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A DUAL ITERATIVE SUBSTRUCTURING METHOD WITH A PENALTY TERM IN THREE DIMENSIONS*

CHANG-OCK LEE† AND EUN-HEE PARK‡

ABSTRACT. The FETI-DP method is one of the most advanced dual substructuring methods, which introduces Lagrange multipliers to enforce the pointwise matching condition on the interface. In our earlier work for two dimensional problems, a dual iterative substructuring method was proposed, which is a variant of the FETI-DP method based on the way to deal with the continuity constraint on the interface. The proposed method imposes the continuity by not only the pointwise matching condition on the interface but also using a penalty term which measures the jump across the interface. In this paper, a dual substructuring method with a penalty term is extended to three dimensional problems. A penalty term with a penalization parameter η is constructed by focusing on the geometric complexity of an interface in three dimensions caused by the coupling among adjacent subdomains. For a large η, it is shown that the condition number of the resultant dual problem is bounded by a constant independent of both the subdomain size H and the mesh size h. From the implementational viewpoint of the proposed method, the difference from FETI-DP method is to solve subdomain problems which contain a penalty term with a penalization parameter η. To prevent a large penalization parameter from making subdomain problems ill-conditioned, special attention is paid to establish an optimal preconditioner with respect to a penalization parameter η. Finally, numerical results are presented.

1. INTRODUCTION

We consider the following Poisson model problem with the homogeneous Dirichlet boundary condition

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega,\\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{1.1}
$$

where Ω is a bounded polyhedral domain in $\mathbb{R}^3$ and f is a given function in $L^2(\Omega)$. For simplicity, we assume that Ω is partitioned into two nonoverlapping subdomains $\{\Omega_i\}_{i=1}^2$ such that $\overline{\Omega} = \bigcup_{i=1}^2 \overline{\Omega}_i$. The problem (1.1) can be rewritten as

$$
\begin{aligned}
&\min_{v_i \,\in\, H^1(\Omega_i,\,\partial\Omega)} \sum_{i=1}^{2} \left(\frac{1}{2} \int_{\Omega_i} |\nabla v_i|^2 \, dx - \int_{\Omega_i} f v_i \, dx \right)\\
&\text{subject to } v_1 = v_2 \text{ on } \partial\Omega_1 \cap \partial\Omega_2.
\end{aligned} \tag{1.2}
$$

Here, $H^1(\Omega_i, \partial\Omega)$ is the usual Sobolev space defined as

$$
H^1(\Omega_i, \partial\Omega) = \{ v_i \in H^1(\Omega_i) \mid v_i = 0 \text{ on } \partial\Omega \cap \partial\Omega_i \},
$$

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where $H^1(\Omega) = \{v \in L^2(\Omega) \mid \partial^\alpha v \in L^2(\Omega),\ |\alpha| \leq 1\}$. In the domain decomposition approach based on the reformulated minimization problem (1.2) with a constraint, a key point is how to convert the constrained minimization problem into an unconstrained one. Most studies (e.g. [1, 10, 12]) for treatment of constrained minimizations started in the field of optimal control problem. There are three most popular methods developed for different purposes: the Lagrangian method, the method of penalty function, and the augmented Lagrangian method. Such various ideas have been introduced for handling constraints as the continuity across the interface in (1.2) (see [8, 9, 11]).

The FETI-DP method is the typical algorithm based on the Lagrangian method, which introduces Lagrange multipliers to enforce the continuity constraint on the interface. Many studies for the augmented Lagrangian method have been done in the frame of domain-decomposition techniques which belong to families of nonoverlapping Schwarz alternating methods, variants of FETI method, etc. (cf. [5, 7, 11, 15]) In our previous work [16] for two dimensional problems, a dual iterative substructuring method was proposed in view of the augmented Lagrangian method, which is a variant of the FETI-DP method. To the Lagrangian functional of the standard FETI-DP, a penalty term is added, which measures the jump across the interface and includes a positive penalization parameter η. In the same way as in most dual substructuring approaches, the saddle-point problem related to the augmented Lagrangian functional is reduced to the dual problem with Lagrange multipliers as unknowns. Then it is solved by the conjugate gradient method (CGM). For the preconditioned FETI-DP with the optimal Dirichlet preconditioner, it is well-known that it is numerically scalable in the sense that the condition number grows asymptotically as $(1 + \log(H/h))^2$ in two dimensions [18]. On the other hand, it was proven that the dual problem in [16] has a constant condition number independently of both of H and h even though it is not accompanied by any preconditioner.

In the development of domain-decomposition algorithms as fast and efficient solvers for large scale problems, it is necessary to extend the well-designed algorithms for two dimensional problems to the three dimensional case. In this paper, we extend the dual substructuring method in [16] to the three dimensional case. In the process of extension to three dimensional problems, there are two things to be mainly considered; one is to construct a strong penalty term in 3D enough to guarantee the same convergence speed as in 2D and the other is how to treat an ill-conditioned property of the subdomain problems due to a large penalization parameter. In both of two key issues, emphasis is placed on the awareness of difference between 2D and 3D in the geometric complexity of an interface. An interface in 3D includes not only faces similar to edges in 2D but also edges which make all nodes on the interface coupled. First, it is noted that the adoption of the same penalty as suggested for two-dimensional problems in [16] gives a dual substructuring algorithm which maintains the same performance in the aspect of the condition number of a dual problem. However, the penalty term makes an unnecessary coupling between functions on face nodes and edges nodes. Since such a coupling causes a considerable decrease on practical efficiency, we suggest a modified penalty term for the three dimensional problem in a manner of reducing a coupling between functions on the interface. Next, unlike the FETI-DP method, subdomain problems containing the penalty term are solved iteratively, of which the condition number becomes large as a penalization parameter η increases. The same type of preconditioner as in 2D might be a satisfactory one to the ill-conditioned problem due to a large η. But, since the preconditioner suggested in [16] contains the coupling among all nodes on the interface in 3D, it is hardly practical in the implementational point of view. Based on such an observation, a more appropriate preconditioner for three-dimension problems is constructed, which is not only optimal with respect to η but also more practical than that used in 2D.

This paper is organized as follows. In Sect. 2, we introduce a dual iterative substructuring method with a penalty term. Sect. 3 presents algebraic condition number estimate of the resultant dual system. In Sect. 4, we deal with a computational issue in the implementational point of view. Subdomain problems is solved iteratively, of which the condition number becomes large as a penalization parameter η increases. To remove such an ill-conditioning property due to a large η, the optimal preconditioners are developed with respect to η. Finally, we show numerical results in Sect. 5.

2. Dual iterative substructuring with a penalty term

In this section, we present a dual iterative substructuring method with a penalty term based on the augmented Lagrangian approach. We start with a minimization problem with the pointwise matching constraint on the interface. The adoption of Lagrange multipliers for dealing with the constraint yields a saddle-point problem for a Lagrangian functional. By augmenting a penalty term to the Lagrangian, we consider a slightly modified saddle-point problem which gives a dual iterative substructuring method with a penalty term.

Let $\mathcal{T}_h$ denote a quasi-uniform triangulation on Ω, where the discretization parameter h stands for the maximal mesh size of $\mathcal{T}_h$. For simplicity, we consider a triangulation of hexahedra and the standard trilinear finite element approximate solution of (1.1): find $u_h \in X_h$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in X_h, \tag{2.3}$$

where

$$a(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h \, dx, \quad (f, v_h) = \int_\Omega f v_h \, dx,$$

and $X_h = \{v_h \in H_0^1(\Omega) \cap \mathcal{C}^0(\overline{\Omega}) \mid \forall \tau \in \mathcal{T}_h,\ v_h|_\tau \in \mathbb{Q}_1(\tau)\}$.

We decompose Ω into N non-overlapping subdomains $\{\Omega_k\}_{k=1}^N$, where a partition $\{\Omega_k\}_{k=1}^N$ of Ω is assumed to be shape-regular. On each subdomain, the triangulation $\mathcal{T}_{h_k}$ is quasi-uniform and the matching grids are taken on the boundaries of neighboring subdomains across the interface Γ. Here the interface Γ is the union of the common interfaces among all subdomains, i.e., $\Gamma = \bigcup_{k<l} \Gamma_{kl}$, where Γ_{kl} denotes the common interface of two adjacent subdomains Ω_k and Ω_l. We define the finite-dimensional subspace X^k on each subdomain Ω_k by

$$X^k = \left\{ v_h^k \in \mathcal{C}^0(\overline{\Omega}_k) \;\middle|\; \forall \tau \in \mathcal{T}_{h_k},\ v_h^k|_\tau \in \mathbb{Q}_1(\tau),\ v_h^k|_{\partial\Omega \cap \partial\Omega_k} = 0 \right\}.$$

By enforcing the continuity at the corner points, we assemble X^k's into X_h^c:

$$X_h^c = \left\{ v = (v_h^k)_k \in \prod_{k=1}^N X^k \;\middle|\; v \text{ is continuous at each corner} \right\}.$$

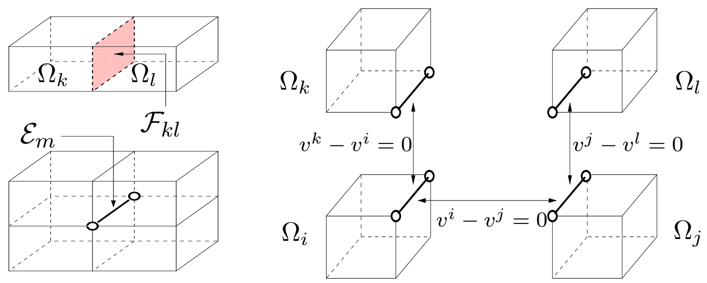

FIGURE 1. Left figure: Geometrical objects (face and edge). Right figure: Choice of three pairs of adjacent subdomains which share an edge.

Before introducing the continuity constraint on the interface nodes except vertices, we define notations related to geometrical objects. The interface Γ is composed of faces which are shared by two subdomains, edges which are shared by more than two subdomains, and vertices. The geometrical objects on the interface are characterized in more details as

(i) $\mathcal{F}_{kl}$ denotes the common face of Ω_k and Ω_l, which is regarded as an open set.
(ii) $\mathcal{E}_m$ where m is an index of an edge is an edge shared by neighboring subdomains, which does not include its end points, vertices.

To enforce the continuity on the interface except vertices, a signed Boolean matrix B is taken in the same way as in the FETI-DP (cf. [8, 13]), that is, $Bv = 0$ implies that

$$v^k - v^l = 0 \quad \text{on } \mathcal{F}_{kl}, \quad k < l,$$
$$v^i - v^j = 0 \quad \text{on } \mathcal{E}_m, \quad (i,j) \in I_{\mathcal{E}_m},$$

where $I_{\mathcal{E}_m}$ is the set of indices of subdomain pairs which have an edge $\mathcal{E}_m$ in common. Note that we do not allow any redundant continuity constraint on all edges, that is, in the case where an edge $\mathcal{E}_m$ is shared by four subdomains, there are four different ways to choose three pairs of adjacent subdomains to impose the continuity on the edge nodes. In Figure 1, one of four possible choices is depicted.

Now, we present a partitioned problem based on the domain-decomposition approach. The finite element problem (2.3) is reformulated as a minimization problem with constraints imposed by the requirement of continuity across the interface Γ:

$$\min_{v \in X_h^c} \left(\frac{1}{2} \sum_{k=1}^{N} \int_{\Omega_k} |\nabla v|^2 \, dx - (f, v) \right) \quad \text{subject to} \quad Bv = 0.$$

Following a well-known techniques for the constrained optimization, we introduce a vector μ of Lagrange multipliers in $\mathbb{R}^M$ and define a Lagrangian functional $\mathcal{L} : X_h^c \times \mathbb{R}^M \to \mathbb{R}$ as

$$\mathcal{L}(v, \mu) = \frac{1}{2} \sum_{k=1}^{N} \int_{\Omega_k} |\nabla v|^2 \, dx - (f, v) + \langle Bv, \mu \rangle,$$

where M represents the number of constraints used for imposing the pointwise matching on the interface and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^M$. Next, we shall slightly change the Lagrangian $\mathcal{L}$ by addition of a penalty term. It is natural to adopt the same penalty term as suggested for the two dimensional problem in [16]:

$$J_\eta(u,v) = \sum_{k<l} \frac{\eta}{h} \int_{\Gamma_{kl}} (u^k - u^l)(v^k - v^l)\, ds, \quad \eta > 0. \tag{2.4}$$

To make a 3D algorithm more efficient in practical sense, it is desirable to minimize a coupling between functions on face nodes and edge nodes. But, the penalty term in (2.4) makes face nodes and edge nodes in each part Γ_{kl} of Γ coupled so that all nodes on the interface are tied. In this context, by considering the interface as a union of two separate objects: faces and edges, we introduce a modified penalty term

$$J_\eta(u,v) = \eta(J_{\mathcal{F}}(u,v) + J_{\mathcal{E}}(u,v)), \quad \eta > 0, \tag{2.5}$$

where

$$J_{\mathcal{F}}(u,v) = \frac{1}{h} \sum_{k<l} \int_{\mathcal{F}_{kl}} (u^k_{\mathcal{F}_{kl}} - u^l_{\mathcal{F}_{kl}})(v^k_{\mathcal{F}_{kl}} - v^l_{\mathcal{F}_{kl}})\, dx$$

and

$$J_{\mathcal{E}}(u,v) = \sum_{\mathcal{E}_m} \sum_{(i,j)\in I_{\mathcal{E}_m}} \int_{\mathcal{E}_m} (u^i - u^j)(v^i - v^j)\, ds.$$

Here, $u^k_{\mathcal{F}_{kl}}$ is a part of u, which is related to the contribution to u^k on $\mathcal{F}_{kl}$ only from the face nodal basis functions except the edge nodal basis functions. We define an augmented Lagrangian $\mathcal{L}_\eta$ with the penalty term J_η

$$\mathcal{L}_\eta(v,\mu) = \mathcal{L}(v,\mu) + \frac{1}{2} J_\eta(v,v).$$

Given the augmented Lagrangian $\mathcal{L}_\eta$, we consider the saddle-point problem:

$$\mathcal{L}_\eta(u_h,\lambda_h) = \max_{\mu_h\in\mathbb{R}^M} \min_{v_h\in X_h^c} \mathcal{L}_\eta(v_h,\mu_h) = \min_{v_h\in X_h^c} \max_{\mu_h\in\mathbb{R}^M} \mathcal{L}_\eta(v_h,\mu_h). \tag{2.6}$$

It is proven that seeking the solution of (2.3) is equivalent to finding the saddle-point of (2.6) (cf. [16]). The problem (2.6) is represented in the algebraic form

$$\begin{bmatrix} A_\eta & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix},$$

where

$$A_\eta = \begin{bmatrix} A_{\Pi\Pi} & A_{\Pi\Delta} \\ A_{\Pi\Delta}^T & A_{\Delta\Delta} + \eta J \end{bmatrix}, \quad B^T = \begin{bmatrix} 0 \\ B_\Delta^T \end{bmatrix}, \quad u = \begin{bmatrix} u_\Pi \\ u_\Delta \end{bmatrix}, \quad F = \begin{bmatrix} f_\Pi \\ f_\Delta \end{bmatrix},$$

where Π indicates the degrees of freedom associated with both the interior nodes and the subdomain corners, Δ those related to the face nodes and the edge nodes on the interface, and λ the Lagrange multipliers introduced for imposing the continuity constraint across the interface. Eliminating u_Π and u_Δ successively, we have a dual system

$$F_\eta \lambda = d_\eta \tag{2.7}$$

where

$$F_\eta = B_\Delta S_\eta^{-1} B_\Delta^T, \quad d_\eta = B_\Delta S_\eta^{-1}(f_\Delta - A_{\Pi\Delta}^T A_{\Pi\Pi}^{-1} f_\Pi)$$

with

$$S_\eta = S + \eta J = (A_{\Delta\Delta} - A_{\Pi\Delta}^T A_{\Pi\Pi}^{-1} A_{\Pi\Delta}) + \eta J.$$

Note that S_η is symmetric positive definite because S is symmetric positive definite [20, Section 6.4] and J is symmetric positive semidefinite. Since F_η is symmetric positive definite, we solve the resultant dual system (2.7) iteratively by the conjugate gradient method.

3. Estimate of condition number

In this section, we provide a sharp estimate for the condition number of the dual system F_η. By letting the vector v_Δ be partitioned as

$$v_\Delta = \begin{bmatrix} v_f \\ v_e \end{bmatrix},$$

the pointwise matching operator B_Δ is represented as

$$B_\Delta = \begin{bmatrix} B_f & 0 \\ 0 & B_e \end{bmatrix}.$$

Let us denote by $D(A)$ a block diagonal matrix such that

$$D(A) = \begin{bmatrix} A & & \\ & \ddots & \\ & & A \end{bmatrix}.$$

Looking at the connection between the operator B_Δ and the penalty term J_η from their definitions, it is obvious that

$$J = \begin{bmatrix} J_F & 0 \\ 0 & J_E \end{bmatrix} = \begin{bmatrix} B_f^T D(J_{B_f}) B_f & 0 \\ 0 & B_e^T D(J_{B_e}) B_e \end{bmatrix}, \tag{3.8}$$

where J_{B_f} and J_{B_e} stand for the 2D mass matrix on each face weighted with $1/h$ and the 1D mass matrix on each edge, respectively. We define by Λ the space of vectors of the degrees of freedom associated with the Lagrange multipliers. To analyze the condition number bound for F_η, based on Lemma 3.1 in [18], it is sufficient to specify a suitable norm $\|\cdot\|_\Lambda$ on Λ and to estimate the constants satisfying the relationship as follows:

$$\begin{aligned} c_1\|\lambda\|_{\Lambda'}^2 \le \langle \lambda, F_\eta \lambda\rangle \le c_2 \|\lambda\|_{\Lambda'}^2 \quad \forall \lambda \in \Lambda, \\ c_3\|\mu\|_\Lambda^2 \le \langle \mu, \mu\rangle \le c_4\|\mu\|_\Lambda^2 \quad \forall \mu \in \Lambda. \end{aligned} \tag{3.9}$$

Taking the structural characteristic of J into consideration, we define the norm $\|\cdot\|_\Lambda$ on Λ by

$$\|\mu\|_\Lambda^2 = \mu^T \begin{bmatrix} D(J_{B_f}) & 0 \\ 0 & D(J_{B_e}) \end{bmatrix} \mu, \quad \forall \mu \in \Lambda. \tag{3.10}$$

The dual norm on Λ is defined by

$$\|\lambda\|_{\Lambda'} = \max_{\substack{\mu \in \Lambda \\ \mu \neq 0}} \frac{|\langle \lambda, \mu \rangle|}{\|\mu\|_\Lambda}, \quad \forall \lambda \in \Lambda.$$

We now mention useful results in deriving bounds on the extreme eigenvalues of F_η. The first proposition states the property related to the norm induced by F_η, which can be easily checked as in [19].

Proposition 3.1. *For any* $\lambda \in \Lambda$,

$$\lambda^T F_\eta \lambda = \max_{v_\Delta \neq 0} \frac{|v_\Delta^T B_\Delta^T \lambda|^2}{\|v_\Delta\|_{S_\eta}^2}$$

where $\|v_\Delta\|_{S_\eta}$ *is the norm induced by the symmetric positive definite matrix* S_η.

Focusing on the fact that $\Lambda = \mathrm{Range}(B_\Delta)$, we have the following characterization of the dual norm (cf. [16]).

Proposition 3.2.

$$\|\lambda\|_{\Lambda'}^2 = \max_{\substack{v_\Delta \perp \mathrm{Ker}(B_\Delta) \\ v_\Delta \neq 0}} \frac{|v_\Delta^T B_\Delta^T \lambda|^2}{v_\Delta^T J v_\Delta}.$$

We state the relationship between S and J, which is proven similarly to Lemma 3.2 of [16].

Proposition 3.3. *For* $S = A_{\Delta\Delta} - A_{\Pi\Delta}^T A_{\Pi\Pi}^{-1} A_{\Pi\Delta}$, *there exists a constant* $C = \lambda_{\max}^S / \lambda_{\min}^J$ *such that*

$$v_\Delta^T S v_\Delta \leq C v_\Delta^T J v_\Delta, \quad \forall v_\Delta \perp \mathrm{Ker}(B_\Delta),$$

where $\lambda_{\max}^S$ *and* $\lambda_{\min}^J$ *are the maximum eigenvalue of* S *and the minimum nonzero eigenvalue of* J, *respectively.*

Since the constant C derived in Proposition 3.3 is one of the main factors in the bound of condition number of F_η, we are concerned with whether $\lambda_{\max}^S$ and $\lambda_{\min}^J$ are independent of the mesh size h and the subdomain size H. The following lemma is used in estimating the bound on $\lambda_{\min}^J$.

Lemma 3.1. *Let* $A_{PN} \in \mathbb{R}^{m \times m}$ *be an* $n \times n$ *block tridiagonal matrix*

$$A_{PN} = \begin{bmatrix} I & -I & 0 & \cdots & 0 \\ -I & 2I & -I & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -I & 2I & -I \\ 0 & \cdots & 0 & -I & I \end{bmatrix}. \tag{3.11}$$

For $v \in \mathbb{R}^m / \mathrm{Ker}(A_{PN})$, *we have*

$$v^T A_{PN} v \geq C \left(\frac{1}{n-1}\right)^2 v^T v.$$

This result follows from noting that A_{PN} in (3.11) is in a similar form to a stiffness matrix for the Poisson problem in one dimension with pure Neumann condition and using the Poincaré inequality [20].

Lemma 3.2. *Let* $\lambda_{\min}^J$ *be the minimum nonzero eigenvalue of* J. *Then, we have*

$$\lambda_{\min}^J \geq Ch$$

where a constant C *is independent of* h *and* H.

Proof. Since J is block diagonal as shown in (3.8), it is sufficient to estimate $\lambda_{\min}^{J_F}$ and $\lambda_{\min}^{J_E}$ which are minimum nonzero eigenvalues of the diagonal blocks J_F and J_E, respectively. Let $\lambda_{\min}^{J_{B_f}}$ and $\lambda_{\min}^{J_{B_e}}$ be minimum eigenvalues of J_{B_f} and J_{B_e}, respectively. It is obvious that both of $\lambda_{\min}^{J_{B_f}}$ and $\lambda_{\min}^{J_{B_e}}$ are bounded below by Ch because J_{B_f} and J_{B_e} are related to the 2D mass matrix on a face weighted with $1/h$ and the 1D mass matrix on an edge, respectively; cf. [20, Theorem B.32].

First, from the fact that $B_f B_f^T = 2I$, it follows

$$\lambda_{\min}^{J_F} = 2\lambda_{\min}^{J_{B_f}}.$$

Hence, we have

$$\lambda_{\min}^{J_F} \geq Ch. \tag{3.12}$$

Next, B_e does not maintain the same property as B_f since an edge is shared by more than two subdomains while only two subdomains have a face in common. We shall estimate of $\lambda_{\min}^{E}$ by focusing on the structural characteristic of J_E. Noting that $J_E = B_e^T D(J_{B_e}) B_e$, we have that for any v_e with $B_e v_e \neq 0$,

$$\begin{aligned} v_e^T J_E v_e &\geq \lambda_{\min}^{J_{B_e}} \lambda_{\min}^{\tilde{B}} v_e^T v_e \\ &\geq Ch \lambda_{\min}^{\tilde{B}} v_e^T v_e, \end{aligned}$$

where $\lambda_{\min}^{\tilde{B}}$ is the minimum nonzero eigenvalue of $B_e^T B_e$. Let N_E be the number of all edges $\mathcal{E}_m$ in the interface. By considering the block structure of B_e as

$$B_e = \left[B_{e_1}, \cdots, B_{e_{N_E}}\right]$$

where B_{e_m} is a block related to subdomains sharing an edge $\mathcal{E}_m$, it follows

$$\lambda_{\min}^{\tilde{B}} = \min_m \lambda_{\min}^{\tilde{B}_m}.$$

Here $\lambda_{\min}^{\tilde{B}_m}$ is the minimum nonzero eigenvalue of $B_{e_m}^T B_{e_m}$. Suppose that an edge $\mathcal{E}_m$ is shared by N_{s,e_m} subdomains. Then, $B_{e_m}^T B_{e_m}$ is an $N_{s,e_m} \times N_{s,e_m}$ block matrix of the form (3.11). By Lemma 3.1, we obtain

$$\lambda_{\min}^{\tilde{B}_m} \geq C \left(\frac{1}{N_{s,e_m} - 1} \right)^2. \tag{3.13}$$

Since a partition $\{\Omega_i\}_i$ of Ω is assumed to be shape-regular, there is a constant $N_{s,\max}$ such that

$$N_{s,e_m} \leq N_{s,\max} \quad \forall m. \tag{3.14}$$

Combination of (3.13) and (3.14) gives $\lambda_{\min}^{\tilde{B}} \gtrsim \left(\frac{1}{N_{s,\max}-1}\right)^2$. Hence, it yields

$$\lambda_{\min}^{J_E} \geq Ch$$

Finally, we have

$$\lambda_{\min}^{J} = \min\{\lambda_{\min}^{J_F}, \lambda_{\min}^{J_E}\} \geq Ch,$$

where a constant C is independent of h and H. □

Thanks to Lemma 3.1 in [18], we have the following estimate of the condition number $\kappa(F_\eta)$.

Theorem 3.1. *For any $\eta > 0$, we have*

$$\kappa(F_\eta) \leq \left(1 + \frac{C}{\eta}\right) C^*$$

where

$$C = \frac{\lambda_{\max}^S}{\lambda_{\min}^J}, \quad C^* = \frac{\max\{\lambda_{\max}^{J_{B_f}}, \lambda_{\max}^{J_{B_e}}\}}{\min\{\lambda_{\min}^{J_{B_f}}, \lambda_{\min}^{J_{B_e}}\}}.$$

Furthermore, the constants C and C^* are independent of the subdomain size H and the mesh size h.

Proof. Proceeding as in Theorem 3.1 of [16], we first get the following bounds:

$$\frac{1}{C+\eta}\|\lambda\|_{\Lambda'}^2 \leq \lambda^T F_\eta \lambda \leq \frac{1}{\eta}\|\lambda\|_{\Lambda'}^2 \quad \forall \lambda \in \Lambda$$

where C is the constant estimated in Proposition 3.3. Combination of Proposition 3.1, 3.2, and 3.3 gives the lower bound, while the upper bound is estimated by using the positive-definiteness of S.

Next, from the definition of the norm $\|\cdot\|_\Lambda$ in (3.10), it follows that

$$\left(\max\{\lambda_{\max}^{J_{B_f}}, \lambda_{\max}^{J_{B_e}}\}\right)^{-1} \|\mu\|_\Lambda^2 \leq \langle \mu, \mu \rangle \leq \left(\min\{\lambda_{\min}^{J_{B_f}}, \lambda_{\min}^{J_{B_e}}\}\right)^{-1} \|\mu\|_\Lambda^2 \quad \forall \mu \in \Lambda.$$

Using Lemma 3.1 in [18], we have

$$\kappa(F_\eta) \leq \left(1 + \frac{C}{\eta}\right) C^*,$$

where

$$C = \frac{\lambda_{\max}^S}{\lambda_{\min}^J}, \quad C^* = \frac{\max\{\lambda_{\max}^{J_{B_f}}, \lambda_{\max}^{J_{B_e}}\}}{\min\{\lambda_{\min}^{J_{B_f}}, \lambda_{\min}^{J_{B_e}}\}}.$$

Finally, we shall check the dependency of the constants C and C^* on the subdomain size H and the mesh size h. From Lemma 4.11 and Lemma B.5 in [20], it follows that $\lambda_{\max}^S \lesssim h$. Then, Lemma 3.2 informs that the constant C is independent of H and h. Moreover, by keeping in mind that J_{B_f} and J_{B_e} are related to the mass matrices in 2D and 1D, it is confirmed that the constant C^* is independent of H and h. □

Corollary 3.1. *For a sufficiently large η, we have*

$$\kappa(F_\eta) \leq C^*,$$

where C^* is the constant estimated in Theorem 3.1.

4. COMPUTATIONAL ISSUE

The dual formulation in (2.7) is intended for the estimate of condition number. To focus on the implementation of the proposed algorithm, we reorder the relevant degrees of freedom. By rearranging u in order $u = [u_r, u_c]^T$ where u_i, u_f, and u_e are assembled into u_r, we obtain the system in the following form

$$K_{rr}^\eta u_r + K_{rc} u_c + B_r^T \lambda = f_r \tag{4.15a}$$

$$K_{rc}^T u_r + K_{cc} u_c = f_c \tag{4.15b}$$

$$B_r u_r = 0 \tag{4.15c}$$

Note that $K_{rr}^{\eta} = K_{rr} + \eta\tilde{J}$ is non-singular because K_{rr} is positive definite (cf. [8]). By substituting

$$u_r = (K_{rr}^{\eta})^{-1}(f_r - K_{rc}u_c - B_r^T\lambda) \tag{4.16}$$

from (4.15a) into (4.15b) and (4.15c), we have

$$\begin{bmatrix} F_{cc} & -F_{rc}^T \\ F_{rc} & F_{rr} \end{bmatrix} \begin{bmatrix} u_c \\ \lambda \end{bmatrix} = \begin{bmatrix} d_c \\ d_r \end{bmatrix} \tag{4.17}$$

where

$$F_{rr} = B_r(K_{rr}^{\eta})^{-1}B_r^T, \quad F_{rc} = B_r(K_{rr}^{\eta})^{-1}K_{rc}, \quad F_{cc} = K_{cc} - K_{rc}^T(K_{rr}^{\eta})^{-1}K_{rc}$$

and

$$d_r = B_r^T(K_{rr}^{\eta})^{-1}f_r, \quad d_c = f_c - K_{rc}^T(K_{rr}^{\eta})^{-1}f_r.$$

Since A_η is invertible, so is F_{cc}, the Schur complement of K_{rr}^{η} in A_η. We can therefore eliminate u_c in (4.17) to get

$$F_\eta\lambda = d_\eta \tag{4.18}$$

where

$$F_\eta = F_{rr} + F_{rc}F_{cc}^{-1}F_{rc}^T, \quad d_\eta = d_r - F_{rc}F_{cc}^{-1}d_c.$$

We iteratively solve the dual problem (4.18) by the conjugate gradient method.

Remark 4.1. *For the comparison of the existing augmented FETI-DP methods* [8, 14] *with the proposed method, see Remark 4.1 in* [16].

In view of implementation, the difference with the FETI-DP method is to invert K_{rr}^{η} that contains the penalization parameter η. To compare our algorithm with the FETI-DP method, we need to make more careful observation of behavior of $(K_{rr}^{\eta})^{-1}$. Note that K_{rr}^{η} is detailed as

$$K_{rr}^{\eta} = K_{rr} + \eta\tilde{J} = \begin{bmatrix} A_{ii} & A_{i\Delta} \\ A_{i\Delta}^T & A_{\Delta\Delta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \eta J \end{bmatrix},$$

where

$$A_{\Delta\Delta} = \begin{bmatrix} A_{ff} & A_{fe} \\ A_{fe}^T & A_{ee} \end{bmatrix}, \quad J = \begin{bmatrix} J_F & 0 \\ 0 & J_E \end{bmatrix}.$$

4.1. **Construction of Preconditioner: Type I.** In this section, we characterize the proposed method in more details by observing the conditioning of K_{rr}^{η} and establishing a preconditioner which plays an auxiliary role in analyzing the preconditioner suggested in Sect. 4.3.

Thanks to the specific type of Poincaré inequality proven in Lemma 5.1 of [6], the standard scaling argument gives the following proposition without major difficulty.

Proposition 4.4. *For any v_r, there exist constants C_1 and C_2 independent of h and H such that*

$$C_1\frac{h^4}{H^3}\|v_r\|^2 \le v_r^T K_{rr} v_r \le C_2 h\|v_r\|^2,$$

that is,

$$\kappa(K_{rr}) \lesssim \left(\frac{H}{h}\right)^3.$$

Using the fact that $\lambda_{\max}^J \lesssim h$, we get the conditioning of K_{rr}^{η} by Proposition 4.4.

Lemma 4.3. *For each $\eta > 0$, we have that*

$$\kappa(K_{rr}^{\eta}) \lesssim \left(\frac{H}{h}\right)^3 (1+\eta).$$

Lemma 4.3 shows how severely η deteriorates the property of K_{rr}^{η} as η increases. Since K_{rr}^{η} is solved iteratively, it is expected that the large condition number of K_{rr}^{η} shown above may cause the computational cost relevant to K_{rr}^{η} to be more expensive. We shall establish good preconditioners for K_{rr}^{η} in order to remove a bad effect of η. First, we introduce a preconditioner M_1 as

$$M_1 = \begin{bmatrix} A_{ii} & 0 \\ 0 & A_{\Delta\Delta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \eta J \end{bmatrix}.$$

Theorem 4.2. *The condition number of the preconditioned system by M_1 grows asymptotically as*

$$\kappa(M_1^{-1}K_{rr}^{\eta}) := \frac{\lambda_{\max}(M_1^{-1}K_{rr}^{\eta})}{\lambda_{\min}(M_1^{-1}K_{rr}^{\eta})} \lesssim \left(\frac{H}{h}\right)^2.$$

Proof. Let

$$\gamma = \sup_{\substack{v_i \neq 0 \\ v_\Delta \neq 0}} \frac{|v_i^T A_{i\Delta} v_\Delta|}{(v_i^T A_{ii} v_i \cdot v_\Delta^T A_{\Delta\Delta}^{\eta} v_\Delta)^{\frac{1}{2}}}$$

where the constant $\gamma < 1$ is referred to as the strengthened Cauchy-Schwarz-Bunyakowski constant (see [2, 3, 17]). Proceeding the same way as in the proof of Theorem 4.2 in [16], we have that

$$\gamma \leq (1 - C^*)^{\frac{1}{2}} \tag{4.19}$$

where $C^* = \inf_{v_\Delta \neq 0} \frac{v_\Delta^T S_{\Delta\Delta} v_\Delta}{v_\Delta^T A_{\Delta\Delta} v_\Delta}$ and $S_{\Delta\Delta} = A_{\Delta\Delta} - A_{i\Delta}^T A_{ii}^{-1} A_{i\Delta}$. Next, we shall estimate C^*. In a similar way as in Lemma 4.11 of [20], it is easy to show that

$$\lambda_{\min}(S_{\Delta\Delta}) \gtrsim \frac{h^2}{H(1+\frac{H}{h})} \tag{4.20}$$

based on the specific type of Poincaré inequality mentioned in Lemma 5.1 of [6]. By using the inverse inequality and Lemma B.5 of [20], it is noted that

$$\lambda_{\max}(A_{\Delta\Delta}) \lesssim h. \tag{4.21}$$

Then, it follows from (4.20) and (4.21) that

$$\gamma \leq (1 - \hat{C})^{\frac{1}{2}} \tag{4.22}$$

where $\hat{C} = O\left(\frac{h}{H(1+\frac{H}{h})}\right)$. Similarly to the proof of Theorem 4.2 in [16], (4.22) yields that

$$\begin{aligned} \kappa(M_1^{-1}K_{rr}^{\eta}) &= \frac{\lambda_{\max}(M_1^{-1}K_{rr}^{\eta})}{\lambda_{\min}(M_1^{-1}K_{rr}^{\eta})} \\ &\leq \frac{2}{1-(1-\hat{C})^{\frac{1}{2}}} \\ &\lesssim \left(\frac{H}{h}\right)^2 \quad \text{for a sufficiently large } \frac{H}{h}. \end{aligned}$$

□

4.2. **Construction of Preconditioner: Type II.** Next, we suggest a preconditioner M_2 as

$$M_2 = \begin{bmatrix} A_{ii} & 0 \\ 0 & \tilde{A}_{\Delta\Delta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \eta J \end{bmatrix} \quad \text{with} \quad \tilde{A}_{\Delta\Delta} = \begin{bmatrix} A_{ff} & 0 \\ 0 & A_{ee} \end{bmatrix}.$$

In M_2, we additionally drop the coupling between faces and edges on the boundary of each subdomain while M_1 cuts off only the connection between interior nodes and boundary nodes in each subdomain. In analyzing the block diagonal type of preconditioners, it is a key issue to measure the orthogonality of relevant subspaces, which is represented in terms of a strengthened Cauchy inequality. We derive a strengthened Cauchy inequality which is useful to show the spectral equivalence between M_1 and M_2. It is proven by adopting the argument used in [4] to estimate a strengthened Cauchy inequality in two dimensions.

Lemma 4.4. *Let V and W be two subspaces of X^k with $V \cap W = \{\mathbf{0}\}$. For any $v \in V$ and $w \in W$, there exists a constant $0 < \gamma_T < 1$ such that on each $T \in \mathcal{T}_{h_k}$,*

$$\left| \int_T \nabla v \cdot \nabla w \, dx \right| \leq \gamma_T \left(\int_T |\nabla v|^2 \, dx \right)^{1/2} \left(\int_T |\nabla w|^2 \, dx \right)^{1/2},$$

where γ_T depends on the types of finite element functions v and w, but is independent of h.

Proof. For any triangle T in $\mathcal{T}_{h_k}$, there exists an affine mapping F_T from the reference triangle T_r onto a triangle T such that

$$F_T : T_r \to T, \quad F_T(x_r) = A_T x_r + a_T,$$

with A_T a linear mapping and a_T a constant vector in $\mathbb{R}^3$. Let $\mathcal{S}_{h_k}$ be the set of triangles T_* with the unit diameter such that T_* is the image of T_r under a linear mapping. Note that an affine mapping F_T is characterized as

$$F_T = B_T \circ B_{T_*}$$

where B_{T_*} is a linear mapping from the reference T_r onto a triangle T_* in $\mathcal{S}_{h_k}$ and B_T maps T_* onto T by a scaling and a translation: $B_T(x_*) = h_T x_* + a_T$. Based on the fact that for v_r and w_r satisfying

$$v_r(x_r) = v \circ F_T(x_r), \quad w_r(x_r) = w \circ F_T(x_r),$$

there exists a constant γ_{T_r} such that

$$\left| \int_{T_r} \nabla v_r \cdot \nabla w_r \, dx_r \right| \leq \gamma_{T_r} \left(\int_{T_r} |\nabla v_r|^2 \, dx_r \right)^{1/2} \left(\int_{T_r} |\nabla w_r|^2 \, dx_r \right)^{1/2},$$

we shall find a constant $0 < C(\gamma_{T_r}) < 1$ such that on each $T \in \mathcal{T}_{h_k}$,

(4.23)
$$\left| \int_T \nabla v \cdot \nabla w \, dx \right| \leq C(\gamma_{T_r}) \left(\int_T |\nabla v|^2 \, dx \right)^{1/2} \left(\int_T |\nabla w|^2 \, dx \right)^{1/2} \quad \forall v \in V, w \in W.$$

First, it is easily noted that a transformation B_T generated by a scaling and a translation has no influence on the constant in (4.23), that is, under the change of variables (4.23) becomes

(4.24)
$$\left| \int_{T_*} \nabla v_* \cdot \nabla w_* \, dx_* \right| \leq C(\gamma_{T_r}) \left(\int_{T_*} |\nabla v_*|^2 \, dx_* \right)^{1/2} \left(\int_{T_*} |\nabla w_*|^2 \, dx_* \right)^{1/2}.$$

Next, to observe the connection between two strengthened Cauchy inequalities in terms of the inner products associated with T_* and T_r, we define $\langle \cdot, \cdot \rangle_{T_*}$ and $\langle \cdot, \cdot \rangle_{T_r}$ by

$$\langle v, w \rangle_{T_*} = \int_{T_*} \nabla v_* \cdot \nabla w_* \, dx_*, \quad \langle v, w \rangle_{T_r} = \int_{T_r} \nabla v_r \cdot \nabla w_r \, dx_r$$

where $v_*(x_*) = v \circ B_T(x_*)$, $w_*(x_*) = w \circ B_T(x_*)$. For $v \in V$ and $w \in W$, let

$$\cos \theta_* := \frac{\langle v, w \rangle_{T_*}}{\|v\|_{T_*} \|w\|_{T_*}}, \quad \cos \theta_r := \frac{\langle v, w \rangle_{T_r}}{\|v\|_{T_r} \|w\|_{T_r}},$$

where $\| \cdot \|_{T_*}$ and $\| \cdot \|_{T_r}$ are the norms induced by the inner products $\langle \cdot, \cdot \rangle_{T_*}$ and $\langle \cdot, \cdot \rangle_{T_r}$, respectively. By following the standard argument in affine-equivalent finite elements, we get the relationship between two norms $\| \cdot \|_{T_*}$ and $\| \cdot \|_{T_r}$:

$$0 < \mu_1 \leq \frac{\langle v, v \rangle_{T_*}}{\langle v, v \rangle_{T_r}} \leq \mu_2 \tag{4.25}$$

where $\mu_1 = 1/\left(|\det(B_{T_*}^{-1})| \|B_{T_*}\|_2^2 \right)$, $\mu_2 = |\det(B_{T_*})| \|B_{T_*}^{-1}\|_2^2$. Using (4.25), it holds from Lemma 4.1 of [4] that

$$|\cos \theta_*| \leq \sqrt{1 - C(\mu_1, \mu_2)(1 - \cos^2 \theta_r)},$$

that is,

$$|\langle v, w \rangle_{T_*}| \leq \sqrt{1 - C(\mu_1, \mu_2)(1 - \cos^2 \theta_r)} \|v\|_{T_*} \|w\|_{T_*},$$

where

$$C(\mu_1, \mu_2) = \left(\frac{\mu_1}{\mu_2} \right)^4 = \left(\frac{1}{\|B_{T_*}\|_2^2 \|B_{T_*}^{-1}\|_2^2} \right)^4, \quad |\cos \theta_r| < 1.$$

Since $\mathcal{S}_{h_k}$ is shape-regular, that is, there exist a constant $\sigma > 0$ such that

$$\forall T_* \in \mathcal{S}_{h_k}, \quad \frac{h_{T_*}}{\rho_{T_*}} \leq \sigma,$$

where ρ_{T_*} is the diameter of the largest ball inscribed in T_*, we have

$$C(\mu_1, \mu_2) \geq \left(\frac{\rho_{T_r}}{h_{T_r}} \right)^8 \left(\frac{1}{\sigma} \right)^8$$

so that $C(\sigma, v, w) := \sqrt{1 - C(\mu_1, \mu_2)(1 - \cos^2 \theta_r)} < 1$ independently of h. Hence, based on the fact confirmed in (4.24), it follows

$$|\langle v, w \rangle_T| \leq \gamma_T \|v\|_T \|w\|_T,$$

where a positive constant $\gamma_T = C(\sigma, v, w) < 1$ depends on both of the shape parameter σ and the types of finite element functions v and w, but is independent of h. $\square$

Lemma 4.5. *Let X_f^k and X_e^k be the subspaces of X^k such that X_f^k and X_e^k consist of the degrees of freedom associated with the face nodes and the edge nodes, respectively. Then, there exists a constant $0 < \gamma < 1$ such that the strengthened Cauchy inequality*

$$\left| \int_{\Omega_k} \nabla v_f \cdot \nabla v_e \, dx \right| \leq \gamma \left(\int_{\Omega_k} |\nabla v_f|^2 \, dx \right)^{1/2} \left(\int_{\Omega_k} |\nabla v_e|^2 \, dx \right)^{1/2}$$

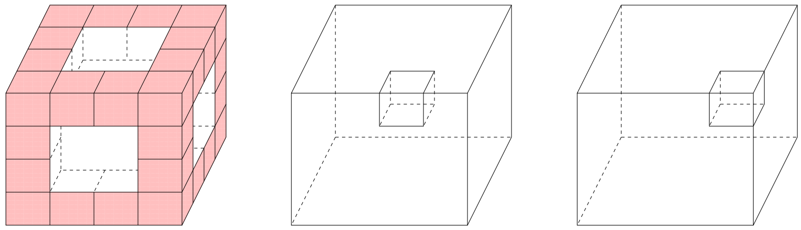

FIGURE 2. Left figure: $\mathcal{T}_\Gamma$. Center figure: Triangle in T_{Γ_1}. Right figure: Triangle in T_{Γ_2}.

holds for all $v_f \in X_f^k$ and $v_e \in X_e^k$. Here γ depends on both of the shape parameter of a triangulation $\mathcal{T}_{h_k}$ but is independent of h, v_f, and v_e.

Proof. We shall derive the strengthened Cauchy inequality based on the triangle-wise computation. Let $\mathcal{T}_\Gamma$ be the set of triangles $T \in \mathcal{T}_{h_k}$ such that T includes at least one point on an edge of Ω_k or one corner. For any $v_f \in X_f^k$ and $v_e \in X_e^k$, it follows from Lemma 4.4 that

$$\begin{aligned}
\left|\int_{\Omega_k} \nabla v_f \cdot \nabla v_e \, dx\right| &= \left|\sum_{T\in\mathcal{T}_\Gamma} \int_T \nabla v_f \cdot \nabla v_e \, dx\right| \\
&\le \sum_{T\in\mathcal{T}_\Gamma} \gamma_T \left(\int_T |\nabla v_f|^2 \, dx\right)^{1/2} \left(\int_T |\nabla v_e|^2 \, dx\right)^{1/2} \\
&\le \gamma \sum_{T\in\mathcal{T}_\Gamma} \left(\int_T |\nabla v_f|^2 \, dx\right)^{1/2} \left(\int_T |\nabla v_e|^2 \, dx\right)^{1/2} \\
&\le \gamma \left(\sum_{T\in\mathcal{T}_\Gamma} \int_T |\nabla v_f|^2 \, dx\right)^{1/2} \left(\sum_{T\in\mathcal{T}_\Gamma} \int_T |\nabla v_e|^2 \, dx\right)^{1/2} \\
&\le \gamma \left(\int_{\Omega_k} |\nabla v_f|^2 \, dx\right)^{1/2} \left(\int_{\Omega_k} |\nabla v_e|^2 \, dx\right)^{1/2},
\end{aligned}$$

where $\gamma = \max_T(\gamma_T)$. In the proof of Lemma 4.4, each constant γ_T is computed by using the angle between $v_{f,r}$ and $v_{e,r}$ measured on a reference triangle T_r. It means that γ_T is dependent on the position of a concerned triangle T. Figure 2 depicts the set $\mathcal{T}_\Gamma$ whose elements are classified into two types T_{Γ_1} and T_{Γ_2} depending on whether a triangle includes a subdomain corner as its vertex. Keeping in mind the types of finite element functions v_f, v_e on T_{Γ_1} and T_{Γ_2}, we estimate γ_T: for $T \in T_{\Gamma_j}$ with $j = 1, 2$,

$$\gamma_T < \sqrt{1 - C(\sigma)(1 - \cos^2\theta_{r_j})},$$

where $|\cos\theta_{r_j}| < 1$, $j = 1, 2$. Therefore, there exists a positive constant γ such that for all $v_f \in X_f^k$ and $v_e \in X_e^k$,

$$\left|\int_{\Omega_k} \nabla v_f \cdot \nabla v_e \, dx\right| \le \gamma \left(\int_{\Omega_k} |\nabla v_f|^2 \, dx\right)^{1/2} \left(\int_{\Omega_k} |\nabla v_e|^2 \, dx\right)^{1/2},$$

where the constant $\gamma < 1$ is independent of h and H. $\square$

Now we are ready to derive the spectral relationship between two preconditioners M_1 and M_2.

Theorem 4.3. *Two preconditioners M_1 and M_2 are spectrally equivalent, that is, there are constants c and C independent of h and H such that*

$$cv_r^T M_2 v_r \le v_r^T M_1 v_r \le C v_r^T M_2 v_r, \quad \forall v_r.$$

Proof. Note that M_1 and M_2 are different only in the second diagonal block and both of $A_{\Delta\Delta}$ and $\tilde{A}_{\Delta\Delta}$ are block diagonal matrices as

$$A_{\Delta\Delta} = \begin{bmatrix} A_{\Delta\Delta,1} & & \\ & \ddots & \\ & & A_{\Delta\Delta,N_s} \end{bmatrix} \quad \tilde{A}_{\Delta\Delta} = \begin{bmatrix} \tilde{A}_{\Delta\Delta,1} & & \\ & \ddots & \\ & & \tilde{A}_{\Delta\Delta,N_s} \end{bmatrix}.$$

Hence, it is sufficient to find constants c_k and C_k such that

$$c_k v_{\Delta_k}^T \tilde{A}_{\Delta\Delta,k} v_{\Delta_k} \le v_{\Delta_k}^T A_{\Delta\Delta,k} v_{\Delta_k} \le C_k v_{\Delta_k}^T \tilde{A}_{\Delta\Delta,k} v_{\Delta_k},$$

where v_{Δ_k} is the restriction of v_Δ to the degrees of freedom in a subdomain Ω_k.

For each k with $k = 1, \cdots, N_s$, it follows from Lemma 4.5 that there exists γ_k such that

$$|v_{f_k}^T A_{fe,k} v_{e_k}| \le \gamma_k \left(v_{f_k}^T A_{ff,k} v_{f_k}\right)^{1/2} \left(v_{e_k}^T A_{ee,k} v_{e_k}\right)^{1/2}. \tag{4.26}$$

Based on the fact

$$\frac{v_{\Delta_k}^T A_{\Delta\Delta,k} v_{\Delta_k}}{v_{\Delta_k}^T \tilde{A}_{\Delta\Delta,k} v_{\Delta_k}} = 1 + \frac{2 v_{f_k}^T A_{fe,k} v_{e_k}}{v_{\Delta_k}^T \tilde{A}_{\Delta\Delta,k} v_{\Delta_k}},$$

the strengthened Cauchy inequality (4.26) gives

$$(1-\gamma_k) v_{\Delta_k}^T \tilde{A}_{\Delta\Delta,k} v_{\Delta_k} \le v_{\Delta_k}^T A_{\Delta\Delta,k} v_{\Delta_k} \le (1+\gamma_k) v_{\Delta_k}^T \tilde{A}_{\Delta\Delta,k} v_{\Delta_k}.$$

Focusing on the difference between M_1 and M_2, we have that

$$(1-\gamma) v_r^T M_2 v_r \le v_r^T M_1 v_r \le (1+\gamma) v_r^T M_2 v_r, \quad \forall v_r,$$

where $\gamma = \max_{k=1,\cdots,N_s} \gamma_k$ depends on the shape of triangulations $\{\mathcal{T}_{h_k}\}_{k=1}^{N_s}$ on $\{\Omega_k\}_{k=1}^{N_s}$ but is independent of h and H. Therefore, the preconditioners M_1 and M_2 are spectrally equivalent. □

Corollary 4.2. *The condition number of the preconditioned system by M_2 grows asymptotically as*

$$\kappa(M_2^{-1} K_{rr}^\eta) \lesssim \left(\frac{H}{h}\right)^2.$$

4.3. **Construction of Preconditioner: Type III.** Finally, by eliminating the coupling between all pairs of faces and edges, we establish a preconditioner M_3 as

$$M_3 = \begin{bmatrix} A_{ii} & 0 \\ 0 & \overline{A}_{\Delta\Delta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \eta J \end{bmatrix} \quad \text{with} \quad \overline{A}_{\Delta\Delta} = \begin{bmatrix} \overline{A}_{ff} & 0 \\ 0 & \overline{A}_{ee} \end{bmatrix}.$$

Here, the matrices $\overline{A}_{ff}$ and $\overline{A}_{ee}$ are block diagonal with a block for each face and for each edge, respectively. Also we rewrite A_{ff} and A_{ee} as block matrices in the same structure as $\overline{A}_{ff}$ and $\overline{A}_{ee}$.

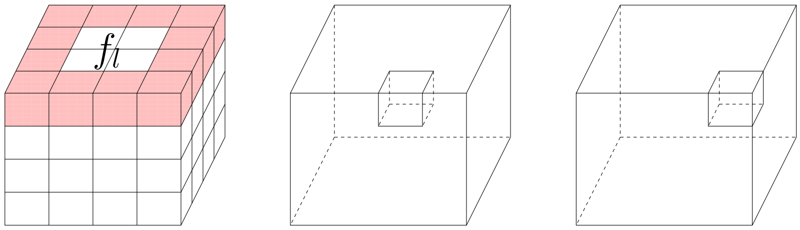

FIGURE 3. Left figure: $\mathcal{T}_{f_l}$. Center figure: Triangle in $\mathcal{T}_{f_l,1}$. Right figure: Triangle in $\mathcal{T}_{f_l,2}$.

Theorem 4.4. *Assume that on each subdomain Ω_k, a triangulation $\mathcal{T}_{h_k}$ satisfies*

$$Volume(T_c) \leq \min\{Volume(T_c^a)\},$$

where $T_c \in \mathcal{T}_{h_k}$ is a triangle containing a subdomain corner as one of its vertices and T_c^a is an adjacent triangle to T_c. Then, the condition number of the preconditioned system by M_3 grows asymptotically as

$$\kappa(M_3^{-1}K_{rr}^{\eta}) \lesssim \left(\frac{H}{h}\right)^2.$$

Proof. Proceeding a similar argument to the proof of Theorem 4.3, we shall estimate a spectral relationship between $\tilde{A}_{\Delta\Delta}$ and $\overline{A}_{\Delta\Delta}$. It suffices to find constants $c_{f_k}, c_{e_k}, C_{f_k}, C_{e_k}$ such that

$$\begin{aligned} c_{f_k} v_{f_k}^T \overline{A}_{ff,k} v_{f_k} \leq v_{f_k}^T A_{ff,k} v_{f_k} \leq C_{f_k} v_{f_k}^T \overline{A}_{ff,k} v_{f_k}, &\quad v_{f_k} \in X_f^k, \\ c_{e_k} v_{e_k}^T \overline{A}_{ee,k} v_{e_k} \leq v_{e_k}^T A_{ee,k} v_{e_k} \leq C_{e_k} v_{e_k}^T \overline{A}_{ee,k} v_{e_k}, &\quad v_{e_k} \in X_e^k. \end{aligned}$$

To observe the contribution from the off-diagonal blocks of A_{ff}, for each face f_l of Ω_k, we define $\mathcal{T}_{f_l}$ the set of triangles $T \in \mathcal{T}_{h_k}$ such that one of faces of T is contained in f_l and at least one of edges of T meets edges in $\overline{f}_l$. Figure 3 shows that $\mathcal{T}_{f_l}$ consists of two parts $\mathcal{T}_{f_l,1}$ and $\mathcal{T}_{f_l,2}$ of which difference is whether a triangle contains a subdomain corner as one of its vertices. Let N_F be the number of faces included in $\partial\Omega_k \cap \Gamma$. First, we note that

$$\frac{v_{f_k}^T A_{ff,k} v_{f_k}}{v_{f_k}^T \overline{A}_{ff,k} v_{f_k}} = 1 + \frac{\sum_{l_1 \neq l_2} v_{f_{l_1}}^T A_{f_{l_1} f_{l_2}} v_{f_{l_2}}}{\sum_{l=1}^{N_F} v_{f_l}^T A_{f_l f_l} v_{f_l}}. \tag{4.27}$$

Focusing on the triangle-wise computation, we get that

$$\begin{aligned} \sum_{l_1 \neq l_2} v_{f_{l_1}}^T A_{f_{l_1} f_{l_2}} v_{f_{l_2}} &= \sum_{l=1}^{N_F} \sum_{f_{l_n}} v_{f_l}^T A_{f_l f_{l_n}} v_{f_{l_n}} \\ &= \sum_{l=1}^{N_F} \int_{\Omega_k} \nabla \left(\sum_{f_{l_n}} v_{f_{l_n}} \right) \cdot \nabla v_{f_l} \, dx \\ &= \sum_{l=1}^{N_F} \sum_{m=1}^{2} \sum_{T \in \mathcal{T}_{f_l,m}} \int_T \nabla \left(\sum_{f_{l_n}} v_{f_{l_n}} \right) \cdot \nabla v_{f_l} \, dx, \end{aligned}$$

where f_{l_n} are adjacent faces to the face f_l. We need to take into consideration that a face f_l has only one adjacent face on each $T \in \mathcal{T}_{f_l,1}$ while on each $T \in \mathcal{T}_{f_l,2}$, a face f_l is affected by possibly two adjacent faces. Then, it follows from Lemma 4.4 that

(4.28)

$$\left|\sum_{l=1}^{N_F}\sum_{m=1}^{2}\sum_{T\in\mathcal{T}_{f_l,m}}\int_T \nabla\left(\sum_{f_{l_n}} v_{f_{l_n}}\right)\cdot\nabla v_{f_l}\,dx\right|$$
$$\leq \sum_{l=1}^{N_F}\frac{1}{2}\sum_{T\in\mathcal{T}_{f_l,1}}\gamma_T\left(\int_T|\nabla v_{f_{l_n}}|^2\,dx+\int_T|\nabla v_{f_l}|^2\,dx\right)$$
$$+\sum_{l=1}^{N_F}\frac{1}{2}\sum_{T\in\mathcal{T}_{f_l,2}}\left(\sum_{f_{l_n}}\gamma_T(f_{l_n})\int_T|\nabla v_{f_{l_n}}|^2\,dx+\left(\sum_{f_{l_n}}\gamma_T(f_{l_n})\right)\int_T|\nabla v_{f_l}|^2\,dx\right),$$

where $\gamma_T(f_{l_n})$ is estimated as in the proof of Lemma 4.5 for the degrees of freedom associated with f_{l_n} and those associated with f_l. Note that it is independent of h and H. Let us look at (4.28) in a different way. First, assembling triangle-wisely computed values in (4.28) face by face, it is clear that

$$\sum_{l=1}^{N_F}\frac{1}{2}\sum_{T\in\mathcal{T}_{f_l,1}}\gamma_T\left(\int_T|\nabla v_{f_{l_n}}|^2\,dx+\int_T|\nabla v_{f_l}|^2\,dx\right)=\sum_{l=1}^{N_F}\sum_{T\in\mathcal{T}_{f_l,1}}\gamma_T\int_T|\nabla v_{f_l}|^2\,dx$$

and

$$\sum_{l=1}^{N_F}\frac{1}{2}\sum_{T\in\mathcal{T}_{f_l,2}}\left(\sum_{f_{l_n}}\gamma_T(f_{l_n})\int_T|\nabla v_{f_{l_n}}|^2\,dx+\left(\sum_{f_{l_n}}\gamma_T(f_{l_n})\right)\int_T|\nabla v_{f_l}|^2\,dx\right)$$
$$=\sum_{l=1}^{N_F}\sum_{T\in\mathcal{T}_{f_l,2}}\left(\sum_{f_{l_n}}\gamma_T(f_{l_n})\right)\int_T|\nabla v_{f_l}|^2\,dx.$$

Next, we assume that on a subdomain Ω_k, a triangulation $\mathcal{T}_{h_k}$ satisfies

$$Volume(T_c)\leq\min\{Volume(T_c^a)\},\quad\forall T_c \tag{4.29}$$

where $T_c\in\mathcal{T}_{h_k}$ is a triangle including a subdomain corner as one of its vertices and $T_c^a\in\mathcal{T}_{h_k}$ is an adjacent triangle to T_c. As a result, we have

$$\left|\sum_{l_1\neq l_2} v_{f_{l_1}}^T A_{f_{l_1}f_{l_2}} v_{f_{l_2}}\right|\leq\gamma_f\sum_{l=1}^{N_F} v_{f_l}^T A_{f_lf_l} v_{f_l}, \tag{4.30}$$

where γ_f is taken as the maximum of γ_T's in (4.28). The combination of (4.27) and (4.30) yields

$$(1-\gamma_f)v_f^T\overline{A}_{ff}v_f\leq v_f^TA_{ff}v_f\leq(1+\gamma_f)v_f^T\overline{A}_{ff}v_f. \tag{4.31}$$

Similarly, under the assumption (4.29), it is confirmed that there exists a constant γ_e such that

$$(1-\gamma_e)v_e^T\overline{A}_{ee}v_e\leq v_e^TA_{ee}v_e\leq(1+\gamma_e)v_e^T\overline{A}_{ee}v_e, \tag{4.32}$$

where a constant $\gamma_e<1$ is independent of h and H. Hence, from (4.31) and (4.32), we have

$$\min\{1-\gamma_f,1-\gamma_e\}v_r^TM_3v_r\leq v_r^TM_2v_r\leq\max\{1+\gamma_f,1+\gamma_e\}v_r^TM_3v_r. \tag{4.33}$$

By combining Theorem 4.2, Theorem 4.3, and (4.33), we have that for a sufficiently large H/h,

$$\kappa(M_3^{-1}K_{rr}^{\eta}) \lesssim \left(\frac{H}{h}\right)^2.$$

$\square$

Remark 4.2. *Note that the block $(A_{\Delta\Delta} + \eta J)$ in M_1 is coupled over all boundary nodes while the block $(\tilde{A}_{\Delta\Delta} + \eta J)$ in M_2 has the coupling among all face nodes and among all edge nodes. Hence, the preconditioners M_1 and M_2 are less practical in the implementational point of view than the preconditioner M_3. But, the spectral relationships shown in Theorem 4.2 and Theorem 4.3 play major roles in analyzing the estimate of the condition number $\kappa(M_3^{-1}K_{rr}^{\eta})$ in Theorem 4.4.*

5. Numerical results

In this section, computational results are presented, which verify the theoretical bounds estimated in previous sections and show the efficiency of the proposed method as an iterative solver. We consider the exact solution

$$u(x,y,z) = \sin(\pi x)\sin(\pi y)z(1-z)$$

for the model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega = (0,1)^3$ is the unit cube. We use the conjugate gradient method with a constant initial guess $(\lambda_0 \equiv 1)$. The stop criterion is the relative reduction of the initial residual by a chosen TOL

$$\frac{\|r_k\|_2}{\|r_0\|_2} \le \text{TOL},$$

where r_k is the dual residual error on the k-th CG iteration. We detail later how to choose TOL in an appropriate way. Here, discretization parameters h, H, and N_s are used, which stand for the mesh size, the subdomain size, and the number of subdomains, respectively. Through numerical tests, Ω is decomposed into N_s cubic subdomains with $N_s = 1/H \times 1/H \times 1/H$. Each subdomain is partitioned into $H/h \times H/h \times H/h$ uniform cubic elements. Computational tests for the proposed method are carried out in two cases with $\eta = \eta^{opt}$ and $\eta = \eta^{big}$, where $\eta^{opt} = 2$ is an optimal value estimated in heuristic way while $\eta^{big} = 10^6$ is chosen to be large enough to show major characteristics of the proposed method. We also test the FETI-DP method since the proposed method is considered as its variant. Let us explain how to choose TOL's used in CGM applied to the dual system. It is noted that a dual residual r_k can be rewritten as jump of the primal solution on the interface (see Remark 4.1 in [16]). The larger a penalization parameter η is chosen, the smaller an initial jump becomes since a penalty term in the proposed method plays a role in reducing jump on the interface. Hence it is expected that $\text{TOL}^{\eta^0} \le \text{TOL}^{\eta^{opt}} \le \text{TOL}^{\eta^{big}}$ where $\text{TOL}^{\eta^0}, \text{TOL}^{\eta^{opt}}, \text{TOL}^{\eta^{big}}$ are three TOLs used in the cases with $\eta = 0, \eta^{opt}, \eta^{big}$, respectively. In addition, each TOL needs to be taken with paying attention to the conditioning property of an associated dual system. In the FETI-DP formulation, the condition number of a dual problem is larger in the case with a fully interior subdomain than in the case when all subdomains meet the

TABLE 1. Convergence behavior

N_s	$\frac{H}{h}$	h	$\eta = \eta^{big}$		$\eta = 0$
			$\frac{\|u-u_h\|_2}{\|u\|_2}$	ratio	$\frac{\|u-u_h\|_2}{\|u\|_2}$
2^3	4	1/8	2.6106e-2	-	2.6106e-2
	8	1/16	6.4577e-3	0.2474	6.4577e-3
	16	1/32	1.6101e-3	0.2493	1.6101e-3
	32	1/64	4.0227e-4	0.2498	4.0227e-4
4^3	4	1/16	6.4577e-3	-	6.4577e-3
	8	1/32	1.6101e-3	0.2493	1.6101e-3
	16	1/64	4.0227e-4	0.2498	4.0227e-4

boundary $\partial\Omega$. On the other hand, the proposed method with a large penalization parameter has a constant condition number bound independently of the type of domain decomposition into subdomains. Based on such observations from the two viewpoints, in our computational tests, $\mathrm{TOL}^{\eta^0}, \mathrm{TOL}^{\eta^{opt}}, \mathrm{TOL}^{\eta^{big}}$ for $N_s = 2^3$ are chosen as $10^{-10}, 10^{-7}, 10^{-6}$ while $10^{-11}, 10^{-8}, 10^{-6}$ for $N_s = 4^3$.

First, to see the convergence behavior, we show in Table 1 the relative errors $\frac{\|u-u_h\|_2}{\|u\|_2}$ estimated in L^2-norm for several H and h. The $O(h^2)$ convergence is observed in Table 1. In order to get rid of the bad effect of a large η on K_{rr}^{η}, a preconditioner M_3 was proposed in Sect. 4.3, which is optimal with respect to η. It is confirmed in Table 2 that the influence of η on $\kappa(K_{rr}^{\eta})$ is completely removed after adopting M_3 and the esimated condition number $\kappa(M_3^{-1}K_{rr}^{\eta})$ grows as $O((H/h)^2)$. Finally, we make a comparison between our proposed methods with $\eta = \eta^{opt}, \eta^{big}$ and the well-known FETI-DP method from the viewpoint of the conditioning of the related matrices F_η and F. Table 3 informs that in both cases with $\eta = \eta^{opt}$, η^{big}, the condition number $\kappa(F_\eta)$ and the CG iteration number (iter. #) for convergence is bounded by a constant even if the mesh is refined when keeping N_s constant. On the contrary, it is numerically confirmed that the condition number in the FETI-DP grows nearly as $O(H/h)^k$ with $k = 1, 2$. In the case with $\eta = \eta^{big}$, the inner preconditioner M_3 is used during CG iterations on subdomain problems. The inner iter. # presented in Table 3 is the number of inner CG iterations on subdomain problems.

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TABLE 2. Performance of preconditioner M_3 for $(K_{rr}^{\eta})^{-1}$ where $N_s = 4^3$

η	$\frac{H}{h}=4$		$\frac{H}{h}=8$		$\frac{H}{h}=16$	
	$\kappa(K_{rr}^{\eta})$	$\kappa(M_3^{-1}K_{rr}^{\eta})$	$\kappa(K_{rr}^{\eta})$	$\kappa(M_3^{-1}K_{rr}^{\eta})$	$\kappa(K_{rr}^{\eta})$	$\kappa(M_3^{-1}K_{rr}^{\eta})$
0	1.5515e+2	-	1.1171e+3	-	8.1714e+3	-
1	5.2631e+1	4.4679e+1	1.8204e+2	1.6804e+2	6.5729e+2	6.2276e+2
10^1	3.9493e+2	4.0769e+1	1.4367e+3	1.5972e+2	5.3247e+3	6.0634e+2
10^2	3.8342e+3	4.0344e+1	1.4035e+4	1.5886e+2	5.2171e+4	6.0468e+2
10^3	3.8227e+4	4.0301e+1	1.4002e+5	1.5875e+2	5.2064e+5	6.0451e+2
10^4	3.8216e+5	4.0297e+1	1.3999e+6	1.5876e+2	5.2054e+6	6.0449e+2
10^5	3.8215e+6	4.0297e+1	1.3999e+7	1.5875e+2	5.2053e+7	6.0449e+2
10^6	3.8215e+7	4.0297e+1	1.3999e+8	1.5876e+2	5.2053e+8	6.0449e+2

TABLE 3. Comparison between the FETI-DP method $(\eta = 0)$ and the proposed methods $(\eta = \eta^{opt}, \eta^{big})$

N_s	$\frac{H}{h}$	$\eta=0$		$\eta=\eta^{opt}$		$\eta=\eta^{big}$		
		iter. #	$\kappa(F)$	iter. #	$\kappa(F_\eta)$	iter. #	inner iter. # (min/max)	$\kappa(F_\eta)$
2^3	4	29	1.2301e+1	12	3.1756	6	18/23	4.8584
	8	37	2.0893e+1	14	4.0153	13	33/35	7.5638
	16	45	3.8373e+1	14	4.2971	16	59/63	8.5576
	32	58	7.3647e+1	14	4.3022	18	108/117	8.8694
4^3	4	73	8.1805e+1	14	3.1992	7	42/44	4.8585
	8	107	3.0183e+2	16	4.0118	14	75/78	7.5688
	16	153	1.1892e+3	16	4.3020	16	132/147	8.5609

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