A FETI-DP Formulation for the Stokes Problem without Primal Pressure Components

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A FETI–DP FORMULATION FOR THE STOKES PROBLEM WITHOUT PRIMAL PRESSURE COMPONENTS *

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Abstract. A scalable FETI–DP algorithm for the Stokes problem that employs a lumped preconditioner is developed and analyzed. A pair of inf-sup stable velocity and pressure finite element spaces is used to obtain a discrete problem. Differently from the previous approaches, no primal pressure unknowns are selected and only velocity primal unknowns at subdomain corners are selected. This leads to a symmetric and positive definite coarse problem matrix in the FETI-DP operator, while a less stable and indefinite coarse problem appears in the previous approaches. In addition, its condition number bound is proved to be the same as the FETI-DP algorithm with a lumped preconditioner for elliptic problems. Numerical results are included.

Key words. FETI-DP, Stokes problem, lumped preconditioner

AMS subject classifications. 65N30, 65N55, 76D07

1. Introduction. FETI-DP (Dual-Primal Finite Element Tearing and Interconnecting) algorithms were originally developed for second order elliptic problems in [2]. These algorithms belong to the dual iterative substructuring domain decomposition methods. A separate set of interface unknowns is assigned to each subdomain. Among the interface unknowns, some are selected as primal unknowns and the continuity at the primal unknowns will be enforced strongly. On the other part of the interface unknowns, the continuity will be enforced weakly using dual Lagrange multipliers. Elimination of unknowns other than the Lagrange multipliers results in a linear system of the dual variables. This linear system will be solved iteratively with a preconditioner. The primal unknowns are closely related to the coarse problem matrix that appears in the FETI-DP algorithms and a proper selection of the primal unknowns is crucial in obtaining a scalable algorithm.

FETI-DP algorithms have been developed for the three dimensional elliptic problems with heterogeneous coefficients, the Stokes problems, and three dimensional compressible elasticity problems, see [6, 11, 12, 14]. Its close connection to the BDDC algorithms was also studied in [1, 7, 16, 18]. Recently, extensions to irregular subdomains and to inexact subdomain solvers have been done in [9, 10, 17].

For the Stokes problem, both the FETI-DP and the BDDC algorithms have been developed in [13, 14, 15]. A FETI-DP algorithm was also developed for nonconforming finite

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element discretizations in [8]. In these algorithms, the compatibility condition on the jump of solutions of local Stokes problems is required so that the velocity averages on edges in addition to the velocity unknowns at the subdomain corners are selected to be primal unknowns in two dimensions. In three dimensions, introduction of face averages and more complicated edge averages is unavoidable. By enforcing the compatibility condition, additional primal unknowns of pressure components, that are constant in each subdomain, appears in these algorithms. This gives an indefinite coarse problem with both velocity and pressure primal unknowns.

In this paper, we develop a new FETI-DP algorithm for the Stokes problem in two dimensions. To reduce complication in implementing the FETI-DP algorithm, we employ only the primal velocity components that are unknowns at the subdomain corners. The primal pressure components will not be introduced contrary to the previous approaches for the Stokes problem. With only the primal velocity unknowns at the corners, the FETI-DP elimination process gives the solutions of the local Stokes problems of which jump across the interface does not satisfy the compatibility condition of the local Stokes problems. The Dirichlet-type preconditioners are no longer relevant to our FETI-DP formulation so that a lumped preconditioner is employed. By relaxing the compatibility condition on the jump of the solutions across the interface and using the lumped preconditioner, edge averages are no longer necessary and all the pressure unknowns can be eliminated. Our new formulation can be considered as an extension of the work in [17] to the Stokes problem. In the work, a FETI-DP algorithm for the elliptic problems with a lumped preconditioner was analyzed and a bound for its condition number was shown to be $C(H/h)(1 + log(H/h))$. We prove the same bound for the Stokes problem with the constant C depending on the inf-sup constant of a certain pair of velocity and pressure spaces. We also prove that the inf-sup constant is independent of any mesh parameters for rectangular subdomain partitions.

This paper is organized as follows. In Section 2, the FETI-DP formulation without any primal pressure unknowns will be derived and in Section 3 some preliminary results will be provided. The analysis of the condition number bound will be carried out in Section 4. In the final section, numerical results will be presented. Throughout this paper, C stands for any positive constants that do not depend on any mesh parameters.

2. FETI–DP formulation.

2.1. A model problem and finite element spaces. We consider the two-dimensional Stokes problem,

$$\begin{aligned} -\triangle \mathbf{u} + \nabla p &= \mathbf{f} \;\text{ in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \;\text{ in } \Omega, \\ \mathbf{u} &= 0 \;\text{ on } \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω is a bounded polygonal domain in $\mathbf{R}^2$ and $\mathbf{f} \in [L^2(\Omega)]^2$.

We introduce an inf-sup stable finite element space $(\widehat{X}, \overline{P})$ for a given triangulation in Ω; the pressure finite element space consists of functions which are discontinuous across the element boundaries, while functions in the velocity finite element space are continuous. From the finite element space $(\widehat{X}, \overline{P})$, we obtain a discrete problem of (2.1):

find $(\widehat{\mathbf{u}}, \overline{p}) \in (\widehat{X}, \overline{P})$ satisfying

$$
\begin{aligned}
&\sum_i \int_{\Omega_i} \nabla \widehat{\mathbf{u}} \cdot \nabla \mathbf{v}\, dx - \sum_i \int_{\Omega_i} \overline{p}\, \nabla \cdot \mathbf{v}\, dx = \sum_i \int_{\Omega_i} \mathbf{f} \cdot \mathbf{v}\, dx, \ \forall \mathbf{v} \in \widehat{X}, \\
&- \sum_i \int_{\Omega_i} \nabla \cdot \widehat{\mathbf{u}}\, \overline{q}\, dx = 0, \ \forall \overline{q} \in \overline{P}.
\end{aligned} \tag{2.2}
$$

For a fast solution of the discrete problem, we will derive an equivalent algebraic system. The equivalent algebraic system leads to a symmetric and positive definite system on dual variables, which will be solved iteratively with a preconditioner.

2.2. A FETI-DP formulation. We first decompose Ω into a non–overlapping subdomain partition $\{\Omega_i\}_{i=1}^N$ in such a way that each subdomain inherits the finite elements equipped for Ω. The subdomain finite element spaces are then given by

$$
X^{(i)} = \widehat{X}|_{\Omega_i}, \ P^{(i)} = \overline{P}|_{\Omega_i},
$$

that are the restrictions of $\widehat{X}$ and $\overline{P}$ to the individual subdomains. Among the subdomain velocity unknowns, we select some unknowns at the subdomain boundary as primal unknowns and we denote each part of the subdomain velocity unknowns by $\mathbf{u}_I^{(i)}$, $\mathbf{u}_\Pi^{(i)}$, and $\mathbf{u}_\Delta^{(i)}$, where I, Π, and Δ denote interior unknowns, the primal unknowns, and the remaining dual unknowns at the subdomain boundary, respectively. In the present work, the velocity unknowns at the subdomain corners are selected to be the primal unknowns. After the separation of unknowns, we introduce the corresponding velocity spaces, $X_I^{(i)}$, $X_\Pi^{(i)}$, and $X_\Delta^{(i)}$. We also introduce a space $X_r^{(i)}$ with the interior and the dual velocity unknowns,

$$
X_r^{(i)} = X_I^{(i)} \times X_\Delta^{(i)},
$$

and use the notation $\boldsymbol{u}_r^{(i)}$ for the velocity unknowns in the space $X_r^{(i)}$.

Throughout the paper, for a given space $W^{(i)}$ on Ω_i we denote by W the product space of $W^{(i)}$ and by $\widetilde{W}$ the subspace of W, where the strong continuity at the primal unknowns is enforced. The subspace of W, where continuity at the all interface unknowns is enforced, will be denoted by $\widehat{W}$. The unknowns at these spaces W, $\widetilde{W}$, and $\widehat{W}$ are then decoupled, partially coupled, and fully coupled across the subdomain interface, respectively. We also allow the same notational convention for the velocity unknowns; $\boldsymbol{u}_r$ denotes $(\boldsymbol{u}_r^{(1)}, \dots, \boldsymbol{u}_r^{(N)})$ and $\widetilde{\boldsymbol{u}}$ denotes velocity unknowns in the space $\widetilde{X}$. We will use the same notation $\boldsymbol{u}$ to denote velocity unknowns and the corresponding finite element function.

We now obtain an equivalent mixed form of the Stokes problem (2.2) in the finite element space $(\widetilde{X}, \overline{P})$ by enforcing the pointwise continuity on the remaining part of the interface unknowns using Lagrange multipliers $\lambda \in M$:

find $((\mathbf{u}_I, \mathbf{u}_\Delta, \widehat{\mathbf{u}}_\Pi), \overline{p}, \lambda) \in \widetilde{X} \times \overline{P} \times M$ such that

$$\begin{pmatrix} K_{II} & K_{I\Delta} & K_{I\Pi} & \overline{B}_I^T & 0 \\ K_{I\Delta}^T & K_{\Delta\Delta} & K_{\Delta\Pi} & \overline{B}_\Delta^T & J_\Delta^T \\ K_{I\Pi}^T & K_{\Delta\Pi}^T & K_{\Pi\Pi} & \overline{B}_\Pi^T & 0 \\ \overline{B}_I & \overline{B}_\Delta & \overline{B}_\Pi & 0 & 0 \\ 0 & J_\Delta & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \\ \widehat{\mathbf{u}}_\Pi \\ \overline{p} \\ \lambda \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ \boldsymbol{f}_\Pi \\ 0 \\ 0 \end{pmatrix}, \tag{2.3}$$

where $\overline{B}_I$, $\overline{B}_\Delta$, and $\overline{B}_\Pi$ are from

$$-\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\mathbf{u}}\, q\, dx, \quad \forall q \in \overline{P},$$

J_Δ is a boolean matrix that computes jump across the subdomain interface Γ_{ij},

$$J_\Delta \mathbf{u}_\Delta|_{\Gamma_{ij}} = \mathbf{u}_\Delta^{(i)} - \mathbf{u}_\Delta^{(j)},$$

and the other terms are from

$$\sum_i \int_{\Omega_i} \nabla \widetilde{\mathbf{u}} \cdot \nabla \widetilde{\mathbf{v}}\, dx.$$

To proceed to a new FETI-DP formulation, we first formulate an extended algebraic system of (2.3). We note that the pressure finite element space $\overline{P}$ can be represented by

$$\overline{P} = (\prod_{i=1}^N P^{(i)}) \bigcap L_0^2(\Omega),$$

where $P^{(i)}$ is the pressure finite element space equipped for the subdomain Ω_i. We denote that

$$P = \prod_i P^{(i)}.$$

We will extend the pressure space $\overline{P}$ of (2.3) to the space P by adding the following condition for any constant c,

$$\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\mathbf{u}}\, q\, dx = 0, \quad q = c. \tag{2.4}$$

Since any $\widetilde{\mathbf{u}}$ in $\widetilde{X}$ satisfies

$$\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\mathbf{u}}\, c\, dx = c \sum_{ij} \int_{\Gamma_{ij}} (\mathbf{u}_\Delta^{(i)} - \mathbf{u}_\Delta^{(j)}) \cdot \boldsymbol{n}_{ij}\, ds,$$

the above condition (2.4) is redundant to

$$J_\Delta \mathbf{u}_\Delta = 0. \tag{2.5}$$

In other words, the condition (2.4) can be obtained from a linear combination of the continuity constraints in (2.5). Therefore, addition of the condition (2.4) gives an extended algebraic system of (2.3).

We write the extended algebraic system with the pressure space P as follows:

find $((\mathbf{u}_I, \mathbf{u}_\Delta, \widehat{\mathbf{u}}_\Pi), p, \lambda) \in (\widetilde{X}, P, M)$ such that

$$\begin{pmatrix} K_{II} & K_{I\Delta} & K_{I\Pi} & B_I^T & 0 \\ K_{I\Delta}^T & K_{\Delta\Delta} & K_{\Delta\Pi} & B_\Delta^T & J_\Delta^T \\ K_{I\Pi}^T & K_{\Delta\Pi}^T & K_{\Pi\Pi} & B_\Pi^T & 0 \\ B_I & B_\Delta & B_\Pi & 0 & 0 \\ 0 & J_\Delta & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \\ \widehat{\mathbf{u}}_\Pi \\ p \\ \lambda \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ \boldsymbol{f}_\Pi \\ 0 \\ 0 \end{pmatrix}. \tag{2.6}$$

Here B_I, B_Δ, and B_Π are from

$$-\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\mathbf{u}}\, q \, dx, \quad \forall q \in P,$$

and the other terms are the same as those in (2.3).

In the new algebraic form, we first eliminate unknowns $(\mathbf{u}_I, \mathbf{u}_\Delta, p)$ and then eliminate $\widehat{\mathbf{u}}_\Pi$ to obtain the resulting algebraic system on λ. We can eliminate the unknowns $(\mathbf{u}_I, \mathbf{u}_\Delta, p)$ by solving independent local Stokes problems since the spaces X_I, X_Δ, and P are the products of independent local spaces, $X_I^{(i)}$, $X_\Delta^{(i)}$, and $P^{(i)}$, respectively. We then obtain algebraic equations for $\widehat{\mathbf{u}}_\Pi$ and λ. Note that with the extended pressure space P, we are able to eliminate all the pressure unknowns p by solving the local Stokes problem so that the coarse problem with only the primal velocity unknowns is obtained. However, the continuity constraints make $\sum_{ij} \int_{\Gamma_{ij}} (\boldsymbol{u}_\Delta^{(i)} - \boldsymbol{u}_\Delta^{(j)}) \cdot \boldsymbol{n}_{ij} \, ds = 0$ redundant, which has been enforced during the elimination of the unknowns $(\boldsymbol{u}_I, \boldsymbol{u}_\Delta, p)$. So that the resulting linear system for λ has the one dimensional null space and is positive definite on a subspace of M. Later we will specify the subspace in detail.

In the previous works on the Stokes problem [8, 14, 15], the primal pressure unknowns are selected and the pressure space P_I with the remaining unknowns is used to solve the local Stokes problem. In more detail,

$$\overline{P} = P_I \bigoplus \overline{P}_\Pi, \quad P_I = \prod_{i=1}^N P_I^{(i)},$$

where $\overline{P}_\Pi$ is the space of primal pressure components that are constant in each subdomain and have their average value zero in Ω, and $P_I^{(i)}$ consists of functions in $P^{(i)}$ which have zero averages, i.e., $P_I^{(i)} = P^{(i)} \bigcap L_0^2(\Omega_i)$. This approach gives the coarse problem with both the velocity and the pressure primal unknowns so that the coarse problem is a mixed problem, that is not the case in our formulation. In fact, our coarse problem is symmetric and positive definite and its size is smaller.

We rewrite (2.6) into

$$\begin{pmatrix} K_{II} & K_{I\Delta} & B_I^T \\ K_{I\Delta}^T & K_{\Delta\Delta} & B_\Delta^T \\ B_I & B_\Delta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \\ p \end{pmatrix} + \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix} \widehat{\mathbf{u}}_\Pi + \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix} \lambda = \begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix}, \tag{2.7}$$

$$\begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T \begin{pmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_\Delta \\ p \end{pmatrix} + K_{\Pi\Pi}\widehat{\boldsymbol{u}}_\Pi = \boldsymbol{f}_\Pi, \tag{2.8}$$

$$J_\Delta \boldsymbol{u}_\Delta = 0. \tag{2.9}$$

Let

$$S = \begin{pmatrix} K_{II} & K_{I\Delta} & B_I^T \\ K_{I\Delta}^T & K_{\Delta\Delta} & B_\Delta^T \\ B_I & B_\Delta & 0 \end{pmatrix}. \tag{2.10}$$

We recall that $X_r = X_I \times X_\Delta$. We can show that (X_r, P) satisfies the following condition: for any nonzero $p \in P$, there exists $\boldsymbol{v}_r \in X_r$ satisfying

$$\sum_{i=1}^N \int_{\Omega_i} \nabla \cdot \boldsymbol{v}_r p \, dx \neq 0. \tag{2.11}$$

So that S is invertible. This assertion will be proved in Lemma 3.1 of the following section. We then eliminate $(\boldsymbol{u}_I, \boldsymbol{u}_\Delta, p)$ from (2.7),

$$\begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \\ p \end{pmatrix} = S^{-1} \left(\begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix} - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix} \widehat{\boldsymbol{u}}_\Pi - \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix} \lambda \right). \tag{2.12}$$

This is solving the local Stokes problem with a Dirichlet boundary condition given with the values $\widehat{\boldsymbol{u}}_\Pi$ at the primal unknowns and a Neumann boundary condition given at the other unknowns of the subdomain boundary.

Substituting $(\mathbf{u}_I, \mathbf{u}_\Delta, p)$ into (2.8) gives

$$S_{\Pi\Pi}\widehat{\boldsymbol{u}}_\Pi = \boldsymbol{f}_\Pi - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T S^{-1} \left(\begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix} \lambda \right), \tag{2.13}$$

where the coarse problem matrix is given by

$$S_{\Pi\Pi} = K_{\Pi\Pi} - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T S^{-1} \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}.$$

Since $X_r \subset \widetilde{X}$, the assertion (2.11) also holds for $(\widetilde{X}, P)$. This fact gives that $S_{\Pi\Pi}$ is invertible, in fact, symmetric and positive definite. We can then eliminate $\widehat{\boldsymbol{u}}_\Pi$ from (2.13). By substituting $\widehat{\mathbf{u}}_\Pi$ into (2.12) and then $\boldsymbol{u}_\Delta$ into (2.9), we obtain the resulting algebraic equations for λ,

$$F_{DP}\lambda = d, \tag{2.14}$$

where

$$F_{DP} = \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}^T S^{-1} \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}^T S^{-1} \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix} S_{\Pi\Pi}^{-1} \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T S^{-1} \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix},$$

and

$$d = \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}^T S^{-1} \left(\begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix} - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix} S_{\Pi\Pi}^{-1} \left(\boldsymbol{f}_\Pi - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T S^{-1} \begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix} \right) \right).$$

The resulting system for $\lambda \in M$ is symmetric and positive semidefinite. We will now find a subspace of M where F_{DP} is positive definite. Since F_{DP} has only one null space component, it suffices to find λ that gives $F_{DP}\lambda = 0$. Then F_{DP} is positive definite on the subspace which is orthogonal to the null space. We recall the algebraic equations (2.6). We can see that $(\mathbf{u}_I, \mathbf{u}_\Delta, \widehat{\mathbf{u}}_\Pi) = 0$, $p = c$, and $\lambda|_{\Gamma_{ij}} = c\zeta_{ij}\boldsymbol{n}_{ij}$ are solutions of (2.6) for the zero force terms $(\boldsymbol{f}_I, \boldsymbol{f}_\Delta, \boldsymbol{f}_\Pi) = 0$. Here c is any constant, $\boldsymbol{n}_{ij}$ is the unit normal to Γ_{ij}, and $\zeta_{ij}(x_l)$ at the node $x_l \in \Gamma_{ij}$ is the integral of the corresponding finite element nodal basis function ϕ_l of $X^{(i)}$ on Γ_{ij}, i.e.,

$$\zeta_{ij}(x_l) = \int_{\Gamma_{ij}} \phi_l(x(s), y(s))\, ds. \tag{2.15}$$

We note that the values of $\lambda|_{\Gamma_{ij}}$ are given at every nodes of Γ_{ij} except the two end points.

We now introduce a subspace of M, which is orthogonal to the null space of F_{DP},

$$M_c = \left\{ \mu \in M \;:\; \sum_{ij} \boldsymbol{\mu}_{ij} \cdot \zeta_{ij}\boldsymbol{n}_{ij} = 0 \right\},$$

where $\boldsymbol{\mu}_{ij} = \mu|_{\Gamma_{ij}}$. Then F_{DP} is positive definite on M_c. Moreover, M_c is in fact the range space of F_{DP} and $d \in M_c$; see the formula for d and Lemma 3.2.

We suggest a preconditioner $\widehat{M}^{-1}$ by omitting the coarse problem term and by replacing S^{-1} in F_{DP} with S,

$$\widehat{M}^{-1} = \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}^T S \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}.$$

We recall the matrix S in (2.10) and obtain the resulting form of the preconditioner

$$\widehat{M}^{-1} = J_{\triangle} K_{\triangle\triangle} J_{\triangle}^T. \tag{2.16}$$

We call it a lumped preconditioner. We note that this preconditioner was introduced for FETI-type algorithms of the elliptic problems to reduce the cost for solving a Dirichlet problem which appears in the optimal preconditioner [3]. Later, FETI-DP algorithms with the lumped preconditioner was analyzed and proved to give a good convergence [17] for the elliptic problems. In our formulation, the lumped preconditioner is natural since the jump of the solutions of the local Stokes problems, $J_{\triangle}\boldsymbol{u}_{\triangle}$, which is obtained from the FETI-DP elimination process, does not satisfy the compatibility condition of the local Stokes problems. We note that, in the previous FETI-DP and BDDC algorithms [14, 15] additional primal velocity unknowns, such as edge averages and face averages, are selected in order to make $J_{\triangle}^T J_{\triangle}\boldsymbol{u}_{\triangle}$ satisfy the compatibility condition of the local Stokes problem.

In more detail, we present the step-by-step summary of the algorithm as follows:

Algorithm Let F_{DP}, d, and TOL be given. Let P_c be the orthogonal projection to the space M_c.

Step 1. Start with initial $\lambda_0 \in M_c$, compute residual $r_0 = d - F_{DP}\lambda_0$, and set $k = 0$.

Step 2. while $(\|r_k\|/\|r_0\| > TOL)$

Step 2.1 $z_k = P_c\widehat{M}^{-1}r_k$

Step 2.2 $k = k + 1$

Step 2.3 if $(k \geq 2)$

$\beta_k = \langle z_{k-1}, r_{k-1}\rangle / \langle z_{k-2}, r_{k-2}\rangle$

$q_k = z_{k-1} + \beta_k q_{k-1}$

else

$\beta_1 = 0,\ q_1 = z_0$

end if

Step 2.4 $\alpha_k = \langle z_{k-1}, r_{k-1}\rangle / \langle F_{DP}q_k, q_k\rangle$

Step 2.5 Compute $\lambda_k = \lambda_{k-1} + \alpha_k q_k$

Step 2.6 Compute $r_k = d - F_{DP}\lambda_k$

end while

Step 3. $\lambda = \lambda_k$ is the required solution.

In Step 2.1, we correct the residual r_k using the lumped preconditioner and then project the corrected residual to the space M_c to obtain z_k. In Step 2.6, we compute the residual with the iterate λ_k. Here we solve the local Stokes problems. The pressure unknowns are eliminated after solving the local Stokes problems and the coarse problem with only velocity unknowns at corners appears when we compute F_{DP}. By using the extended pressure space, we are able to eliminate all the pressure unknowns but we need to perform the iteration on the subspace M_c of M. This can be done easily by projecting the corrected residual to the space M_c during the iteration.

3. Preliminary results. In this section, we provide some preliminary results to analyze a condition number bound of the new FETI-DP algorithm equipped with the lumped preconditioner for the Stokes problem.

LEMMA 3.1. *The space (X_r, P) satisfies that for any nonzero $p \in P$, there exists $\boldsymbol{v}_r \in X_r$ such that*

$$\int_{\Omega} \nabla \cdot \boldsymbol{v}_r p \, dx \neq 0.$$

Proof. For a nonzero p, we can select $p^{(i)} = p|_{\Omega_i}$ such that $p^{(i)} \neq 0$. We then decompose it into

$$p^{(i)} = p_I + p_c,$$

where $p_I \in P_I^{(i)} (:= P^{(i)} \bigcap L_0^2(\Omega_i))$ and p_c is a constant. At least, one of them should be nonzero.

When $p_I \neq 0$, there exists $\boldsymbol{v}_I \in X_I^{(i)}$ such that

$$\int_{\Omega_i} \nabla \cdot \boldsymbol{v}_I p_I \, dx > 0,$$

because $(X_I^{(i)}, P_I^{(i)})$ is inf-sup stable. If $p_I = 0$, then we choose $\boldsymbol{v}_I = 0$.

When $p_c \neq 0$, we can find $\boldsymbol{v}_\Delta \in X_\Delta^{(i)}$ such that

$$\int_{\Omega_i} \nabla \cdot \boldsymbol{v}_\Delta p_c \, dx > 0,$$

since p_c is constant on Ω_i. If $p_c = 0$, then we choose $\boldsymbol{v}_\Delta = 0$. For the given $\boldsymbol{v}_\Delta$, we solve the local Stokes problem with $\boldsymbol{f}^{(i)} = 0$ using the pair $(X_I^{(i)}, P_I^{(i)})$ to obtain $\boldsymbol{w}_I \in X_I^{(i)}$ satisfying

$$\int_{\Omega_i} \nabla \cdot (\boldsymbol{w}_I + \boldsymbol{v}_\Delta) q_I \, dx = 0, \quad \forall q_I \in P_I^{(i)}. \tag{3.1}$$

Let $\boldsymbol{v}_r|_{\Omega_i} = \boldsymbol{v}_I + \boldsymbol{v}_\Delta + \boldsymbol{w}_I$ and $\boldsymbol{v}_r = 0$ on the other subdomains Ω_j, $j \neq i$. Using (3.1) and

$$\int_{\Omega_i} \nabla \cdot \boldsymbol{v}_I p_c \, dx = \int_{\Omega_i} \nabla \cdot \boldsymbol{w}_I p_c \, dx = 0,$$

we obtain

$$\begin{aligned} \sum_{j=1}^{N} \int_{\Omega_j} \nabla \cdot \boldsymbol{v}_r p \, dx &= \int_{\Omega_i} \nabla \cdot (\boldsymbol{v}_I + \boldsymbol{v}_\Delta + \boldsymbol{w}_I)(p_I + p_c) \, dx \\ &= \int_{\Omega_i} \nabla \cdot \boldsymbol{v}_I p_I \, dx + \int_{\Omega_i} \nabla \cdot \boldsymbol{v}_\Delta p_c \, dx. \end{aligned} \tag{3.2}$$

Since one of the two terms in (3.2) is positive, the desired result then follows. □

We recall the matrix S in (2.10).

LEMMA 3.2. *Let* $(\boldsymbol{u}_I, \boldsymbol{u}_\Delta, p)$ *be the solution of*

$$S \begin{pmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_\Delta \\ p \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix}, \tag{3.3}$$

with $\boldsymbol{f}_I$ *and* $\boldsymbol{f}_\Delta$ *arbitrarily given. Then* $J_\Delta \boldsymbol{u}_\Delta$ *belongs to* M_c.

Proof. We will show that $J_\Delta \boldsymbol{u}_\Delta$ is orthogonal to the null space component of F_{DP}, i.e., $\lambda|_{\Gamma_{ij}} = \zeta_{ij}\boldsymbol{n}_{ij}$, for all Γ_{ij}.

Since $(\boldsymbol{u}_I, \boldsymbol{u}_\Delta, p)$ is the solution of (3.3),

$$\sum_i \int_{\Omega_i} \nabla \cdot (\boldsymbol{u}_I + \boldsymbol{u}_\Delta)\, dx = 0.$$

We write the above equation into

$$\sum_i \int_{\partial\Omega_i} \boldsymbol{u}_\Delta \cdot \boldsymbol{n}_i \, ds = \sum_{ij} (\boldsymbol{u}_\Delta^{(i)} - \boldsymbol{u}_\Delta^{(j)}) \cdot \zeta_{ij}\boldsymbol{n}_{ij} = 0,$$

where $\boldsymbol{n}_i$ is the outward unit normal vector to $\partial\Omega_i$ and $\boldsymbol{n}_{ij}$ is the unit normal vector from Ω_i to Ω_j on their common part Γ_{ij}. Using $J_\Delta \boldsymbol{u}_\Delta|_{\Gamma_{ij}} = \boldsymbol{u}_\Delta^{(i)} - \boldsymbol{u}_\Delta^{(j)}$, the result then follows. □

We now consider the pair of velocity and pressure finite element spaces,

$$(\widehat{E}_{I,\Pi}, \overline{P}),$$

where $\widehat{E}_{I,\Pi} = X_I + \widehat{E}_\Pi$. Here $\widehat{E}_\Pi$ is an enriched primal velocity space that is constructed as follows. For the given values at the subdomain corners and the given average values on edges, we can find a discrete harmonic extension to the space $\widehat{X}$. These functions are continuous across the subdomain interface and become basis elements of the enriched primal velocity space. That is, the space $\widehat{E}_\Pi$ is determined by the velocity unknowns at the subdomain corners and the velocity averages on the edges. The space $\overline{P}$ is decomposed into

$$\overline{P} = P_I \bigoplus \overline{P}_\Pi.$$

We will first prove that (X_I, P_I) and $(\widehat{E}_\Pi, \overline{P}_\Pi)$ are inf-sup stable with the constants β_I and β_Π, respectively, and then we will prove that $(\widehat{E}_{I,\Pi}, \overline{P})$ is inf-sup stable. Since the finite element spaces $(X_I^{(i)}, P_I^{(i)})$ are inf-sup stable with the constant β_I, it is easy to see that (X_I, P_I) is also inf-sup stable with the constant β_I; see Remark 3.3. Here β_I does not depend on any mesh parameters. We introduce the notations

$$|\boldsymbol{u}|_1^2 = \sum_{i=1}^N |\boldsymbol{u}|^2_{H^1(\Omega_i)}, \quad \|p\|_0^2 = \sum_{i=1}^N \|p\|^2_{L^2(\Omega_i)},$$

where $|\cdot|_{H^1(\Omega_i)}$ and $\|\cdot\|_{L^2(\Omega_i)}$ are the semi H^1-norm and the L^2-norm in Ω_i, respectively.

REMARK 3.3. *Since $(X_I^{(i)}, P_I^{(i)})$ are inf-sup stable, for $p_I = (p_I^{(1)}, p_I^{(2)}, \cdots, p_I^{(N)}) \in P_I$, there is $\boldsymbol{u}_I^{(i)} \in X_I^{(i)}$ such that*

$$\int_{\Omega_i} \nabla \cdot \boldsymbol{u}_I^{(i)} p_I^{(i)}\, dx = \|p_I^{(i)}\|^2_{L^2(\Omega_i)}, \quad |\boldsymbol{u}_I^{(i)}|^2_{H^1(\Omega_i)} \le \frac{1}{\beta_I^2} \|p_I^{(i)}\|^2_{L^2(\Omega_i)}.$$

See [4, Remark 1.4]. We let $\boldsymbol{u}_I|_{\Omega^{(i)}} = \boldsymbol{u}_I^{(i)}$. We then see easily that

$$\frac{\left(\sum_{i=1}^N \int_{\Omega_i} \nabla \cdot \boldsymbol{u}_I\, p_I\, dx \right)^2}{|\boldsymbol{u}_I|_1^2 \|p_I\|_0^2} \ge \beta_I^2.$$

Using the result in [13, Proof of Theorem 4 in Section 4.3], for the pair $(\widehat{E}_\Pi, \overline{P}_\Pi)$ the inf-sup stability can be proved with an inf-sup constant β_Π such that

$$\beta_\Pi^2 \ge C \left(1 + \log \frac{H}{h}\right)^{-1}, \tag{3.4}$$

where C is independent of any mesh parameters. We note that the functions in $\widehat{E}_\Pi$ has the smaller semi H^1-norm than the functions in the space W_Π considered in [13] when the same average values on edges and the same values at the corners are given. When the subdomains are rectangular, we can improve the bound without the log-factor. In the numerical results, we will report the computed values of β_Π when H/h increases on a uniform rectangular subdomain partition. We introduce a pair (Q_2, Q_0) on the rectangular subdomain partition such that Q_2 is the space of piecewise biquadratic functions and Q_0 is the space of piecewise constant functions with its average zero. Since the (Q_2, P_1) element is inf-sup stable, so is (Q_2, Q_0); see [4].

LEMMA 3.4. *For a rectangular subdomain partition, the pair $(\widehat{E}_\Pi, \overline{P}_\Pi)$ is inf-sup stable with the constant β_Π satisfying*

$$\beta_\Pi^2 \ge C \max \left\{ \beta_Q^2 - \left(\frac{h}{H}\right)^2, \left(1 + \log \frac{H}{h}\right)^{-1} \right\}, \tag{3.5}$$

where C is a constant that does not depend on any mesh parameters and β_Q is the inf-sup constant of the pair (Q_2, Q_0). Therefore, β_Π is independent of H/h and any mesh parameters.

Proof. We will prove that for any $p \in \overline{P}_\Pi$ there exists $\widehat{\boldsymbol{u}}_E \in \widehat{E}_\Pi$ that satisfies

$$\frac{|\sum_i \int_{\Omega_i} \nabla \cdot \widehat{\boldsymbol{u}}_E\, p\, dx|^2}{|\widehat{\boldsymbol{u}}_E|_1^2 \|p\|_0^2} \ge C \left(\beta_Q^2 - \left(\frac{h}{H}\right)^2 \right).$$

Combined with the bound (3.4), we then obtain the desired bound. By considering the case when $(\frac{h}{H})^2 \le \frac{1}{2}\beta_Q^2$ with the first term of (3.5) and the other case with the second term of

(3.5), we can see that the constant β_Π depends only on β_Q but not on H/h and any mesh parameters.

Since subdomains form a rectangular partition of the original domain Ω, we can consider an inf-sup stable finite element pair (Q_2, Q_0) on the subdomain partition and denote by β_Q its inf-sup constant. We note that Q_0 is identical to $\overline{P}_\Pi$. For any $p \in \overline{P}_\Pi$, there exists $\boldsymbol{u} \in Q_2$ such that

$$\frac{|\sum_i \int_{\Omega_i} \nabla \cdot \boldsymbol{u}\, p\, dx|^2}{|\boldsymbol{u}|_1^2 \|p\|_0^2} \geq \beta_Q^2. \tag{3.6}$$

We now consider a nodal interpolant of $\boldsymbol{u}$ to the finite element space $\widehat{X}$ and denote it by $I^h(\boldsymbol{u})$. Then $I^h(\boldsymbol{u})$ satisfies the following approximation properties,

$$\begin{aligned} |I^h(\boldsymbol{u}) - \boldsymbol{u}|_1^2 &\leq Ch^2 |\boldsymbol{u}|_2^2 \leq Ch^2 H^{-2} |\boldsymbol{u}|_1^2, \\ |I^h(\boldsymbol{u})|_1 &\leq C|\boldsymbol{u}|_1, \end{aligned} \tag{3.7}$$

where h and H denote the mesh size and the subdomain diameter, respectively, and an inverse inequality is used for the bound, $|\boldsymbol{u}|_2^2 \leq CH^{-2}|\boldsymbol{u}|_1^2$.

From $I^h(\boldsymbol{u})$, we obtain $\widehat{\boldsymbol{u}}_E \in \widehat{E}_\Pi$ that is the discrete harmonic extension to the space $\widehat{X}$ with the given values at the subdomain corners and the given average values on the subdomain edges, which are the same as those of $I^h(\boldsymbol{u})$. Then $\widehat{\boldsymbol{u}}_E$ satisfies that

$$\begin{gathered} \sum_i \int_{\Omega_i} \nabla \cdot \widehat{\boldsymbol{u}}_E\, p\, dx = \sum_i \int_{\Omega_i} \nabla \cdot I^h(\boldsymbol{u})\, p\, dx, \ p \in \overline{P}_\Pi, \\ |\widehat{\boldsymbol{u}}_E|_1 \leq |I^h(\boldsymbol{u})|_1. \end{gathered}$$

Combined with (3.7), we obtain

$$\frac{|\int_\Omega \nabla \cdot \widehat{\boldsymbol{u}}_E\, p\, dx|^2}{|\widehat{\boldsymbol{u}}_E|_1^2 \|p\|_0^2} \geq C \frac{|\int_\Omega \nabla \cdot I^h(\boldsymbol{u})\, p\, dx|^2}{|\boldsymbol{u}|_1^2 \|p\|_0^2}. \tag{3.8}$$

We now consider

$$\int_\Omega \nabla \cdot I^h(\boldsymbol{u})\, p\, dx = \int_\Omega \nabla \cdot (I^h(\boldsymbol{u}) - \boldsymbol{u})\, p\, dx + \int_\Omega \nabla \cdot \boldsymbol{u}\, p\, dx$$

to obtain

$$\left| \int_\Omega \nabla \cdot I^h(\boldsymbol{u})\, p\, dx \right|^2 \geq \frac{1}{2} \left| \int_\Omega \nabla \cdot \boldsymbol{u}\, p\, dx \right|^2 - \left| \int_\Omega \nabla \cdot (I^h(\boldsymbol{u}) - \boldsymbol{u})\, p\, dx \right|^2.$$

For the second term of the above inequality, by the Schwarz inequality and (3.7) we have that

$$\begin{aligned} \left| \int_\Omega \nabla \cdot (I^h(\boldsymbol{u}) - \boldsymbol{u})\, p\, dx \right|^2 &\leq |I^h(\boldsymbol{u}) - \boldsymbol{u}|_1^2 \|p\|_0^2 \\ &\leq C \left(\frac{h}{H} \right)^2 |\boldsymbol{u}|_1^2 \|p\|_0^2. \end{aligned}$$

We then obtain

$$\left|\int_\Omega \nabla\cdot I^h(\boldsymbol{u})\,p\,dx\right|^2 \geq \frac{1}{2}\left|\int_\Omega \nabla\cdot\boldsymbol{u}\,p\,dx\right|^2 - C\left(\frac{h}{H}\right)^2 |\boldsymbol{u}|_1^2\|p\|_0^2.$$

This gives that

$$\begin{aligned}\frac{|\int_\Omega \nabla\cdot I^h(\boldsymbol{u})\,p\,dx|^2}{|\boldsymbol{u}|_1^2\|p\|_0^2} &\geq \frac{1}{2}\frac{|\int_\Omega \nabla\cdot\boldsymbol{u}\,p\,dx|^2}{|\boldsymbol{u}|_1^2\|p\|_0^2} - C\left(\frac{h}{H}\right)^2 \\ &\geq \frac{1}{2}\beta_Q^2 - C\left(\frac{h}{H}\right)^2.\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we prove that

$$\frac{|\int \nabla\cdot\widehat{\boldsymbol{u}}_E\,p\,dx|^2}{|\widehat{\boldsymbol{u}}_E|_1^2\|p\|_0^2} \geq C\left(\beta_Q^2 - \left(\frac{h}{H}\right)^2\right).$$

□

We will now prove that the inf-sup constant of $(\widehat{E}_{I,\Pi},\overline{P})$ depends on β_I and β_Π.

LEMMA 3.5. *The pair $(\widehat{E}_{I,\Pi},\overline{P})$ is inf-sup stable with the constant β such that*

$$\beta^2 \geq C\min\left\{\frac{\beta_I^2}{\beta_I^2+1}\beta_\Pi^2, \beta_I^2\right\},$$

where β_I and β_Π are inf-sup constants of the pairs (X_I,P_I) and $(\widehat{E}_\Pi,\overline{P}_\Pi)$, respectively.

Proof. We will prove that for any nonzero $p\in\overline{P}$ there exists $\boldsymbol{u}\in\widehat{E}_{I,\Pi}$ such that

$$\frac{\left(\sum_{i=1}^N\int_{\Omega_i}\nabla\cdot\boldsymbol{u}p\,dx\right)^2}{|\boldsymbol{u}|_1^2\|p\|_0^2} \geq C\min\left\{\frac{\beta_I^2}{\beta_I^2+1}\beta_\Pi^2, \beta_I^2\right\}.$$

We first decompose $p\in\overline{P}$ into

$$p = p_I + p_\Pi,$$

where $p_I\in P_I$ and $p_\Pi\in\overline{P}_\Pi$.

Case 1) We consider the case when p_I and p_Π satisfy that

$$\|p_I\|_0^2 \leq \|p_\Pi\|_0^2. \tag{3.10}$$

Since $(\widehat{E}_{I,\Pi},\overline{P}_\Pi)$ is inf-sup stable with the constant β_Π, for $p_\Pi\in\overline{P}_\Pi$ there exists $\widehat{\boldsymbol{u}}_E\in\widehat{E}_{I,\Pi}$ such that

$$\frac{\left(\sum_{i=1}^N\int_{\Omega_i}\nabla\cdot\widehat{\boldsymbol{u}}_E p_\Pi\,dx\right)^2}{|\widehat{\boldsymbol{u}}_E|_1^2\|p_\Pi\|_0^2} \geq \beta_\Pi^2. \tag{3.11}$$

In addition, for such $\widehat{\boldsymbol{u}}_E$ we solve the local Stokes problem to find $(\boldsymbol{u}_I^{(i)},p_I^*)\in X_I^{(i)}\times P_I^{(i)}$ such that

$$\begin{aligned}\int_{\Omega_i}\nabla\boldsymbol{u}_I^{(i)}\cdot\nabla\boldsymbol{v}_I\,dx - \int_{\Omega_i}\nabla\cdot\boldsymbol{v}_I p_I^*\,dx = 0, \quad \forall\boldsymbol{v}_I\in X_I^{(i)} \\ -\int_{\Omega_i}\nabla\cdot\boldsymbol{u}_I^{(i)}q_I\,dx = \int_{\Omega_i}\nabla\cdot\widehat{\boldsymbol{u}}_E q_I\,dx, \quad \forall q_I\in P_I^{(i)}.\end{aligned}$$

We note that $(X_I^{(i)}, P_I^{(i)})$ is also inf-sup stable with the constant β_I. Applying Lemma 4.1 of Section 4 with $\mu = 1$ and $\alpha = \infty$, we obtain

$$|\boldsymbol{u}_I^{(i)}|^2_{H^1(\Omega_i)} \leq C \frac{1}{\beta_I^2} |\widehat{\boldsymbol{u}}_E|^2_{H^1(\Omega_i)}. \tag{3.12}$$

We solve the local Stokes problem for each subdomain Ω_i and let

$$\boldsymbol{u} = \boldsymbol{u}_I + \widehat{\boldsymbol{u}}_E,$$

where $\boldsymbol{u}_I|_{\Omega_i} = \boldsymbol{u}_I^{(i)}$ for each Ω_i. We then have

$$\begin{aligned} \frac{\left(\sum_{i=1}^N \int_{\Omega_i} \nabla \cdot \boldsymbol{u}\, p\, dx\right)^2}{|\boldsymbol{u}|_1^2 \|p\|_0^2} &= \frac{\left(\sum_{i=1}^N \int_{\Omega_i} \nabla \cdot (\boldsymbol{u}_I + \widehat{\boldsymbol{u}}_E)(p_I + p_\Pi)\, dx\right)^2}{|\boldsymbol{u}_I + \widehat{\boldsymbol{u}}_E|_1^2 \|p_I + p_\Pi\|_0^2} \\ &= \frac{\left(\sum_{i=1}^N \int_{\Omega_i} \nabla \cdot (\boldsymbol{u}_I + \widehat{\boldsymbol{u}}_E)\, p_\Pi\, dx\right)^2}{|\boldsymbol{u}_I + \widehat{\boldsymbol{u}}_E|_1^2 \|p_I + p_\Pi\|_0^2} \\ &\geq C \frac{\left(\sum_{i=1}^N \int_{\Omega_i} \nabla \cdot \widehat{\boldsymbol{u}}_E\, p_\Pi\, dx\right)^2}{(|\boldsymbol{u}_I|_1^2 + |\widehat{\boldsymbol{u}}_E|_1^2)(\|p_I\|_0^2 + \|p_\Pi\|_0^2)}. \end{aligned} \tag{3.13}$$

Combining (3.10) and (3.12) with the above inequality (3.13) and using (3.11) we obtain

$$\begin{aligned} \frac{\left(\sum_{i=1}^N \int_{\Omega_i} \nabla \cdot \boldsymbol{u} p\, dx\right)^2}{|\boldsymbol{u}|_1^2 \|p\|_0^2} &\geq C \frac{\left(\sum_{i=1}^N \int_{\Omega_i} \nabla \cdot \widehat{\boldsymbol{u}}_E p_\Pi\, dx\right)^2}{\left(1 + \frac{1}{\beta_I^2}\right) |\widehat{\boldsymbol{u}}_E|_1^2 \|p_\Pi\|_0^2} \\ &\geq C \frac{\beta_I^2}{\beta_I^2 + 1} \beta_\Pi^2. \end{aligned} \tag{3.14}$$

Case 2) We now consider the other case when

$$\|p_\Pi\|_0^2 \leq \|p_I\|_0^2. \tag{3.15}$$

Similarly as before, since (X_I, P_I) is inf-sup stable with the constant β_I, for the given p_I we obtain $\boldsymbol{u}_I \in X_I$ satisfying

$$\frac{\left(\sum_{i=1}^N \int_{\Omega_i} \nabla \cdot \boldsymbol{u}_I p_I\, dx\right)^2}{|\boldsymbol{u}_I|_1^2 \|p_I\|_0^2} \geq \beta_I^2. \tag{3.16}$$

We let $\boldsymbol{u} = \boldsymbol{u}_I$ and using (3.15) and (3.16) we obtain

$$\begin{aligned} \frac{\left(\sum_{i=1}^N \int_{\Omega_i} \nabla \cdot \boldsymbol{u}\, p\, dx\right)^2}{|\boldsymbol{u}|_1^2 \|p\|_0^2} &= \frac{\left(\sum_{i=1}^N \int_{\Omega_i} \nabla \cdot \boldsymbol{u}_I (p_I + p_\Pi)\, dx\right)^2}{|\boldsymbol{u}_I|_1^2 (\|p_I\|_0^2 + \|p_\Pi\|_0^2)} \\ &\geq \frac{\left(\sum_{i=1}^N \int_{\Omega_i} \nabla \cdot \boldsymbol{u}_I\, p_I\, dx\right)^2}{|\boldsymbol{u}_I|_1^2 \|p_I\|_0^2} \\ &\geq \beta_I^2. \end{aligned} \tag{3.17}$$

The proof for the bound of the inf-sup constant β completes with the bounds in (3.14) and (3.17). □

REMARK 3.6. *Since β_I and β_Π are independent of any mesh parameters, the same holds for β.*

4. Condition number analysis. In this section, we will provide a condition number bound of the FETI-DP operator with the lumped preconditioner by proving the following inequalities:

$$C_1\beta^2\langle\widehat{M}\lambda,\lambda\rangle \leq \langle F_{DP}\lambda,\lambda\rangle \leq C_2\left(1+\log\frac{H}{h}\right)\left(\frac{H}{h}\right)\langle\widehat{M}\lambda,\lambda\rangle, \quad \forall\lambda\in M_c,$$

where β is the inf-sup constant of the pair $(\widehat{E}_{I,\Pi},\overline{P})$. This leads to a condition number bound

$$\kappa(\widehat{M}^{-1}F_{DP}) \leq C\frac{1}{\beta^2}\left(1+\log\frac{H}{h}\right)\left(\frac{H}{h}\right),$$

which is slightly weaker than that of the optimal Dirichlet preconditioner in the other FETI-DP formulations.

4.1. Lower bound analysis. We provide the analysis for the lower bound of the proposed FETI-DP algorithm. The following lemma is in [5, Lemma 2.3].

LEMMA 4.1. *Consider the discrete saddle point problem*

$$\begin{pmatrix}\mu A & B^T\\ B & -1/\alpha C\end{pmatrix}\begin{pmatrix}\boldsymbol{u}\\ p\end{pmatrix}=\begin{pmatrix}\boldsymbol{f}\\ g\end{pmatrix},$$

where A and C are positive definite and, if $\alpha=\infty$, B has full row rank. Let $\beta\geq 0$ be the best inf-sup constant such that

$$p^TBA^{-1}B^Tp \geq \beta^2p^TCp, \quad \forall p.$$

Then,

$$\|\boldsymbol{u}\|_A \leq 1/\mu\|\boldsymbol{f}\|_{A^{-1}}+\frac{1}{\sqrt{\beta^2+\mu/\alpha}}\|g\|_{C^{-1}},$$
$$\|p\|_C \leq \frac{1}{\sqrt{\beta^2+\mu/\alpha}}\|\boldsymbol{f}\|_{A^{-1}}+\frac{\mu}{\beta^2+\mu/\alpha}\|g\|_{C^{-1}}.$$

We introduce a matrix K, which gives the H^1-seminorm for $\boldsymbol{u}=(\boldsymbol{u}_I,\boldsymbol{u}_\Delta,\widehat{\boldsymbol{u}}_\Pi)\in\widetilde{X}$, i.e.,

$$\sum_{i=1}^N|\boldsymbol{u}|^2_{H^1(\Omega_i)}=\boldsymbol{u}^TK\boldsymbol{u},$$

where K is given by the block matrices in (2.3),

$$K=\begin{pmatrix}K_{II} & K_{I\Delta} & K_{I\Pi}\\ K_{I\Delta}^T & K_{\Delta\Delta} & K_{\Delta\Pi}\\ K_{I\Pi}^T & K_{\Delta\Pi}^T & K_{\Pi\Pi}\end{pmatrix}.$$

With the help of the pair $(\widehat{E}_{I,\Pi}, \overline{P})$, we obtain the following lemma.

LEMMA 4.2. *For any $\mu \in M_c$, there exists $\boldsymbol{u} \in \widetilde{X}$ such that*
1. $J_\Delta \boldsymbol{u}_\Delta = \mu$,
2. $\sum_i \int_{\Omega_i} \nabla \cdot \boldsymbol{u}\, q\, dx = 0, \quad \forall q \in P$,
3. $\langle K\boldsymbol{u}, \boldsymbol{u}\rangle \leq C\frac{1}{\beta^2}\langle K_{\Delta\Delta} J_\Delta^T J_\Delta \boldsymbol{u}_\Delta, J_\Delta^T J_\Delta \boldsymbol{u}_\Delta\rangle$*, where β is the inf-sup constant of the pair $(\widehat{E}_{I,\Pi}, \overline{P})$.*

Proof. For any $\mu \in M_c$, we let

$$\boldsymbol{w}_\Delta^{(i)}|_{\Gamma_{ij}} = \frac{1}{2}\mu|_{\Gamma_{ij}}, \quad \boldsymbol{w}_\Delta^{(j)}|_{\Gamma_{ij}} = -\frac{1}{2}\mu|_{\Gamma_{ij}}. \tag{4.1}$$

For the given $\boldsymbol{w}_\Delta$, we find $\boldsymbol{w}_I \in X_I$, $\widehat{\boldsymbol{w}}_E \in \widehat{E}_\Pi$, and $p \in \overline{P}$ such that

$$\begin{aligned}
\sum_i \int_{\Omega_i} \nabla(\boldsymbol{w}_I + \boldsymbol{w}_\Delta + \widehat{\boldsymbol{w}}_E)\cdot \nabla \boldsymbol{v}_I\, dx - \sum_i \int_{\Omega_i} \nabla \cdot \boldsymbol{v}_I p\, dx = 0, \quad \forall \boldsymbol{v}_I \in X_I,\\
\sum_i \int_{\Omega_i} \nabla(\boldsymbol{w}_I + \boldsymbol{w}_\Delta + \widehat{\boldsymbol{w}}_E)\cdot \nabla \widehat{\boldsymbol{v}}_E\, dx - \sum_i \int_{\Omega_i} \nabla \cdot \widehat{\boldsymbol{v}}_E p\, dx = 0, \quad \forall \widehat{\boldsymbol{v}}_E \in \widehat{E}_\Pi,\\
-\sum_i \int_{\Omega_i} \nabla \cdot (\boldsymbol{w}_I + \boldsymbol{w}_\Delta + \widehat{\boldsymbol{w}}_E) q\, dx = 0, \quad \forall q \in \overline{P}.
\end{aligned} \tag{4.2}$$

We let $\boldsymbol{u} = \boldsymbol{w}_I + \boldsymbol{w}_\Delta + \widehat{\boldsymbol{w}}_E$ and will show that $\boldsymbol{u}$ satisfies the three requirements.

We represent $\widehat{\boldsymbol{w}}_E$ with a vector of unknowns in the space $\widetilde{X}$,

$$\widehat{\boldsymbol{w}}_E = (\boldsymbol{z}_I, \boldsymbol{z}_\Delta, \widehat{\boldsymbol{w}}_\Pi),$$

and obtain $\boldsymbol{u}$ as the vector of unknowns of the form,

$$\boldsymbol{u} = (\boldsymbol{w}_I + \boldsymbol{z}_I, \boldsymbol{w}_\Delta + \boldsymbol{z}_\Delta, \widehat{\boldsymbol{w}}_\Pi),$$

so that we have

$$\boldsymbol{u}_\Delta = \boldsymbol{w}_\Delta + \boldsymbol{z}_\Delta.$$

Since $\widehat{\boldsymbol{w}}_E \in \widehat{X}$, $J_\Delta \boldsymbol{z}_\Delta = 0$. This gives that

$$J_\Delta \boldsymbol{u}_\Delta = J_\Delta \boldsymbol{w}_\Delta = \mu \tag{4.3}$$

and

$$\begin{aligned}
\sum_{i=1}^N \int_{\Omega_i} \nabla \cdot \boldsymbol{u} &= \sum_{ij} \int_{\Gamma_{ij}} (\boldsymbol{w}_\Delta^{(i)} - \boldsymbol{w}_\Delta^{(j)}) \cdot \boldsymbol{n}_{ij}\, ds\\
&= \sum_{ij} (\zeta_{ij}\boldsymbol{\mu}_{ij}) \cdot \boldsymbol{n}_{ij} = \sum_{ij} \boldsymbol{\mu}_{ij} \cdot (\zeta_{ij}\boldsymbol{n}_{ij}),
\end{aligned}$$

where $\boldsymbol{\mu}_{ij} = \mu|_{\Gamma_{ij}}$ and ζ_{ij} is given by (2.15). Here we used that $\boldsymbol{w}_\Delta^{(i)} - \boldsymbol{w}_\Delta^{(j)}|_{\Gamma_{ij}} = \boldsymbol{\mu}_{ij}$. Since $\mu \in M_c$, the above equation is zero so that the second requirement is proved for $\boldsymbol{u}$, combined with (4.2).

We write the weak form in (4.2) into the algebraic equations,

$$\begin{pmatrix} K_{II} & K_{IE} & B_I^T \\ K_{EI} & K_{EE} & B_E^T \\ B_I & B_E & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{w}_I \\ \widehat{\boldsymbol{w}}_E \\ p \end{pmatrix} = \begin{pmatrix} -K_{I\Delta}\boldsymbol{w}_\Delta \\ -K_{E\Delta}\boldsymbol{w}_\Delta \\ -B_\Delta \boldsymbol{w}_\Delta \end{pmatrix}. \tag{4.4}$$

Let

$$A = \begin{pmatrix} K_{II} & K_{IE} \\ K_{EI} & K_{EE} \end{pmatrix}, \quad B = \begin{pmatrix} B_I & B_E \end{pmatrix}.$$

We let C be the mass matrix which gives the L^2-norm of functions in the space $\overline{P}$, i.e.,

$$\langle Cq, q\rangle = \|q\|^2_{L^2(\Omega)}, \quad \text{for } q \in \overline{P}.$$

Since $(\widehat{E}_{I,\Pi}, \overline{P})$ is inf-sup stable, the pair (A, B) satisfies the inf-sup condition with the constant β and the matrix B has full row rank. We apply Lemma 4.1 to the mixed problem (4.4) with $\mu = 1$ and $\alpha = \infty$ to obtain

$$\left\| \begin{pmatrix} \boldsymbol{w}_I \\ \widehat{\boldsymbol{w}}_E \end{pmatrix} \right\|_A^2 \le 2 \left\| \begin{pmatrix} K_{I\Delta}\boldsymbol{w}_\Delta \\ K_{E\Delta}\boldsymbol{w}_\Delta \end{pmatrix} \right\|_{A^{-1}}^2 + \frac{2}{\beta^2} \|B_\Delta \boldsymbol{w}_\Delta\|^2_{C^{-1}}. \tag{4.5}$$

Here $\|\boldsymbol{v}\|_A^2 = \langle A\boldsymbol{v}, \boldsymbol{v}\rangle$.

The first term of (4.5) is bounded by

$$\begin{aligned} \left\| \begin{pmatrix} K_{I\Delta}\boldsymbol{w}_\Delta \\ K_{E\Delta}\boldsymbol{w}_\Delta \end{pmatrix} \right\|_{A^{-1}}^2 &= \max_{\boldsymbol{v}_I + \widehat{\boldsymbol{v}}_E \in \widehat{E}_{I,\Pi}} \frac{\left(\sum_i \int_{\Omega_i} \nabla \boldsymbol{w}_\Delta \cdot \nabla(\boldsymbol{v}_I + \widehat{\boldsymbol{v}}_E)\, dx\right)^2}{\sum_i \int_{\Omega_i} |\nabla(\boldsymbol{v}_I + \widehat{\boldsymbol{v}}_E)|^2\, dx} \\ &\le \sum_i \int_{\Omega_i} |\nabla \boldsymbol{w}_\Delta|^2\, dx \\ &= \langle K_{\Delta\Delta}\boldsymbol{w}_\Delta, \boldsymbol{w}_\Delta\rangle. \end{aligned} \tag{4.6}$$

The second term of (4.5) is estimated by

$$\begin{aligned} \|B_\Delta \boldsymbol{w}_\Delta\|^2_{C^{-1}} &= \max_{q \in \overline{P}} \frac{\left(\sum_i \int_{\Omega_i} \nabla \cdot \boldsymbol{w}_\Delta q\, dx\right)^2}{\sum_i \int_{\Omega_i} q^2\, dx}, \\ &\le C \sum_i \int_{\Omega_i} |\nabla \boldsymbol{w}_\Delta|^2\, dx, \\ &= C\langle K_{\Delta\Delta}\boldsymbol{w}_\Delta, \boldsymbol{w}_\Delta\rangle. \end{aligned} \tag{4.7}$$

Since K gives the H^1-seminorm for $\boldsymbol{u} = \boldsymbol{w}_I + \boldsymbol{w}_\Delta + \widehat{\boldsymbol{w}}_E$,

$$\langle K\boldsymbol{u}, \boldsymbol{u}\rangle \le 2\langle K_{\Delta\Delta}\boldsymbol{w}_\Delta, \boldsymbol{w}_\Delta\rangle + 2 \left\| \begin{pmatrix} \boldsymbol{w}_I \\ \widehat{\boldsymbol{w}}_E \end{pmatrix} \right\|_A^2.$$

This bound combined with (4.5)-(4.7) gives

$$\langle K\boldsymbol{u},\boldsymbol{u}\rangle \le \frac{C}{\beta^2}\langle K_{\Delta\Delta}\boldsymbol{w}_\Delta,\boldsymbol{w}_\Delta\rangle. \tag{4.8}$$

From (4.1) and (4.3), $\boldsymbol{w}_\Delta$ satisfies that

$$\boldsymbol{w}_\Delta = \frac{1}{2}J_\Delta^T J_\Delta \boldsymbol{w}_\Delta = \frac{1}{2}J_\Delta^T J_\Delta \boldsymbol{u}_\Delta,$$

and (4.8) combined with this equality gives the desired bound for $\langle K\boldsymbol{u},\boldsymbol{u}\rangle$. ∎

THEOREM 4.3. *For any $\lambda \in M_c$, we have*

$$C_1\beta^2\langle\widehat{M}\lambda,\lambda\rangle \le \langle F_{DP}\lambda,\lambda\rangle,$$

where β is the inf-sup constant of the pair $(\widehat{E}_{I,\Pi},\overline{P})$ and C_1 is a positive constant that does not depend on any mesh parameters.

Proof. We recall that

$$P = \prod_i P^{(i)}.$$

We introduce

$$\widetilde{X}(\mathrm{div}) = \left\{\boldsymbol{v}\in\widetilde{X}\,:\,\int_{\Omega_i}\nabla\cdot\boldsymbol{v}q\,dx = 0,\quad \forall q\in P\right\}. \tag{4.9}$$

We then have the identity,

$$\langle F_{DP}\lambda,\lambda\rangle = \max_{\boldsymbol{v}\in\widetilde{X}(\mathrm{div})}\frac{\langle J_\Delta\boldsymbol{v}_\Delta,\lambda\rangle^2}{\langle K\boldsymbol{v},\boldsymbol{v}\rangle}. \tag{4.10}$$

For any $\mu\in M_c$, we can select $\boldsymbol{u}$ satisfying the three conditions in Lemma 4.2. It gives that

$$\begin{aligned}\langle F_{DP}\lambda,\lambda\rangle &\ge \frac{\langle J_\Delta\boldsymbol{u}_\Delta,\lambda\rangle^2}{\langle K\boldsymbol{u},\boldsymbol{u}\rangle}\\ &\ge C\beta^2\frac{\langle\mu,\lambda\rangle^2}{\langle K_{\Delta\Delta}J_\Delta^T\mu,J_\Delta^T\mu\rangle}.\end{aligned}$$

Since μ is arbitrary, we obtain

$$\begin{aligned}\langle F_{DP}\lambda,\lambda\rangle &\ge C\beta^2\max_{\mu\in M_c}\frac{\langle\mu,\lambda\rangle^2}{\langle\widehat{M}^{-1}\mu,\mu\rangle}\\ &= C\beta^2\langle\widehat{M}\lambda,\lambda\rangle.\end{aligned}$$

∎

4.2. Upper bound analysis. We refer the following result in Li and Widlund [17, Lemma 4]:

LEMMA 4.4. *Let Ω_i be a two-dimensional subdomain. For any $u^{(i)}\in X^{(i)}$,*

$$\|u^{(i)} - I^{H,\Gamma_{ij}}u^{(i)}\|^2_{L^2(\Gamma_{ij})} \le CH\left(1+\log\frac{H}{h}\right)|u^{(i)}|^2_{H^1(\Omega_i)},$$

where Γ_{ij} is an edge of the subdomain Ω_i and $I^{H,\Gamma_{ij}}u^{(i)}$ is the linear function on Γ_{ij} with its values at the two end points equal to those of $u^{(i)}$.

LEMMA 4.5. *There exists a constant C such that*

$$\langle K_{\Delta\Delta} J_\Delta^T J_\Delta \boldsymbol{u}_\Delta, J_\Delta^T J_\Delta \boldsymbol{u}_\Delta \rangle \le C \frac{H}{h}\left(1+\log\frac{H}{h}\right)\langle K\boldsymbol{u},\boldsymbol{u}\rangle, \quad \textit{for any } \boldsymbol{u}\in\widetilde{X}.$$

Proof. From the inverse inequality, we obtain

$$\langle K_{\Delta\Delta} J_\Delta^T J_\Delta \boldsymbol{u}_\Delta, J_\Delta^T J_\Delta \boldsymbol{u}_\Delta \rangle \le Ch^{-1}\sum_{i,j}\|\boldsymbol{u}_\Delta^{(i)}-\boldsymbol{u}_\Delta^{(j)}\|^2_{L^2(\Gamma_{ij})}, \tag{4.11}$$

where Γ_{ij} is the common interface of Ω_i and Ω_j. Let $\boldsymbol{u}^{(i)}=\boldsymbol{u}|_{\Omega_i}$. Since $\boldsymbol{u}$ is continuous at the subdomain corners, we have

$$\begin{aligned}\|\boldsymbol{u}_\Delta^{(i)}-\boldsymbol{u}_\Delta^{(j)}\|^2_{L^2(\Gamma_{ij})} &= \|\boldsymbol{u}^{(i)}-\boldsymbol{u}^{(j)}\|^2_{L^2(\Gamma_{ij})}\\ &\le 2\|\boldsymbol{u}^{(i)}-I^{H,\Gamma_{ij}}\boldsymbol{u}^{(i)}\|^2_{L^2(\Gamma_{ij})}+2\|\boldsymbol{u}^{(j)}-I^{H,\Gamma_{ij}}\boldsymbol{u}^{(j)}\|^2_{L^2(\Gamma_{ij})}.\end{aligned}\tag{4.12}$$

Here $I^{H,\Gamma_{ij}}\boldsymbol{u}$ is the linear function on the interface Γ_{ij} with its values at the two end points equal to those of $\boldsymbol{u}$. From (4.11), (4.12), and by applying Lemma 4.4 to the two terms in (4.12), the desired bound then follows. □

THEOREM 4.6. *For any $\lambda\in M_c$, we have*

$$\langle F_{DP}\lambda,\lambda\rangle \le C_2\frac{H}{h}\left(1+\log\frac{H}{h}\right)\langle\widehat{M}\lambda,\lambda\rangle,$$

where C_2 is a positive constant that does not depend on any mesh parameters.

Proof. We recall the space $\widetilde{X}(\text{div})$ in (4.9) and the identity in (4.10). Any $\boldsymbol{v}=(\boldsymbol{v}_I,\boldsymbol{v}_\Delta,\boldsymbol{v}_\Pi)\in\widetilde{X}(\text{div})$ satisfies the bound in Lemma 4.5 so that

$$\langle F_{DP}\lambda,\lambda\rangle \le C\frac{H}{h}\left(1+\log\frac{H}{h}\right)\max_{\boldsymbol{v}\in\widetilde{X}(\text{div})}\frac{\langle J_\Delta\boldsymbol{v}_\Delta,\lambda\rangle^2}{\langle K_{\Delta\Delta}J_\Delta^T J_\Delta\boldsymbol{v}_\Delta, J_\Delta^T J_\Delta\boldsymbol{v}_\Delta\rangle}.$$

We note that $J_\Delta\boldsymbol{v}_\Delta\in M_c$ for any $\boldsymbol{v}=(\boldsymbol{v}_I,\boldsymbol{v}_\Delta,\boldsymbol{v}_\Pi)\in\widetilde{X}(\text{div})$; see Lemma 3.2. Therefore, we obtain

$$\langle F_{DP}\lambda,\lambda\rangle \le C\frac{H}{h}\left(1+\log\frac{H}{h}\right)\max_{\mu\in M_c}\frac{\langle\mu,\lambda\rangle^2}{\langle\widehat{M}^{-1}\mu,\mu\rangle}.$$

The desired bound then follows. □

5. Numerical results. We consider a model Stokes problem in a unit rectangular domain Ω with the exact solution given by

$$\boldsymbol{u}=\begin{pmatrix}\sin^3(\pi x)\sin^2(\pi y)\cos(\pi y)\\ -\sin^2(\pi x)\sin^3(\pi y)\cos(\pi x)\end{pmatrix},\ p=x^2-y^2.$$

N	$\beta_\Pi\,(H/h=4)$	$\beta_\Pi\,(H/h=8)$	H/h	$\beta_\Pi\,(N=4^2)$	$\beta_\Pi\,(N=8^2)$
4^2	0.3435	0.3548	4	0.3435	0.2838
8^2	0.2838	0.2880	8	0.3548	0.2880
16^2	0.2529	0.2551	16	0.3599	0.2898
32^2	0.2342	0.2356	32	0.3622	0.2905
64^2	0.2221	0.2230	64	0.3634	0.2909

TABLE 1

Left three columns: the values of β_Π for the space $(\widehat{E}_\Pi, \overline{P}_\Pi)$ when the number of subdomains N increases with a fixed local problem size $(H/h = 4, 8)$. Right three columns: the values of β_Π for the space $(\widehat{E}_\Pi, \overline{P}_\Pi)$ when the local problem size H/h increases in a fixed subdomain partition $(N = 4^2,\ 8^2)$.

For this model problem, we test our FETI-DP algorithm with the lumped preconditioner. We perform the conjugate gradient iteration with the preconditioner until the relative residual norm is reduced by a factor of 10^6. We consider a pair of the inf-sup stable finite element $(P_1(h), P_0(2h))$ for a given triangulation in Ω and then divide the unit rectangular domain Ω into uniform rectangular subdomain partitions. These partitions align with the given triangulation in Ω, i.e., they do not cut triangles equipped for Ω. The finite element of each subdomain is then inherited from the finite element space provided for Ω.

For such discrete model problems, we first compute inf-sup constant β_Π of the corresponding pair $(\widehat{E}_\Pi, \overline{P}_\Pi)$. We note that in Lemma 3.4 we obtained an improved estimate for β_Π when the subdomains are rectangular. In Table 1, those numbers are reported and they seem to be stable as H/h increases and as N increases. We observe that the values of β_Π^2 tend to follow the bound $C(\beta_Q^2 - (h/H)^2)$ as h/H gets smaller. As we proved in the condition number analysis, from these results we then expect that the convergence of the FETI-DP algorithm depends on $C(H/h)(1+\log(H/h))$ with the factor C independent of the mesh parameters. In other words, the values of β_Π do not deteriorate the convergence of our FETI-DP algorithm.

We now perform our FETI-DP algorithm to verify the condition number bound. In Tables 2 and 3, we report the condition numbers and the number of iterations as the domain Ω is divided into more subdomains with their problem size fixed, and as the local problem size increases in a fixed subdomain partition, respectively. In addition, the L^2-errors are presented. We can see that the condition numbers are consistent with our theory and the new formulation gives the approximate solutions with the optimal order of L^2-errors.

We note that a FETI-DP algorithm for the elliptic problems with a lumped preconditioner was developed and a bound of its condition number was analyzed in [17]. Our FETI-DP algorithm is an extension of the FETI-DP algorithm to the Stokes problem. Moreover, we proved the same condition number bound, $C(H/h)(1 + \log(H/h))$ for the Stokes problem. In Table 4, we compare the numerical results of the elliptic problem from [17] and the Stokes

N	Iter	κ	λ_{min}	λ_{max}	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$	$\|p-p_h\|_0$
2^2	9	4.3143e+00	2.5985	1.1211e+01	8.4678e-03	1.1932e-01
4^2	16	1.1722e+01	2.5452	2.9835e+01	2.2282e-03	6.5222e-02
8^2	21	1.3676e+01	2.5040	3.4244e+01	5.6482e-04	3.3344e-02
12^2	21	1.4023e+01	2.4975	3.5022e+01	2.5172e-04	2.2319e-02
16^2	22	1.4138e+01	2.4943	3.5264e+01	1.4172e-04	1.6763e-02

TABLE 2

Scalability as the increase of the number of subdomains N with a fixed local problem size ($H/h = 8$): the number of iterations Iter, the condition numbers κ, the minimum eigenvalues λ_{min}, the maximum eigenvalues λ_{max}, L^2-errors for the velocity $\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$, and L^2-errors for the pressure $\|p-p_h\|_0$

H/h	Iter	κ	λ_{min}	λ_{max}	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$	$\|p-p_h\|_0$
4	12	5.0922e+00	2.6398	1.3442e+01	8.4678e-03	1.1932e-01
8	16	1.1722e+01	2.5452	2.9835e+01	2.2282e-03	6.5222e-02
16	24	2.7844e+01	2.5415	7.0766e+01	5.6482e-04	3.3344e-02
20	26	3.6449e+01	2.5690	9.3638e+01	3.6214e-04	2.6745e-02
26	29	5.0406e+01	2.5792	1.3001e+02	2.1456e-04	2.0612e-02
32	32	6.5079e+01	2.5859	1.6833e+02	1.4172e-04	1.6763e-02

TABLE 3

Scalability as the increase of the local problem size H/h in a fixed subdomain partition ($N = 4^2$): the number of iterations Iter, the condition numbers κ, the minimum eigenvalues λ_{min}, the maximum eigenvalues λ_{max}, L^2-errors for the velocity $\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$, and L^2-errors for the pressure $\|p-p_h\|_0$

problem with the lumped preconditioner. We can observe similar behaviors of the condition numbers and the number of iterations.

In the FETI-DP algorithm for the Stokes problem by Li and Widlund [15], both the primal velocity and the primal pressure components are introduced and the Dirichlet preconditioner is used. In their work, the velocity averages on the edges in addition to the velocity unknowns at the corners are selected as the primal velocity components and the optimal condition number bound, $C(1 + \log(H/h))^2$ was proved.

In our FETI-DP formulation, only the velocity unknowns at the subdomain corners are employed and no primal pressure component is used. Thus, compared to the FETI-DP algorithm in [15], we have a smaller and more stable coarse problem, that is symmetric and positive definite, than an indefinite mixed problem appeared in the FETI-DP formulation in [15]. At each iteration, our algorithm solves one local Stokes problem in each subdomain, while the previous algorithm solves two Stokes problems in each subdomain; one with a Neumann boundary condition and the other with a Dirichlet boundary condition. So that our algorithm results in a less computing time at each iteration.

	Elliptic		Stokes			Elliptic		Stokes	
H/h	κ	Iter	κ	Iter	N	κ	Iter	κ	Iter
4	3.3	9	5.1	12	4^2	8.3	12	11.7	16
8	8.3	12	11.7	16	8^2	10.8	19	13.6	21
16	19.6	16	27.8	24	12^2	11.2	19	14.0	21
32	56.7	22	65.1	32	16^2	11.3	19	14.1	22

TABLE 4

Performance of the FETI-DP algorithm with a lumped preconditioner when the local problem size H/h increases in a subdomain partition with $N = 4^2$ and when the number of subdomains N increases with a fixed local problem size ($H/h = 8$): κ (condition numbers), Iter (the number of iterations)

Introduction of additional primal unknowns such as the edge averages was necessary to achieve the optimal convergence with a Dirichlet preconditioner. It often makes the implementation much harder especially for three dimensional problems. In the present work, a new FETI-DP algorithm for the Stokes problem that employs a lumped preconditioner and a more stable coarse problem is developed and analyzed. The new approach makes the implementation much simpler in three dimensions. Extension to the three dimensional Stokes problem will be addressed in the forthcoming work.

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