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BK21 Research Report

09 - 10

June 5, 2009

DEPARTMENT OF MATHEMATICAL SCIENCES

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June 4, 2009

Abstract

Let M be the closed, simply connected, 4-manifold with nonnegative sectional curvature, called a *nonnegatively curved* 4*-manifold*, with an effective and isometric $\mathbf{Z}_m$-action for a positive integer $m \geq 61^7$. Assume that $\mathbf{Z}_m$ acts trivially on the homology of M. The goal of this short paper is to prove that if the fixed point set of any nontrivial element of $\mathbf{Z}_m$ has at most one two-dimensional component, then M is homeomorphic to S^4, $\#_{i=1}^l S^2 \times S^2$, $l = 1$, 2, or $\#_{j=1}^k \pm \mathbf{CP}^2$, k=1, 2, 3, 4, 5. The main strategy of this paper is to give an upper bound of the Euler characteristic $\chi(M)$ under the homological assumption of a $\mathbf{Z}_m$-action as above by using the Lefschetz fixed point formula.

1 Introduction and Main Results

In this paper, we study the classification problem of closed simply connected 4-manifolds with nonnegative sectional curvature. The only known examples of closed 4-manifolds with positive sectional curvature are S^4 and $\mathbf{CP}^2$. In

Key words and phrases: nonnegatively curved 4-manifolds, effective finite group action.

2000 Mathematics Subject Classification: primary 53C15.

[9], Hsiang and Kleiner showed that every orientable closed 4-manifold with positive sectional curvature and an isometric S^1-action is homeomorphic to S^4 or $\mathbf{CP}^2$. On the other hand, the only known examples of closed simply connected 4-manifolds with nonnegative sectional curvature are S^4, $\mathbf{CP}^2$, $S^2 \times S^2$, or $\mathbf{CP} \# \pm \mathbf{CP}^2$. Yang and Searl and independently Kleiner showed in [17] and [11] that every closed simply connected nonnegatively curved 4-manifold with an isometric S^1-action is homeomorphic to

$$S^4,\ \mathbf{CP}^2,\ S^2 \times S^2 \text{ or } \mathbf{CP} \# \pm \mathbf{CP}^2.$$

Recently, by using the techniques of Fintushel and Pao in [4, 5] and [14] the first named author showed in [10] how to prove that the homeomorphism classifications by Hsiang–Kleiner and Yang–Searl are indeed diffeomorphism ones. Such a diffeomorphism classfication was made possible essentially due to the recent resolution of the Poincaré conjecture by Perelman (cf. Section 3.2 in [12]). However, it still remains open whether or not they are the only closed orientable 4-manifolds with positive sectional curvature or nonnegative sectional curvature. Clearly, this question is closely related to the well-known conjecture of Hopf.

The aim of this paper is to investigate the topology of closed simply connected nonnegatively curved 4-manifold with an isometric $\mathbf{Z}_m$-action. In fact, Yang has already started this problem in his paper [16] and, among other things, he has shown that the second Betti number b_2 satisfies the bound $0 \leq b_2 \leq 5$ under some additional conditions. To be precise, let C_4 be the Gromov's upper bound for the total Betti number of the nonnegatively curved 4-manifold M. If $m \geq C_4$ or $m \geq 41$ (m prime) and there is a smooth S^1-action on M such that $\mathbf{Z}_m$ acts on M as a subgroup of S^1, then it was shown in [16] that the nonnegatively curved 4-manifold M should be homeomorphic to S^4, $\#_{i=1}^{l} S^2 \times S^2$ ($l = 1, 2$), or $\#_{j=1}^{k} \pm \mathbf{CP}^2$ ($1 \leq k \leq 5$) by using the well-known classification of Freedman in [6]. We like to emphasize that the smooth S^1-action means that the elements typically do not act as isometries.

In a similar context, in this paper we obtain the following result, as follows.

Theorem 1.1. *Let M be a closed simply connected nonnegatively curved manifold of dimension 4 with an isometric $\mathbf{Z}_m$-action for a positive integer $m \geq 61^7$. Assume that the $\mathbf{Z}_m$-action acts trivially on the homology of M and that the fixed point set of any nontrivial element of $\mathbf{Z}_m$ has at most one two-dimensional fixed point component. Then M is homeomorphic to S^4, $\#_{i=1}^{l} S^2 \times S^2$, $1 \leq l \leq 2$, or $\#_{j=1}^{k} \pm \mathbf{CP}^2$, $1 \leq k \leq 5$.*

Here, it is important to note that the $\mathbf{Z}_m$-action may not be extended to a circle action on M. Moreover, according to [8], Gromov constant C_n is $((n+1)J)^{100^n}$, where $J = 2^M$ and $M = 8^n 10^{n^2+4n}$. Thus, compared to the estimation $p \geq C_4$ that is the assumption in Theorem 1.(a) in [16], it seems that our lower bound 61^7 is quite reasonable. In view of these respects, Theorem 1.1 can be considered as a significant improvement of Theorem 1 in [16]. The significance of the lower bound 61^7 in the statement of Theorem 1.1 will be obvious in Proposition 2.1 and the proof of Theorem 1.1 in Section 2.

To describe our main results precisely, we first need to set up the following definitions. A point $x \in M$ is called an *isolated fixed point* under the cyclic group G-action if $G \cdot x = x$ and the stabilizer $G_v = \{1\}$ for any v in the unit sphere on T_xM. On the other hand, a point $x \in M$ is called a *non-isolated fixed point* under the cyclic group G-action if $G \cdot x = x$, $G \cdot v \neq \{v\}$ for any v in the unit sphere S on T_xM, and there exists $v \in S$ such that $G_v \neq \{1\}$.

The main strategy of the proof of Theorem 1.1 is to give an upper bound of the Euler characteristic $\chi(M)$. It then follows of the Lefschetz fixed point formula that under the homological assumption of $\mathbf{Z}_m$-action we have

$$\chi(\mathrm{Fix}(M, \mathbf{Z}_m)) = \chi(M).$$

It can be shown as in Proposition 2.1 (or Proposition 3.3 in [3]) that if m is greater than or equal to 61 then the $\mathbf{Z}_m$ can have at most 5 isolated fixed points. In particular, if the fixed point set is isolated, then $\chi(M)$ is just equal to the number of isolated fixed points which is less than or equal to 5. The well-known homeomorphism classification of Freedman about simply connected topological manifolds of dimension 4 in [6] will then complete the proof of Theorem 1.1.

We organize this paper as follows. Section 2 is devoted to counting the number of isolated fixed points. In the same section, we will also give an upper bound for the Euler characteristic $\chi(M)$ of M by using $\mathbf{Z}_m$-action which acts trivially on the homology of M and whose fixed point set of a nontrivial element has at most one 2-dimensional fixed point component. This upper bound will immediately complete the proof of Theorem 1.1.

2 Proof of Theorem 1.1

The aim of this section is to give a proof of Theorem 1.1. We do so by giving an upper bound for the Euler characteristic $\chi(M)$ of M by using certain $\mathbf{Z}_m$-action which acts trivially on the homology of M.

Throughout this section, we assume that $\mathbf{Z}_m$ acts trivially on the homology of M and that the fixed point set of any non-trivial element of $\mathbf{Z}_m$ has at most one 2-dimensional fixed point component.

In [9], Hsiang and Kleiner showed that any isometric circle action on a positively curved 4-manifold can have at most three isolated fixed points. In 1995, Grove and Markvorsen also considered the above statement in a general dimension (cf. pp. 2 in [7]). Furthermore, Yang counted the number of isolated fixed points of an isometric $\mathbf{Z}_m$-action on nonnegatively curved 4-manifolds under some assumption on the $\mathbf{Z}_m$-action, where m is a prime (see the paper [16] by Yang). Recently Fang proved the same result without any assumption that m is a prime (cf. Proposition 3.3 in [3]). So, we have:

Proposition 2.1. *Let M be a closed simply connected oriented manifold of dimension 4 with nonnegative sectional curvature. If $\mathbf{Z}_m$ acts on M by isometries and m is greater than or equal to 61, then $\mathbf{Z}_m$-action has at most 5 isolated fixed points.*

The proof of Proposition 2.1 easily follows from Corollary 2.6 in [3] and Proposition 2.7 in [3]. Actually, by Corollary 7 in [16] Proposition 2.1 holds even for any prime $m \geq 41$.

Since the action of $\mathbf{Z}_m$ is assumed to be homologically trivial in this section, it follows from the Lefschetz fixed point formula and Proposition 2.1 that, when $\mathrm{Fix}(M, \mathbf{Z}_m)$ is the set of isolated fixed points, the Euler characteristic $\chi(M)$ of a simply connected nonnegatively curved 4-manifold M satisfies $2 \leq \chi(M) \leq 5$. Thus the classification result of Freedman in [6] implies that in our case M is indeed homeomorphic to

$$S^4,\ \mathbf{CP}^2,\ S^2 \times S^2,\ \#^k \pm \mathbf{CP}^2 \quad 1 \leq k \leq 3.$$

On the other hand, even in case that there is a 2-dimensional fixed point component, we want to show that a similar estimate as in Proposition 2.3 below holds. To do so, we first begin with the following simple lemma.

Lemma 2.2. *Suppose that a finite group action on M has a fixed point component N of dimension 2. Then N is diffeomorphic to S^2 or T^2.*

Proof. Since M is orientable, N is totally geodesic and N is nonnegatively curved, therefore N is homeomorphic (and so diffeomorphic) to S^2 or T^2. □

However, the assumption that $\mathbf{Z}_m$ acts trivially on the homology of M implies that N is homeomorphic to S^2 by a result of McCooey (cf. [13]). So we may assume that N is homeomorphic to S^2. From now on, we shall denote by N such a 2-dimensional fixed point component.

In order to prove Theorem 1.1, it suffices to prove the following proposition.

Proposition 2.3. *Under the same assumptions as in Theorem 1.1, the following estimate holds:*

$$2 \leq \chi(M) \leq 7.$$

Proof. Note first that since $H_3(M;\mathbf{Z}) = H_1(M;\mathbf{Z}) = 0$, we have

$$\chi(M) = 2 + b_2(M) \geq 2.$$

If one of prime factors of m is greater than or equal to 61, there exists an element $g \in \mathbf{Z}_m$ whose order is greater than or equal to 61. Hence it follows form Proposition 2.1 that the number of isolated fixed points of $\langle g \rangle$ is less than or equal to 5. Since the action is homologically trivial and there is at most one 2-dimensional fixed point component of $\langle g \rangle$ which is just a 2-sphere, we have

$$\chi(\mathrm{Fix}(M, g)) = \chi(M) \leq 7.$$

So we are done in this case.

From now on, we assume that all the prime factors of m are strictly less than 61. Then we can prove the following lemma.

Lemma 2.4. *Let g denote an element of $\mathbf{Z}_m$ with m greater than or equal to 61. Assume that $\mathrm{Fix}(M, \langle g \rangle)$ contains a 2-dimensional fixed point component N. Then we have $2 \leq \chi(M) \leq 7$.*

Proof. For the proof, let $P_1, \cdots, P_n$ be the points outside N whose isotropy group under the action of $\langle g \rangle$ is non-trivial. If all the points P_i are isolated then n should be less than or equal to 5 by Proposition 2.1. Hence the Euler characteristic of M is less than or equal to 7.

So we assume that some P_i is non-isolated. In this case there is a 2-dimensional fixed point set F passing through P_i under the action of $\langle g^k \rangle \subset \langle g \rangle$. If $F \cap N$ is empty, $\mathrm{Fix}(M, g^k)$ would contain both F and N. But, it contradicts to the assumption that there is at most one 2-dimensional fixed point component for any non-trivial element of $\mathbf{Z}_m$. On the other hand, if there is a point $q \in F \cap N$, then $F \cap N$ should contain a fixed point component of $\langle g^k \rangle$-action containing q. Since a fixed point component has always even codimension, the connected component of $F \cap N$ containing q is either q or $F = \Sigma$. But the first case implies that $\langle g^k \rangle$ trivially acts on T_qM which is generated by T_qF and T_qN, which does not make sense for any effective action. Moreover, it is easy to see that the second case does not occur, either. Indeed, the second case implies that $P_i \in F = N$ is not a non-isolated point any more. It again contradicts to the choice of P_i, since P_i was chosen to be non-isolated.

This completes the proof of Lemma 2.4. □

In order to finish the proof of Theorem 1.1, we also need the following lemma.

Lemma 2.5. *As before, let g denote an element of $\mathbf{Z}_m$ with m greater than or equal to* 61*. Assume that* $\mathrm{Fix}(M, \langle g \rangle)$ *is zero-dimensional. Then we have* $2 \leq \chi(M) \leq 7$*.*

Proof. If all the fixed points of g are isolated, it follows from Proposition 2.1 that we have $\chi(M) \leq 5$. So we are done. Otherwise, we can assume that there exists a non-isolated fixed points of the $\mathbf{Z}_m$-action. Let $H_i = \langle g^{k_i} \rangle$ be the isotropy group which fixes a 2-dimensional connected surface F_i passing through non-isolated fixed points for $1 \leq i \leq l$.

If we let c_i be the order of $H_i = \langle g^{k_i} \rangle$ and c_i is no less than 61, then we are reduced to Lemma 2.4, and thus we are done again.

Therefore, we may assume without loss of generality that all c_i's satisfy the inequality $c_i < 61$ $(1 \leq i \leq l)$. Since F_i is homeomorphic to S^2 by Lemma 2.2, we can conclude that F_i contains two non-isolated fixed points.

Since $\mathbf{Z}_m$ is cyclic, then H_i is a normal subgroup of $\mathbf{Z}_m$. If $l \leq 5$, then it is clear that

$$|\mathbf{Z}_m/(H_1 \cdots H_l)| \geq 61^7/61^5 \geq 61^2,$$

since $c_i = |H_i| < 61$ for each i. Note also that $\mathbf{Z}_m/(H_1 ... H_l)$ has only isolated fixed points. Hence, by Proposition 2.1, we obtain $\chi(M) \leq 5$.

Finally, we claim that $l \leq 5$. Otherwise, we have

$$|\mathbf{Z}_m/(H_1 ... H_6)| > 61$$

so that Proposition 2.1 would imply that $\mathbf{Z}_m/(H_1 ... H_6)$ has at most 5 isolated fixed points. However, if $l \geq 6$, then there should exist at least 6 non-isolated fixed points of the $\mathbf{Z}_m$-action. Thus $\mathbf{Z}_m/(H_1 ... H_6)$ should have at least 6 isolated fixed points. But this is clearly a contradiction.

This completes the proof of Lemma 2.5. □

It is immediate to see that combining Lemma 2.4 with Lemma 2.5 will complete the proof of Proposition 2.1. □

Finally, we close this paper with the following simple but interesting observation.

Lemma 2.6. *Let H_i and F_i $(0 \leq i \leq 4)$ be the same as in Lemma 2.5 (with different indices). If $Q_{i \pmod 5}$ and $Q_{i+1 \pmod 5}$ are non-isolated fixed points in F_i for $0 \leq i \leq 4$, then the fixed point set*

$$\mathrm{Fix}(M, \mathbf{Z}_m/(H_0 H_1 ... H_4))$$

consists of only isolated fixed points.

Before giving the proof, we briefly recall the definition of q-extent as follows. The q-extent $xt_q(X)$, $q \geq 2$, of a compact metric space (X, d) is, by definition, given by the following formula:

$$xt_q(X) = \binom{q}{2}^{-1} \max \left\{ \sum_{1 \leq i < j \leq q} d(x_i, x_j) \ : \ \{x_i\}_{i=1}^q \right\}.$$

Proof. To prove the lemma, we essentially use the ideas in (Case 2) of the proof of Theorem 1 in [17]. To do so, first recall from the proof of Lemma 2.5 that F_i contains non-isolated fixed points Q_i, Q_{i+1} for $0 \leq i \leq 4$ mod 5, and then suppose that $\mathrm{Fix}(M, \mathbf{Z}_m/(H_0 H_1 ... H_4)) = N \cup \{Q_0, Q_1, ..., Q_4\}$ where N has codimension 2 in M. Then we will derive a contradiction.

To be precise, note that we may assume without loss of generality that

$$\mathrm{dist}(Q_i, N) \geq \mathrm{dist}(Q_0, N) = \mathrm{dist}(Q_0, Q_5)$$

for some $Q_5 \in N$. For $0 \leq i, j \leq 5$ and $i \neq j$, we then let l_{ij} =dist(Q_i, Q_j) and let

$$C_{ij} = \{\gamma : [0, l_{ij}] \to M | \gamma \text{ is a minimizing normal geodesic from } Q_i \text{ to } Q_j\}.$$

For each triple $0 \leq i, j, k \leq 5$, we also set

$$\alpha_{ijk} = \min\{\angle(\gamma_j'(0), \gamma_k'(0)) \,|\, \gamma_j \in C_{ij}, \ \gamma_k \in C_{ik}\}.$$

By Corollary 2.6 in [3], the 5-extent $xt_5(X)$ is less than $\frac{\pi}{3}$, where X is the quotient space of the unit sphere in $T_{Q_0}M$ by the $\mathbf{Z}_m/(H_0 H_1 ... H_4)$-action. Then it easily follows from an argument of (Case 2) in the proof of Theorem 1 in [17] that we have

$$\alpha_{0i5} \geq \frac{\pi}{2}$$

for all $1 \leq i \leq 4$. Moreover, since $\gamma_1'(0) \perp \gamma_4'(0)$ for $\gamma_1 \in C_{01}$ and $\gamma_4 \in C_{04}$, we have $\alpha_{014} \geq \frac{\pi}{2}$, and it is also easy to see that

$$\alpha_{012} + \alpha_{024} \geq \alpha_{014} \geq \frac{\pi}{2}, \text{ and } \alpha_{013} + \alpha_{034} \geq \alpha_{014} \geq \frac{\pi}{2}.$$

So, we have

$$\alpha_{012} + \alpha_{024} + \alpha_{014} + \alpha_{013} + \alpha_{034} \geq \frac{3\pi}{2}.$$

Therefore, we have

$$\begin{aligned}&\sum_{1\leq i<j\leq 4} \alpha_{0ij} + \sum_{1\leq i\leq 4} \alpha_{0i5} \\ &> \frac{3\pi}{2} + 2\pi = \frac{7\pi}{2}.\end{aligned}$$

But this is a contradiction, since $\frac{7\pi}{2} > \binom{5}{2}\frac{\pi}{3} > \binom{5}{2}xt_5(X)$, as noted earlier. This completes the proof of Lemma 2.6 □

Acknowledgements: The authors are very grateful to the anonymous reader for his careful reading and valuable suggestions. This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2007-313-C00062). The second author was supported by the BK21 program of KAIST.

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