0}},$$

uniformly on m. Therefore, the product $\hat{\Phi}_{n;m,m-1}(w)$ converges uniformly on m.
(2): Fix $w \notin \cup_{j=1}^{\infty} 2^j(\pm\pi/2 + 2\pi\mathbb{Z})$ and let $\varepsilon > 0$. By (1), we can choose j_1 (independent of m) so that

$$\left| \hat{\Phi}_{n;m,m-1}(w) - \prod_{j=1}^{j_1} M_{n;m,m-1}(\frac{w}{2^j}) \right| < \varepsilon,$$

and

$$\left| \hat{\varphi}_{SH}(w) - \prod_{j=1}^{j_1} m_{SH}(\frac{w}{2^j}) \right| < \varepsilon.$$

Therefore, we have

$$|\hat{\Phi}_{n;m,m-1}(w) - \hat{\varphi}_{SH}(w)| \leq \left| \hat{\Phi}_{n;m,m-1}(w) - \prod_{j=1}^{j_1} M_{n;m,m-1}(\frac{w}{2^j}) \right|$$
$$+ \left| \prod_{j=1}^{j_1} M_{n;m,m-1}(\frac{w}{2^j}) - \prod_{j=1}^{j_1} m_{SH}(\frac{w}{2^j}) \right| + \left| \prod_{j=1}^{j_1} m_{SH}(\frac{w}{2^j}) - \hat{\varphi}_{SH}(w) \right|$$
$$< 2\varepsilon + \left| \prod_{j=1}^{j_1} M_{n;m,m-1}(\frac{w}{2^j}) - \prod_{j=1}^{j_1} m_{SH}(\frac{w}{2^j}) \right|.$$

Note that $w/2^j \notin \pm\pi/2 + 2\pi\mathbb{Z}$ for any $j \geq 1$. Since $M_{n;m,m-1}(w/2^j) \to m_{SH}(w/2^j)$ as $m \to \infty$ for $j = 1, 2, \cdots, j_1$, we can choose $m_0 \in \mathbb{N}$ so that

$$\left| \prod_{j=1}^{j_1} M_{n;m,m-1}(w/2^j) - \prod_{j=1}^{j_1} m_{SH}(w/2^j) \right| < \varepsilon \quad \text{for} \quad m \geq m_0.$$

Therefore, $\hat{\Phi}_{n;m,m-1}(w) \to \hat{\varphi}_{SH}(w)$ pointwise as $n \to \infty$ for $w \notin \cup_{j=1}^{\infty} 2^j(\pm\pi/2 + 2\pi\mathbb{Z})$.
(3) and (4) can also be proved similarly. □

Now, we state and prove our main results in this section.

Theorem 6.5 (1) *For* $1 \le p < \infty$, $||\hat{\Phi}_{n;m,m-1} - \hat{\varphi}_{SH}||_{L^p(\mathbb{R})} \to 0$ *as* $m \to \infty$.

(2) *For* $2 \le q \le \infty$, $||\Phi_{n;m,m-1} - \varphi_{SH}||_{L^q(\mathbb{R})} \to 0$ *as* $m \to \infty$.

(3) *For* $1 \le p < \infty$, $||\hat{\Phi}_{n;m,\ell} - \hat{\varphi}_{SH}||_{L^p(\mathbb{R})} \to 0$ *as* $n \to \infty$.

(4) *For* $2 \le q \le \infty$, $||\Phi_{n;m,\ell} - \varphi_{SH}||_{L^q(\mathbb{R})} \to 0$ *as* $n \to \infty$.
In particular, $\Phi_{n;m,m-1} \to \varphi_{SH}$ *uniformly on* $\mathbb{R}$ *as* $m \to \infty$, *and* $\Phi_{n;m,\ell} \to \varphi_{SH}$ *uniformly on* $\mathbb{R}$ *as* $n \to \infty$.

Proof. We estimate the decay of $\hat{\Phi}_{n;m,m-1}$ for $m \ge 2$:

$$\begin{aligned} |\hat{\Phi}_{n;m,m-1}(w)| &= \prod_{j\in\mathbb{N}} \left|M_{n;m,m-1}(w/2^j)\right| \le \prod_{j\in\mathbb{N}} m_{aux}(w/2^j) \\ &= |\hat{\varphi}_{aux}(w)| \le C(1+|w|)^{-3+\log_2 3} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \end{aligned}$$

where we used Lemma 6.3. Therefore (1) follows from Lemma 6.4 by the Lebesgue dominated convergence theorem. The claim (2) follows from (1) by Hausdorff-Young inequality:

$$||f||_{L^q(\mathbb{R})} \le ||\hat{f}||_{L^p(\mathbb{R})}, \quad \text{for} \quad 1 \le p \le 2,$$

where q is the conjugate exponent to p.
(3) and (4) can also be proved similarly. □

7 Spectral decomposition

For each order $(n;m,\ell)$, the mask $M_{n;m,\ell}$ gives rise to a family of masks $\mu_{n;m,\ell}$, via Riesz factorization,

$$|\mu_{n;m,\ell}(w)|^2 := M_{n;m,\ell}(w).$$

The mask $\mu_{n;m.\ell}$ is a rational trigonometric function of w and defines the corresponding refinable function $\varphi_{n;m,\ell}$, via

$$\hat{\varphi}_{n;m,\ell}(w) := \prod_{j=1}^{\infty} \mu_{n;m,\ell}(w/2^j),$$

which can be regarded as an extension of pseudo-spline of type I ($n = 1$) with order (m,ℓ) [7]. The corresponding analysis of $\varphi_{n;m,\ell}$ can be obtained analogously from that of $\Phi_{n;m,\ell}$ as follows. For example, we have the following whose proofs are similar to the corresponding proof of $\Phi_{n;m,\ell}$ and are omitted.

(1) $s_\infty(\varphi_{n;m,\ell}) = \frac{1}{2} s_\infty(\Phi_{n;m,\ell}) = \frac{m}{2}\log_2(1+3^n) - \frac{1}{2}\log_2 P_{n;m,\ell}\left(\frac{3}{4}\right)$.

(2) The projection operator $\mathcal{P}_j$ from $L^2(\mathbb{R})$ onto $V_j = \overline{\text{span}}\{\varphi_{j,k} : k \in \mathbb{Z}\}$, defined as

$$\mathcal{P}_j f = \sum < f, \varphi_{j,k} > \varphi_{j,k},$$

where $\varphi = \varphi_{n;m,\ell}$ and $\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$, provides approximation order $\min\{nm, 2n(\ell + 1)\}$.

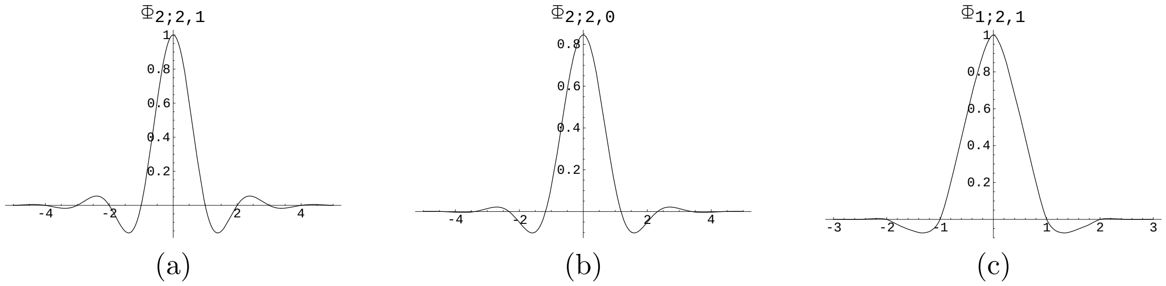

Figure 1: $\Phi_{n;m,\ell}$ (a) Case $n = 2$, $m = 2$ and $\ell = 1$, (b) Case $n = 2$, $m = 2$ and $\ell = 0$, (c) Case $n = 1$, $m = 2$ and $\ell = 1$ (D-D interpolatory refinable function)

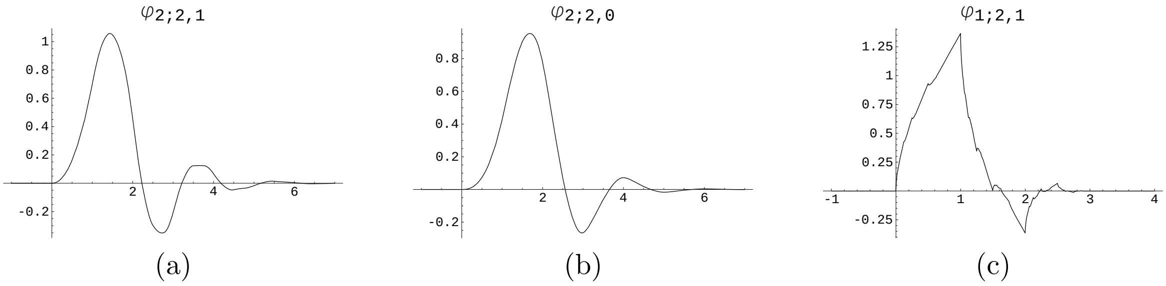

Figure 2: $\varphi_{n;m,\ell}$ (a) Case $n = 2$, $m = 2$ and $\ell = 1$, (b) Case $n = 2$, $m = 2$ and $\ell = 0$, (c) Case $n = 1$, $m = 2$ and $\ell = 1$ (Daubechies D_2 refinable function)

(3) If we associate a wavelet $\psi_{n;m,\ell}$ to the refinable function $\varphi_{n;m,\ell}$ via

$$\hat{\psi}_{n;m,\ell}(2w) := e^{-iw}\overline{\mu_{n;m,\ell}(w+\pi)}\hat{\varphi}_{n;m,\ell}(w),$$

then $\psi_{n;m,\ell}$ generates a Riesz wavelet.

We note that $\mu_{n;m,m-1}$ gives rise to an orthonomal refinable function which reduces to the Daubechies orthonormal refinable function D_m when $n = 1$ [3, 4]. We give some examples $\Phi_{n;m,\ell}$ and $\varphi_{n;m,\ell}$ with their graphs in Figures 1 and 2 and leave a more complete analysis on $\varphi_{n;m,\ell}$'s in another paper.

Example 7.1 *Case $n = 2, m = 2$ and $\ell = 1$. The pseudo-Butterworth mask $M_{2;2,1}(w)$ is*

$$M_{2;2,1}(w) = \frac{\cos^8(w/2)\left(\cos^4(w/2) + 3\sin^4(w/2)\right)}{\left(\cos^4(w/2) + \sin^4(w/2)\right)^3}.$$

We can choose a mask $\mu_{2;2,1}$ with real coefficients from via the Riesz factorization $|\mu_{n;m,\ell}(w)|^2 := M_{n;m,\ell}(w)$. That is,

$$\mu_{2;2,1}(w) := \frac{\left(e^{iw}+1\right)^4\left(e^{2iw} - (z_0+\bar{z}_0)e^{iw} + z_0\bar{z}_0\right)}{\sqrt{2z_0\bar{z}_0}\left((\sqrt{2}+1)e^{2iw} + (\sqrt{2}-1)\right)^3}$$

where $z_0 = \left(\frac{1}{2} - \sqrt[4]{3}\frac{\sqrt{3}-1}{2\sqrt{2}}\right) + i\left(-\frac{\sqrt{3}}{2} + \sqrt[4]{3}\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$. *The corresponding refinable functions are* $\Phi_{2;2,1}$ *(interpolatory) and* $\varphi_{2;2,1}$ *(orthonormal). See Figure 1 (a) and Figure 2 (a).*

Example 7.2 *Case* $n = 2, m = 2$ *and* $\ell = 0$. *The corresponding mask are*

$$M_{2;2,0}(w) = \frac{\cos^8(w/2)}{\left(\cos^4(w/2) + \sin^4(w/2)\right)^2}.$$

and

$$\mu_{2;2,0}(w) := \frac{\left(e^{iw} + 1\right)^4}{2\left((\sqrt{2}+1)e^{2iw} + (\sqrt{2}-1)\right)^2}.$$

See Figure 1 (b) and Figure 2 (b), for the corresponding refinable functions.

Example 7.3 *Case* $n = 1, m = 2$ *and* $\ell = 1$. *The* $M_{1;2,1}(w)$

$$\begin{aligned} M_{1;2,1}(w) &= \cos^4(w/2)\left(\cos^2(w/2) + 3\sin^2(w/2)\right) \\ &= \cos^4(w/2)\left(1 + 2\sin^2(w/2)\right), \end{aligned}$$

is the mask for D-D interpolatory refinable function. The corresponding Riesz factorization $\mu_{1;2,1}(w)$ *is the mask for orthonormal refinable function Daubechies* D_2*:*

$$\mu_{1;2,1}(w) := \frac{1+\sqrt{3}}{8}e^{3iw} + \frac{3+\sqrt{3}}{8}e^{2iw} + \frac{3-\sqrt{3}}{8}e^{iw} + \frac{1-\sqrt{3}}{8}.$$

See Figure 1 (c) and Figure 2 (c).

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