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Invariance property of a conservation law without convexity ⋆

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Abstract

The main goal of this paper is to investigate the mechanism of a conservation law that gives the N-wave like asymptotics. It turns out that the positivity of the flux function provides certain invariance of solutions which singles out the right asymptotics among two parameter family of N-waves. Two kinds of optimal convergence orders in L^1-norm to the N-wave are proved using a potential comparison technique. The first one is of the magnitude of the N-wave itself and the second one is of order $1/t$. One may easily see that these asymptotic convergence orders are related to space and time translation of potentials.

Key words: asymptotics, characteristics, convergence order, N-wave, similarity

1 Introduction

This paper is devoted to a study of the long time asymptotics of bounded L^1 solutions to a general scalar conservation law,

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x), \quad x \in \boldsymbol{R},\ t > 0, \tag{1}$$

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where $u_0 \in L^1(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R})$ is compactly supported. We assume that the flux is continuously differentiable and that

$$f(0) = f'(0) = 0. \tag{2}$$

One can get this normalization assumption without loss of generality after a suitable change of variables. In this paper we consider a non-convex flux that satisfies the following hypotheses:

$$\begin{aligned} &f(u) \geq 0 \text{ for all } u \in \boldsymbol{R}, \\ &f(u) \text{ has a finite number of inflection points}, \\ &f(u)/|u| \to \infty \text{ as } |u| \to \infty. \end{aligned} \tag{H}$$

Notice that the first assumption implies that $u = 0$ is a global minimum point of the flux. The other two hypotheses are technical. The second one about a finite number of inflection points has been used to construct source-type solutions in [11] and that is why we have it here. If one considers a bounded solution, the value of the flux at $|u|$ near infinity does not make any difference and therefore one may assume the third one. In fact we assume that the divergence is monotone for $|u|$ large.

It is well known that N-waves are the long time asymptotics of sign-changing solutions to a general conservation law with a convex flux. In this paper we will see that the first hypothesis in (H) is the essential feature of a conservation law that produce the N-wave like long time asymptotics. The asymptotic convergence order depends on the structure of initial value and one can not expect any L^1 convergence order with the generality of L^1 initial value. In this paper we consider compactly supported initial value such that

$$\int u_0(x)dx = M < \infty, \quad \operatorname{spt}(u_0) \subset [-L, L], \;\; L \in \boldsymbol{R}. \tag{3}$$

Under the hypotheses in (H) the well-posedness of the fundamental or source-type solutions are obtained in [18]. One can easily find the explicit formula of an N-wave for a convex case. For a general non-convex case under (H) fundamental solutions have been recently constructed in [11]. N-waves are two parameters family of functions and we denote them by $n_{p,q}(x,t)$. The N-waves are non-negative $n_{p,q}(x,t) \geq 0$ for $x \geq 0$ and non-positive $n_{p,q}(x,t) \leq 0$ for $x \leq 0$. We set two parameters of the N-wave as

$$p = -\int_{-\infty}^{0} n_{p,q}(y,t)dy, \qquad q = \int_{0}^{\infty} n_{p,q}(y,t)dy. \tag{4}$$

Notice Hypothesis (H) provides certain invariance property to the solutions and makes the integrals in (4) be constant for all $t > 0$.

In this paper we show two kinds of convergence orders of a general solution $u(x,t)$ to the N-wave $n_{p,q}(x,t)$. First we show that if the initial value satisfies

$$p = -\inf_x \int_{-\infty}^{x} u_0(y)dy > 0, \qquad q = M + p > 0, \tag{5}$$

the solution $u(x,t)$ converges to the N-wave $n_{p,q}(x,t)$ as $t \to \infty$ with the convergence order

$$\|u(t) - n_{p,q}(t)\|_1 = O\Big(\|n_{p,q}(t)\|_\infty\Big) \ \text{ as } \ t \to \infty. \tag{6}$$

One may expect a higher convergence order by placing the N-wave at the correct location. In fact under an extra condition on the initial value (15), we will show that there exists $c \in \boldsymbol{R}$ such that

$$\|u(t) - n^c_{p,q}(t)\|_1 = O(\|f(n_{p,q}(t))\|_\infty) \ \text{ as } \ t \to \infty, \tag{7}$$

where $n^c_{p,q}$ is a space translation $n^c_{p,q}(x,t) = n_{p,q}(x-c,t)$. This convergence order turns out to be order $O(1/t)$ under a general assumption

$$\liminf_{u \to 0} \frac{uf'(u)}{f(u)} = \gamma > 1. \tag{$H1$}$$

Similar convergence orders in L^1-norm can be found from the literature. For example the Barenblatt-type solution is a source solution of a nonlinear diffusion equation and decays with certain order depending on the dimension and the flux. The L^1 convergence of exactly this order can be found in various cases [3,4,9,16,20]. The convergence order $O(1/t)$ has been obtained for radial solutions, for solutions to fast diffusion equations [5,14,19,21] and for solutions to its linearized problems [7,22].

For solutions to scalar conservation laws the L^1 convergence order in (6) has been observed for convex cases [2,8,15,12,23]. Convergence order of (7) are found in [10,12]. For the case with a non-convex flux one can find well-posedness and other estimates from [1,6,24]. However, the behavior of the solution is not well understood. Recently N-waves for the non-convex case has been suggested in [11] and the convergence orders in (6,7) have been obtained for positive solutions [13].

The rest of the paper consists as followings. In Section 2 several preliminaries are given including the definition of entropy solutions, their potentials and the potential comparison principle. The main results are given in Theorem 1 which consists of three parts. In the succeeding three sections each of these three are proved. In Section 3 the invariance property of conservation laws is shown under Hypotheses in (H). The convergence orders in (6) and (7) are obtained in Sections 4 and 5, respectively.

2 Preliminaries and main results

We consider a weak solution $u(x,t)$ of (1) that satisfies

$$\int\int(u\phi_t + f(u)\phi_x)dxdt + \int u_0(x)\phi(x,0)dx = 0 \tag{8}$$

for any test function $\phi \in C_0^\infty(\boldsymbol{R}\times[0,\infty))$. If a weak solution has a discontinuity at $x = \xi(t)$, then its propagation speed is given by the Rankine-Hugoniot jump condition

$$\xi'(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r}, \qquad u_l = \lim_{y\uparrow x} u(y,t),\ \ u_r = \lim_{y\downarrow x} u(y,t). \tag{9}$$

Since a weak solution is not unique, one should consider a weak solution with a suitable admissibility condition to single out the physically right one. For a non-convex flux the Oleinik entropy condition is the one which satisfies

$$l(u) \le f(u) \text{ for all } u_l < u < u_r, \text{ and } \quad l(u) \ge f(u) \text{ for all } u_r < u < u_l, \tag{10}$$

where $l(u)$ is the linear function connecting two states u_r and u_l, i.e.,

$$l(u) = f(u_l) + \frac{f(u_l) - f(u_r)}{u_l - u_r}(u - u_l).$$

It is well known that the problem is well-posed under the entropy admissibility condition in the class of bounded and measurable solutions (see [1]) and we consider this unique solution only. It is also known that $u(x,t)$ is a solution if and only if it satisfies the conditions (9-10) at discontinuities and the conservation law in smooth regions.

Our approach for the asymptotic convergence is based on a potential comparison technique. We take the primitive of the solution,

$$U(x,t) = \int_{-\infty}^{x} u(y,t)dy, \quad U_0(x) = \int_{-\infty}^{x} u_0(y)dy, \tag{11}$$

as the potential of the solution $u(x,t)$. The potential of the N-wave $n_{p,q}(x,t)$ is similarly given by

$$N_{p,q}(x,t) = \int_{-\infty}^{x} n_{p,q}(y,t)dy. \tag{12}$$

Notice that N-waves are usually denoted using the capital letter N. However, we denote an N-wave as $n_{p,q}(x,t)$ to denote its potential with a capital letter. Now we are ready to state the main results of this paper:

Theorem 1 *Let $u(x,t)$ be the entropy solution of (1) with initial value $u_0 \in L^1(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R})$ that satisfies (3,5). Let $n_{p,q}(x,t)$ be the N-wave satisfying (4) and $-p = \int_{-\infty}^{c} u_0(y)dy$ for certain $c \in \boldsymbol{R}$. If the smooth flux f satisfies (2) and (H), then the followings hold.*

(i) For all $t > 0$,

$$-p = \inf_x \int_{-\infty}^{x} u(y,t)dy = \int_{-\infty}^{c} u(y,t)dy, \quad q = \int_{c}^{\infty} u(y,t)dy. \tag{13}$$

(ii)

$$\|u(x,t) - n_{p,q}(x,t)\|_1 = O(\max_x |n_{p,q}(x,t)|\ ) \quad as \quad t \to \infty. \tag{14}$$

(iii) Furthermore, if the point $c \in \boldsymbol{R}$ satisfying $-p = \int_{-\infty}^{c} u_0(y)dy$ is unique, $p, q > 0$ and there exist constants $\alpha, \varepsilon > 0$ satisfying

$$\begin{aligned} u_0(x+c) &\geq n_{p,q}(x,\alpha), & 0 \leq x \leq \varepsilon, \\ u_0(x+c) &\leq n_{p,q}(x,\alpha), & -\varepsilon \leq x \leq 0, \end{aligned} \tag{15}$$

then there exist constants $T, C > 0$ such that

$$\|u(t) - n_{p,q}(\cdot + c, t)\|_1 \leq C\|f(n_{p,q}(t))\|_\infty \quad for \quad t > T. \tag{16}$$

The proof of the theorem is based on a potential comparison technique, which has been developed for nonlinear diffusion [14] and then applied to positive solution of conservation laws. The proof of the following comparison principle is given in [13] for positive solutions and it can be directly employed for sign changing cases. In the following we present the proof briefly.

Proposition 2 (Potential comparison) *Let $U_i(x,t)$, $i=1,2$, be the potentials of two integrable solutions u_i, $i=1,2$, respectively. If $U_1(x,0)\le U_2(x,0)$ for all $x\in\boldsymbol{R}$, then $U_1(x,t)\le U_2(x,t)$ for all $x\in\boldsymbol{R}, t>0$.*

PROOF. Roughly speaking, after an integration of (1) on interval $-\infty<y<x$, one obtains $U_t+f(u)=0$ in a weak sense and, hence, $E(x,t)=U_1(x,t)-U_2(x,t)$ is a weak solution of

$$E_t+a(x,t)E_x=0,\quad a(x,t)=(f(u_1)-f(u_2))/(u_1-u_2),$$

where $a(x,t)$ is understood as the derivative of the smooth flux if $u_1=u_2$. Hence the characteristic for E is same as the ones of solutions u_1 and u_2 if $u_1=u_2$ and, otherwise, it is between them. Since E is constant along the characteristics and $E(x,0)\ge 0$, we have $E(x,t)\ge 0$, i.e., $U_1(x,t)\le U_2(x,t)$ for all $x\in\boldsymbol{R}, t>0$. □

3 Invariance property

Theorem 1(i) claims two invariant quantities, which are the global minimum value p of the potential $U(\cdot,t)$ and the minimum point $x=c$. For the proof we study how does a local extremum of a potential evolve. For the convex flux case the invariance has been shown in [17] using the fact that discontinuities that satisfy the Oleinik entropy condition are decreasing ones. Without the convexity the solution has more complicate structures.

Proof of Theorem 1(i): Let $t_0<t_1$ and $x\in\boldsymbol{R}$ be given. Then, since the wave speed is finite and the initial support is compact, there exists $x_0<x$ such that $u(y,s)=0$ for all $y<x_0$ and $s<t_1$. Let $\Omega:=[x_0,x]\times[t_0,t_1]$ and consider the characteristic function $\phi(y,t)=\chi|_\Omega$. Since ϕ is not smooth, we may not directly apply ϕ to (8). However, using classical approximation arguments with smooth functions, $\phi_\varepsilon\to\phi$, one may obtain

$$U(x,t_1)-U(x,t_0)=-\int_{t_0}^{t_1} f(u(x,s))ds. \tag{17}$$

Then, the Lebesgue differentiation theorem implies that, if u is continuous at a point (x,t), then

$$U_t(x,t)=-f(u(x,t)).$$

The derivative of the potential function with respect to the space variable is simply

$$U_x(x,t)=u(x,t)$$

as long as u is continuous at the point (x, t).

Suppose that $x = \xi(t)$ is the (global) minimum point of $U(\cdot, t)$ and $u(\cdot, t)$ is continuous at the point. Then $U_x(\xi, t) = u(\xi, t) = 0$ and hence $x = \xi(t)$ is a characteristic line carrying the zero value, which implies that $\xi'(t) = f'(0) = 0$. Therefore, the minimum $-p(t) = U(\xi(t), t)$ satisfies

$$-p'(t) = \frac{d}{dt}U(\xi(t), t) = \xi'(t)\, u(\xi(t), t) - f(u(\xi(t), t)) = 0,$$

which implies that the minimum value p is constant. Notice that the invariance of p does not depend on the assumptions (H) if the solution is continuous at the minimum point of the potential. Furthermore, since $\xi'(t) = 0$ and p is constant, we have $U(c, t) = p$ for $c = \xi(0)$ as long as u is continuous at the point.

Now we consider the case when $u(\cdot, t)$ has a discontinuity at the minimum point $x = \xi(t)$. Let u_l and u_r be the left and the right hand side limits, respectively, and $l(u)$ be their linear connection. Then, since $U(\cdot, t)$ has minimum at the point $\xi(t)$, it is clear that $u_l \leq 0 \leq u_r$ and $u_l \neq u_r$. Then since $u = 0$ is a global minimum point of the flux, one can easily see that the Oleinik entropy condition (10) holds only if $u_l \neq u_r$ are global minimum points and hence $f(u_{l,r}) = f'(u_{l,r}) = 0$. Therefore, we still have $p'(t) = \xi'(t) = 0$.

Since the total mass M is preserved, the other quantity q in (5) is also constant and the proof of Theorem $1(i)$ is complete. □

The previous approach also shows how does a local extremum of the potential evolve. Let $U(x, t)$ have a local extremum at $x = \xi(t)$ and $u(\cdot, t)$ be continuous at the point. Then, $u(\xi(t), t) = 0$ and

$$\frac{d}{dt}U(\xi(t), t) = \xi'(t)\, u(\xi(t), t) - f(u(\xi(t), t)) = 0.$$

Therefore, the local extremum is constant until it meets a shock discontinuity. In the previous proof it is shown that the minimum point of the potential meets only a harmless shock discontinuity under the assumption that the flux has a global minimum at $u = 0$. If the minimum point is assumed to be unique (i.e., $f(u) > 0$ for all $u \neq 0$), then the minimum does not meet a shock discontinuity and hence one may say that the entropy condition basically prohibits discontinuities at a local minimum point of the potential under the hypothesis (H).

Suppose that the solution $u(\cdot, t)$ has a shock discontinuity at $x = \xi(t)$ and hence $U(\cdot, t)$ has a local maximum at the point. Let

$$u_l(t) = \lim_{y \uparrow \xi(t)} u(y, t), \qquad u_r(t) = \lim_{y \downarrow \xi(t)} u(y, t).$$

Then, the limits satisfy $u_l \geq 0 \geq u_r$ and $u_l \neq u_r$. The wave speed of the discontinuity is given by the Rankine-Hugoniot jump condition which is

$$\xi'(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

Clearly, there exists $\varepsilon > 0$ small such that $u(\cdot, t)$ is continuous on $(\xi_\varepsilon(t), \xi(t))$, where $\xi_\varepsilon(t) = \xi(t) - \varepsilon$. Then, $\xi_\varepsilon'(t) = \xi'(t)$ and

$$\frac{d}{dt}U(\xi_\varepsilon(t), t) = U_x(\xi_\varepsilon(t), t)\xi'(t) + U_t(\xi_\varepsilon(t), t) = u(\xi_\varepsilon(t), t)\xi'(t) - f(u(\xi_\varepsilon(t), t)).$$

Taking $\varepsilon \to 0$ gives

$$\frac{d}{dt}U(\xi(t), t) = u_l \frac{f(u_l) - f(u_r)}{u_l - u_r} - f(u_l) = \frac{f(u_l)u_r - f(u_r)u_l}{u_l - u_r} \leq 0.$$

Therefore, the local maximum decreases. One can easily check that $\frac{d}{dt}U(\xi(t), t) = 0$ if one of the one sided limits is zero. Suppose that the point $(\xi(t), t)$ is a local minimum point of the potential and u is discontinuous at the point. Then, $u_l \leq 0 \leq u_r$ and the local minimum also decreases. However, the Oleinik entropy condition excludes such a case. Now we summarize the properties of the critical values of a potential in the following Proposition:

Proposition 3 *Let $u(x, t)$ be the solution to (1-3) and $U(x, t)$ be its potential function, where the flux $f(u)$ satisfies (H). Then,*

(i) If the potential U has a local minimum at $(\xi(t), t)$, then u is continuous at the point, $\xi'(t) = 0$ and $\frac{d}{dt}U(\xi(t), t) = 0$, i.e., the local minimum is constant as long as it survives.

(ii) If the potential U has a local maximum at $(\xi(t), t)$ and $u_l := \lim_{y \uparrow \xi(t)} u(y, t) \neq u_r := \lim_{y \downarrow \xi(t)} u(y, t)$, then $u_l \geq 0 \geq u_r$ and the maximum decreases as

$$\frac{d}{dt}U(\xi(t), t) = \frac{f(u_l)u_r - f(u_r)u_l}{u_l - u_r} \leq 0. \tag{18}$$

Remark 4 *The non-negativity of the flux is essential for the invariance property. If $u = 0$ is not a global minimum point of the flux f, then the solution may have a discontinuity at the global minimum point of the potential U such that $u_l < 0 < u_r$. Then the derivative in (18) can be strictly positive and hence the global minimum of the potential may strictly increase. Therefore, one may conclude that the non-negativity of the flux is equivalent to the invariance property of a conservation law.*

4 The convergence order $O(\max_x |n_{p,q}(x,t)|)$

Now we show Theorem 1(ii). Since the global minimum point of the potential U is invariant, we may assume U has its minimum at the origin ($c = 0$), i.e.,

$$U(0,t) = -p, \quad U(x,t) \geq -p \quad \text{for all} \quad x \in \boldsymbol{R}, \ t > 0$$

after an appropriate space shift. Then, as discussed earlier, $u(t)$ is continuous at $x = 0$ and $u(0,t) = 0$ for all $t > 0$.

Consider the convex envelope of the flux given by

$$h(u) := \sup_{\eta \in A} \eta(u), \quad A := \{\eta : \eta''(u) \geq 0, \ \eta(u) \leq f(u) \ \text{ for } \ u \in \boldsymbol{R}\}. \tag{19}$$

Since there are only finite number of inflection points, the convex envelope is obtained by simply connecting the humps of the graph of the flux with tangent lines. The convex envelope $h(u)$ is continuously differentiable and is linear on intervals on which $f(u) \neq h(u)$.

It is clear that h' is not invertible. However, one may consider a function $g(x)$ given by an inverse relation

$$g(0) = 0, \qquad h'(g(x)) = x, \qquad x \in \boldsymbol{R}. \tag{20}$$

Then $g(x)$ is piecewise continuous. Under (2) and the hypotheses in (H) there exists a maximal open interval $0 \in (-a, b)$ such that

$$f(u) = h(u), \quad \text{for} \ \ u \in (-a, b).$$

Since the long time asymptotics of a solution depends on the structure of the flux near the origin, the interval $(-a, b)$ will play an important role asymptotically and hence will appear several times in the rest of the paper. Now define N-wave like functions as

$$\tilde{n}_{p,q}(x,t) = \begin{cases} g(x/t) \ , & -a_p(t) < x < b_q(t), \\ \quad 0 \quad , & \text{otherwise}, \end{cases} \tag{21}$$

where $a_p(t), b_q(t) > 0$ satisfy

$$p = -\int_{-a_p(t)}^{0} g(y/t)dy \, , \quad q = \int_{0}^{b_q(t)} g(y/t)dy \, . \tag{22}$$

One can easily check that $\tilde{n}_{p,q}(x,t)$ is a weak solution of both of the conservation laws $u_t + f(u)_x = 0$ and $u_t + h(u)_x = 0$. However, $\tilde{n}_{p,q}(x,t)$ does not satisfy the entropy condition (10) in general and hence it is not a solution. On the other hand, under Hypothesis (H), it is shown in [11] that $n_{p,q}(x,t) = \tilde{n}_{p,q}(x,t)$ for $t > 0$ small or $t \gg 1$ large. In particular there exists $T > 0$ such that

$$n_{p,q}(x,t) = \tilde{n}_{p,q}(x,t) \in (-a,b) \quad \text{for all} \quad x \in \boldsymbol{R}, \; t > T. \tag{23}$$

Since our interest in this paper is the long time behavior of the solution, we employ the explicit formula for the N-wave like function $\tilde{n}_{p,q}(x,t)$ for $t > T$.

Lemma 5 *There exists $T > 0$ such that, for all $t > T$,*

$$u(x,t) \le n_{p,q}(x,t), \quad 0 < x < b_q(t), \;\; \textit{and} \;\; u(x,t) \ge 0, \;\; x > b_q(t), \tag{24}$$

$$u(x,t) \ge n_{p,q}(x,t), -a_p(t) < x < 0, \;\; \textit{and} \;\; u(x,t) \le 0, \;\; x < -a_p(t), \tag{25}$$

$$\|n_{p,q}(t) - u(t)\|_1 \le 4\|N_{p,q}(t) - U(t)\|_\infty. \tag{26}$$

PROOF. We take $T > 0$ that satisfies (23). Since $a_p(t), b_q(t) \to \infty$ as $t \to \infty$, we may assume $a_p(t), b_q(t) > L$ by taking larger $T > 0$ if needed. Consider the backward characteristic $\xi(s), 0 < s < t$, that emanates from a continuity point (x_0, t) with $0 < x_0 < b_q(t)$ and $t > T$. Then $\xi(s)$ conveys the value $u(x_0,t) \in (-a,b)$ with speed $\xi'(s) = f'(u(x_0,t))$. A backward characteristic does not intersect a discontinuity if the flux is convex, which is not our case. However, one can easily check that if the value $u(x_0,t)$ is in the convex region $(-a,b)$, then the characteristic line does not intersect a shock curve.

Suppose it does and the discontinuity connects the value $u(x_0,t)$ to $\bar{u}$ at the intersection point. If $\bar{u} > u(x_0,t)$, then the Oleinik entropy condition implies that $u_r := u(x_0,t)$ and $u_l := \bar{u}$ are right and left hand side limits, respectively. The convexity of the flux on $(-a,b)$ also implies that

$$f'(u(x_0,t)) < (f(u_r) - f(u_l))/(u_r - u_l).$$

Clearly, it is not possible that the slower backward characteristic line intersects a faster shock curve from the right hand side. One can derive similar contradiction if $\bar{u} < u(x_0,t)$ and hence one can conclude that $\xi(s)$ does not intersect a discontinuity for all $0 < s < t$. Furthermore, the invariance property implies that $x = 0$ is a characteristic line of $u(x,t)$ and hence $\xi(0) \ge 0$.

Now consider another backward characteristic $\tilde{\xi}(s), 0 < s < t$, that emanates from the same point (x_0,t) related to the N-wave $n_{p,q}(x,t)$. Then $\tilde{\xi}(s)$ is a line with speed $f'(n_{p,q}(x_0,t))$ and $\tilde{\xi}(0) = 0$ since the N-wave is a rarefaction wave centered at $x = 0$. Therefore, $f'(u(x_0,t)) \le f'(n_{p,q}(x_0,t))$ and hence $u(x_0,t) \le n_{p,q}(x_0,t)$. Since the characteristics for negative values have negative speed and $b_q(t) > L$ we have $u(x,t) \geq 0$ for $x > b_q(t)$, which completes (24). One may similarly obtain (25).

The relations in (24) and (25) imply that

$$\|u(t) - n_{p,q}(t)\|_1 = -\int_{-\infty}^{-a_p} u(x,t)dx + \int_{-a_p}^{0} \Big(u(x,t) - n_{p,q}(x,t)\Big)dx + \int_{0}^{b_q} \Big(n_{p,q}(x,t) - u(x,t)\Big)dx + \int_{b_q}^{\infty} u(x,t)dx.$$

Since $U(0,t) = N_{p,q}(0,t) = p$, one can easily see that

$$\|u(t) - n_{p,q}(t)\|_1 = -2\int_{-\infty}^{-a_p} u(x,t)dx + 2\int_{b_q}^{\infty} u(x,t)dx.$$

Since $-\int_{-\infty}^{-a_p} u(x,t)dx, \int_{b_q}^{\infty} u(x,t)dx \leq \|U(t) - N_{p,q}(t)\|_\infty$, the inequality in (26) is now clear. $\square$

Lemma 6 (Trapped between space translations) *There exists $T > 0$ such that, for all $x \in \boldsymbol{R}$ and $t > T$,*

$$N_{p,0}(x+L,t) + N_{0,q}(x-L,t) \leq U(x,t) \leq N_{p,q}(x,t). \tag{27}$$

PROOF. One can easily check that

$$N_{p,0}(x+L,0) + N_{0,q}(x-L,0) \leq U(x,0),$$

and hence the comparison principle gives the first inequality in (27). Now let

$$\bar{p} = \max_{x<0} U(x,0), \qquad \bar{q} = \max_{x>0} U(x,0).$$

Then, since $U(x,0) \to 0$ as $x \to -\infty$ and $U(x,0) \to M$ as $x \to \infty$, we clearly have $\bar{p} \geq 0$ and $\bar{q} \geq M$. Consider a summation of three N-waves

$$n_{0,\bar{p}}(x+L,t) + n_{\bar{p}+p,\bar{q}+p}(x,t) + n_{\bar{q}-M,0}(x-L,t).$$

Then the N-waves have disjoint supports for $t > 0$ small, say $0 < t < t_0$. Let $n(x,t)$ be a solution with this summation of N-waves as its initial value and $N(x,t)$ as its potential. Then clearly, $U(x,0) \leq N(x,0)$ and hence the potential comparison principle implies that $U(x,t) \leq N(x,t)$ for all $t > 0$. Therefore, the inequality (27) is completed if it is shown that $n(x,t) = n_{p,q}(x,t)$ for all $t > T$, where $T > 0$ is the one in Lemma 5 with $n(x,t)$ in the place of $u(x,t)$.

Suppose that there exists $x_0 > b_q(t)$ such that $n(x_0,t) > 0$. Then, since $n(x_0,t) \in (-a,b)$, the backward characteristic $\xi(s), 0 < s < t$, that emanates from the point (x_0,t) does not intersect a shock curve and $\xi(0) \geq 0$ as discussed in the proof of Lemma 5. Since $n_{\bar{q}-M,0}(x-L,t)$ is negative and $n_{\bar{p}+p,\bar{q}+p}(x,t)$ has rarefaction waves centered at the origin, we have $\xi(0) = 0$. Therefore, for any characteristic $\bar{\xi}(s)$ that emanates from a point (x,t) with $0 < x < b_q(t)$, we have $\bar{\xi}(0) = 0$ and hence comparison of characteristic speed gives $n(x,t) = n_{p,q}(x,t)$ for all $0 < x < b_q(t)$. Therefore $\lim_{x\to\infty} N(x,t) > \lim_{x\to\infty} N_{p,q}(x,t) = M$, which is a contradiction. Therefore, $n(x,t) = 0$ for all $x > b_q(t)$ and hence $n(x,t) = n_{p,q}(x,t)$ for all $x > 0$. One can show the equality for $x < 0$ similarly and obtain $N(x,t) = N_{p,q}(x,t)$ for all $x \in \boldsymbol{R}$ and $t > T$. □

Notice that the inequality (26) transfers the convergence order between two potentials to the one between their derivatives. This is one of the essential steps that make the potential comparison technique work. Now we show the second part of Theorem 1 as a corollary of previous lemmas.

Proof of Theorem 1(ii) : Let $t > T$. Then, Lemma 6 implies that

$$\begin{aligned}|N_{p,q}(x,t) - U(x,t)| &\leq |N_{p,q}(x,t) - N_{p,0}(x+L,t) - N_{0,q}(x-L,t)| \\ &= \begin{cases} |\int_x^{x+L} n_{p,0}(y,t)dy|, & x < 0, \\ |\int_{x-L}^{x} n_{0,q}(y,t)dy|, & x > 0. \end{cases}\end{aligned}$$

Therefore,

$$\|N_{p,q}(t) - U(t)\|_\infty \leq L \max_x (n_{p,q}(x,t)), \quad t > T,$$

and (26) in Lemma 5 implies that

$$\|n_{p,q}(t) - u(t)\|_1 \leq 4L \max_x (n_{p,q}(x,t)), \quad t > T,$$

which completes the proof of Theorem 1(ii). □

5 The convergence order 1/t

Now we show Theorem 1(iii). Remember that we assume U has its minimum at the origin after an appropriate space shift ($c = 0$). Furthermore, since the minimum point is unique, we may set

$$U(0,t) = -p, \quad U(x,t) > -p \quad \text{for all} \;\; x \neq 0, \;\; t > 0.$$

Lemma 7 (Trapped between time translations) *Let u be the solution of (1), $n_{p,q}$ be the N-wave satisfying (4), and $U, N_{p,q}$ be their potentials, respectively. If the flux f satisfies (H) and the initial value $u(x,0)$ satisfies the conditions in (15) (with $c = 0$), then there exist $T, T_1 > 0$ such that, for all $t \geq T$,*

$$N_{p,q}(x, T_1 + t) \leq U(x,t) \leq N_{p,q}(x,t). \tag{28}$$

PROOF. The second inequality in (28) has been shown in Lemma 6 and we show the first one in the followings. Due to the invariance property in Theorem 1(i), $U(0,t) + p = N_{p,q}(0,t) + p = 0$ for all $t > 0$. Therefore, we may split the domain for $x > 0$ and $x < 0$ and show the inequality on each domains separately.

One can clearly see that

$$p + N_{p,q}(L,t) = \int_0^L n_{p,q}(x,t)dx \to 0 \quad \text{as} \ \ t \to \infty.$$

Therefore, there exists $T_1 > \alpha$ such that $p + N_{p,q}(L, T_1) \leq p + U(\varepsilon, 0)$. Since $U(x,0)$ has a unique minimum point at $x = 0$, we may assume that $U(x,0) \geq U(\varepsilon, 0)$ for all $x > \varepsilon$ by taking smaller $\varepsilon > 0$ if needed. Therefore, $N_{p,q}(x, T_1) \leq U(x,0)$ for all $0 < x < L$. Furthermore, since $p + U(x,0) = q$ for all $x \geq L$ and $p + N_{p,q}(x, T_1) \leq q$ for all $x \in \boldsymbol{R}$, we obtain $N_{p,q}(x, T_1) \leq U(x,0)$ for all $x > 0$. For $x < 0$ we may similarly obtain the estimate and obtain the initial comparison $N_{p,q}(x, T_1) \leq U(x,0)$. Therefore, the comparison principle completes that $N_{p,q}(x, T_1 + t) \leq U(x,t)$ for all $x \in \boldsymbol{R}, t > T$, which is the first inequality of the lemma. □

Theorem 1(iii) is obtained as a corollary of the potential comparison principle. Note once again that for our convenience we set the constant c in (15) be $c = 0$.

Proof of Theorem 1(iii) : Using the comparison inequality (28) and the evolution equation for potentials (17), we obtain

$$\begin{aligned}|U(x,t) - N_{p,q}(x,t)| &\leq |N_{p,q}(x, T_1 + t) - N_{p,q}(x,t)| \\ &= \int_t^{t+T_1} f(n_{p,q}(x,s))ds \leq T_1 \| f(n_{p,q}(t)) \|_\infty.\end{aligned}$$

Since the right hand side is independent of $x \in \boldsymbol{R}$, the estimate is uniform. This uniform estimate is naturally transferred to the L^1 estimate of the difference between solutions using (26), i.e.,

$$\|u(x+c,t) - n_{p,q}(x,t)\|_1 \le 4T_1 \|f(n_{p,q}(t))\|_\infty. \tag{29}$$

Therefore, the proof of Theorem 1*(iii)* is completed with $C = 4T_1$. $\square$

In Lemma 7 the potential $U(x,t)$ has been sandwiched between $N_{p,q}(x,t)$ and its time delay $N_{p,q}(x, T_1 + t)$ with $T_1 \ge \alpha$. Basically we may take $T_1 = \alpha$ after takeing larger $T > 0$ if needed. In the followings we consider a brief sketch of it.

First we may assume

$$\max_x n_{p,q}(x,T) \le \max_{0<x<\varepsilon} n_{p,q}(x,\alpha)$$

by taking larger $T > 0$ if needed. Consider a backward characteristic $\xi(s), 0 < s < t$, related to the N-wave $n_{p,q}(x, t+\alpha)$ that emanates from a continuity point (x_0, t) with $0 < x_0 < b_q(t)$ and $t > T$. As discussed in the proof of Lemma 6 it does not meet a discontinuity all the way to $s = 0$ and hence it is a straight line with speed $f'(n_{p,q}(x_0, t+\alpha))$. Similarly consider another backward characteristic $\tilde{\xi}(s), 0 < s < t$, that emanates from the same point (x_0, t) related to the solution $u(x,t)$. Then $\tilde{\xi}(s)$ is also a line with speed $f'(u(x_0,t))$. By taking larger $T > 0$ if needed we may expect that $0 \le \tilde{\xi}(0) \le \varepsilon$ if $u(x_0, t) \ne 0$.

Now we show the order between $\xi(0)$ and $\tilde{\xi}(0)$. Suppose that $\xi(0) < \tilde{\xi}(0)$. Then the speed of the characteristic lines should be ordered by $f'(u(x_0,t)) < f'(n_{p,q}(x_0, t+\alpha))$. Since f is convex near $u = 0$ (or $u \in (-a, b)$) and solutions are constant along the characteristics, we have $u(\tilde{\xi}(0), 0) < n_{p,q}(\xi(0), \alpha)$. Since $n_{p,q}(x, \alpha)$ is an increasing function on the interval $(-a, b)$, we have

$$u(\tilde{\xi}(0), 0) < n_{p,q}(\xi(0), \alpha) < n_{p,q}(\tilde{\xi}(0), \alpha)$$

which contradicts to the initial condition (15). Therefore we have $\xi(0) \ge \tilde{\xi}(0)$ and hence $u(x_0, t) > n_{p,q}(x_0, t)$ if $u(x_0, t) \ne 0$. Therefore,

$$\int_0^x u(y,t)dy \ge \int_0^x n_{p,q}(y, t+\alpha)dy.$$

One may obtain similar estimate for $x < 0$ and may complete the comparison

$$N_{p,q}(x, t+\alpha) \le U(x,t).$$

Therefore, we may take $T_1 = \alpha$ which is reasonable in the sense that the α measures the age of the initial value and hence it should control the convergence speed.

In the followings we compute the order of the supremum norm $\|f(n_{p,q}(t))\|_\infty$ for t large to obtain an algebraic convergence order, which turns out to be the order $O(1/t)$. For that purpose we take a hypothesis

$$\liminf_{u\to 0}\frac{uf'(u)}{f(u)}=\gamma>1. \tag{$H1$}$$

Corollary 8 (Convergence order $O(1/t)$) *If the flux function f satisfies (H1), then*

$$\lim_{t\to\infty} t\,\|u(x,t)-n_{p,q}(x,t)\|_1\le 4T_1\max(p,q)/(\gamma-1). \tag{30}$$

PROOF. Since $|n_{p,q}(\cdot,t)|$ has its supremum at $x=-a_p(t)$ or $x=b_q(t)$ for t large, we only need to check the order of $|f(n_{p,q}(\cdot,t))|$ at these two points to estimate $\|f(n_{p,q}(t))\|_\infty$. Let $u_r=g(b_q(t)/t)$ and hence $f'(u_r)=b_q(t)/t$. One can easily check that $g(x/t)$ and $tf'(x)$ satisfy the inverse relation for any fixed t. Therefore,

$$\int_0^{b_q(t)} g(x/t)dx+\int_0^{u_r} tf'(x)dx=u_rb_q(t).$$

Using these relations one can easily see that

$$\begin{aligned} q&=\textstyle\int_0^{b_q(t)} g(\frac{x}{t})dx=u_rb_q(t)-\int_0^{u_r} tf'(x)dx=t(u_rf'(u_r)-f(u_r))\\ &=t\Big(\frac{u_rf'(u_r)}{f(u_r)}-1\Big)f(u_r).\end{aligned} \tag{31}$$

This equality shows that $u_rf'(u_r)/f(u_r)>1$. Therefore, the flux that satisfies the assumptions in (H) satisfies $\liminf_{u\to 0}\frac{uf'(u)}{f(u)}=:\gamma\ge 1$. Under the extra hypothesis *(H1)*, one obtains from (31) that

$$\lim_{t\to\infty} tf(u_r)\le q/(\gamma-1).$$

We may similarly estimate that $\lim_{t\to\infty} tf(u_l)\le p/(\gamma-1)$ for $u_l=g(-a_p(t)/t)$ and obtain

$$\lim_{t\to\infty} t\|f(n_{p,q}(t))\|_\infty\le \max(p,q)/(\gamma-1).$$

Therefore, the estimate (29) gives the convergence order $O(1/t)$ in (30). □

Even if it is natural to ask that if the assumptions in (H) imply $(H1)$, we do not have a proof nor a counter example. However, there are many examples that satisfy $(H1)$. First, the power law $f(u)=|u|^\gamma,\gamma>1$ is a typical example.

Suppose that f is C^2 and $f''(0) \neq 0$. Then, using the Lehopital's rule, one obtains

$$\lim_{u\to 0} \frac{uf'(u)}{f(u)} = 1 + \lim_{u\to 0} \frac{u}{f'(u)} f''(u) = 2.$$

Suppose f is C^2 and $f''(0) = 0$. Then, one can easily see that $f'(u)/u < f''(u)$ for $|u|$ small, i.e., $1 < uf''(u)/f'(u)$. Therefore, if the flux is C^2 and $f''(0) = 0$, then one has

$$\liminf_{u\to 0} \frac{uf'(u)}{f(u)} \geq 2.$$

If $f(u) = \exp(\frac{-1}{|u|})$ for $|u| < 1$, then one can easily check that $uf'(u)/f(u) \to \infty$ as $u \to 0$. This example indicates that, if the flux f is very flat near the origin, the ratio $uf'(u)/f(u)$ may diverge. However, the hypothesis $(H1)$ is satisfied and we still have the convergence order $O(1/t)$.

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