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Applied Mathematics

Research Report

07-07

October 25, 2007

DEPARTMENT OF MATHEMATICAL SCIENCES

KAIST 한국과학기술원

Korea Advanced Institute of Science and Technology

An algorithm for constructing symmetric dual filters *

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October 25, 2007

Abstract

The symmetric dual filters are essential for the construction of biorthogonal multiresolution analyses and wavelets. We propose an algorithm to seek for dual symmetric trigonometric filters $\tilde{m}_0$ for the given symmetric trigonometric filter m_0 and illustrate our algorithm by examples.

2000 Mathematics Subject Classification: 42C15, 42C40.

Key words and phrases: Dual filters, wavelet, scaling function, multiresolution analysis.

1 Introduction

Two trigonometric polynomials m_0 and $\tilde{m}_0$ with

$$m_0(0) = \tilde{m}_0(0) = 1, \quad m_0(\pi) = \tilde{m}_0(\pi) = 0 \tag{1.1}$$

are called dual filters each other if

$$\overline{m_0(\cdot)}\tilde{m}_0(\cdot) + \overline{m_0(\cdot + \pi)}\tilde{m}_0(\cdot + \pi) = 1. \tag{1.2}$$

A pair of dual filters are used as a pair of analysis filter and synthesis filter in signal processing. Also they are essential to construct biorthogonal multiresolution analyses and biorhogonal wavelets.

*This work was supported by Korea Research Foundation under Grant KRF-2006-311-D00190 (the first author) and by the Yeungnam University Research Grants in 2006 (the second author).

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Let us recall how to construct pairs of biorthogonal scaling functions $(\varphi, \tilde{\varphi})$ and biorthogonal wavelets $(\psi, \tilde{\psi})$ from a pair of dual filters $(m_0, \tilde{m}_0)$. Let m_0 and $\tilde{m}_0$ be trigonometric filters with

$$m_0(0) = \tilde{m}_0(0) = 1, \quad m_0(\pi) = \tilde{m}_0(\pi) = 0. \tag{1.3}$$

We define the scaling functions φ and $\tilde{\varphi}$ in terms of their Fourier transforms as follows:

$$\hat{\varphi}(\xi) := \prod_{j=1}^{\infty} m_0(2^{-j}\xi), \quad \hat{\tilde{\varphi}}(\xi) := \prod_{j=1}^{\infty} \tilde{m}_0(2^{-j}\xi). \tag{1.4}$$

These infinite products in (1.4) converge absolutely and uniformly on compact sets and are the Fourier transforms of compactly supported functions or distributions φ and $\tilde{\varphi}$ with their support widths given by the filter lengths [4, 5, 6]. A necessary condition for φ and $\tilde{\varphi}$ to satisfy the duality condition in $L^2(\mathbb{R})$, i.e.

$$\langle \varphi, \tilde{\varphi}(\cdot - \ell) \rangle = \delta_{0,\ell}, \quad \ell \in \mathbb{Z}, \tag{1.5}$$

is the duality condition (1.2). The duality condition (1.2) with Cohen condition [1, 4] is also sufficient for (1.5).

Given a pair of dual scaling functions φ and $\tilde{\varphi}$ with their associated filters $m_0(\xi)$ and $\tilde{m}_0(\xi)$, the functions ψ and $\tilde{\psi}$ are defined via the relation

$$\hat{\psi}(\xi) = m_1(\xi/2)\hat{\varphi}(\xi/2), \quad \hat{\tilde{\psi}}(\xi) = \tilde{m}_1(\xi/2)\hat{\tilde{\varphi}}(\xi/2),$$

where $m_1(\xi) = e^{-i\xi}\overline{\tilde{m}_0(\xi + \pi)}, \quad \tilde{m}_1(\xi) = e^{-i\xi}\overline{m_0(\xi + \pi)}$. A sufficient condition for ψ and $\tilde{\psi}$ to be biorthogonal wavelets is found in [4].

In applications, such as image processing, symmetric filters are widely used, since they make it easier to deal with the boundaries of the image [5]. Cohen, Daubechies and Feauveau [4] found a necessary and sufficient condition for the dual filters m_0 and $\tilde{m}_0$, which are symmetric, i.e., $m_0(\xi) = m_0(-\xi)$ and $\tilde{m}_0(\xi) = \tilde{m}_0(-\xi)$. Han proposed the construction by cosets (CBC) algorithm to construct the dual filters, which are interpolatory [7]. Given a pair of dual filters, this algorithm was further generalized to the construction of a family of another dual filters with arbitrary vanishing moments [2, 8, 9]. In this paper, we propose a simple algorithm constructing general symmetric dual filters. The material here is an elaborated version of some part of the thesis [11] of the third author under the supervision of the first author.

We conclude the introduction by stating a result from [4]; it will form the basis for the algorithm in this paper:

Proposition 1.1 *Suppose that m_0 and $\tilde{m}_0$ are symmetric trigonometric filters with real coefficients satisfying Condition* (1.3)*. Then the following hold:*

(a) *Both filters* m_0 *and* $\tilde{m}_0$ *can be written as*

$$m_0(\xi) = (\cos^2 \frac{\xi}{2})^{\ell} P(\sin^2 \frac{\xi}{2}), \;\; \tilde{m}_0(\xi) = (\cos^2 \frac{\xi}{2})^{\tilde{\ell}} \tilde{P}(\sin^2 \frac{\xi}{2}), \tag{1.6}$$

where P *and* $\tilde{P}$ *are polynomials with* $P(1) \neq 0 \neq \tilde{P}(1)$ *and* $\ell, \tilde{\ell} \in \mathbb{N}$*;*

(b) *Condition* (1.2) *is equivalent to*

$$P(y)\tilde{P}(y) = P_N(y) + y^N R(y - \frac{1}{2}), \tag{1.7}$$

where $N := \ell + \tilde{\ell}$*,* R *is an odd polynomial and*

$$P_N(y) := \sum_{k=0}^{N-1} \binom{N-1+k}{k} y^k. \tag{1.8}$$

2 Algorithm

In this section, we propose an algorithm to seek for $\tilde{P}$ from P so that P and $\tilde{P}$ satisfy (1.7). That is, an algorithm to seek for $\tilde{m}_0$ from m_0 so that m_0 and $\tilde{m}_0$ be dual to each other, *i.e.*, (1.2) be satisfied.

Suppose m_0 is a symmetric trigonometric polynomial satisfying (1.3). From Proposition 1.1 (a), there exist $\ell \geq 1$ and a polynomial P such that

$$m_0(\xi) = (\cos^2 \frac{\xi}{2})^{\ell} P(\sin^2 \frac{\xi}{2}).$$

Fix $\tilde{\ell} \in \mathbb{N}$, *i.e.*, we will find $\tilde{m}_0$ in the form

$$\tilde{m}_0(w) = (\cos^2 \frac{w}{2})^{\tilde{\ell}} \tilde{P}(\sin^2 \frac{w}{2})$$

satisfying (1.7). From a long division P_N by P, we can find polynomials Q and S with $\deg Q < N$ so that

$$P(y)Q(y) = P_N(y) + y^N S(y). \tag{2.1}$$

Note that such Q is unique. In fact, suppose that Q_1 and Q_2 with $\deg Q_1 < N$ and $\deg Q_2 < N$ both satisfy (2.1). Then, for some S_1 and S_2,

$$P(y)Q_1(y) = P_N(y) + y^N S_1(y);$$
$$P(y)Q_2(y) = P_N(y) + y^N S_2(y).$$

The difference of these two equations lead to

$$P(y)\,\{Q_1(y) - Q_2(y)\} = y^N\{S_1(y) - S_2(y)\}.$$

Since $P(0) = 1 \neq 0$ by (1.3), we have either $Q_1 \equiv Q_2$ or $\deg(Q_1 - Q_2) \geq N$. Since $\deg Q_1 < N$ and $\deg Q_2 < N$, $Q_1 \equiv Q_2$.

For any polynomial F, we note that (2.1) is equivalent to

$$P(y)\left\{Q(y) + y^N F(y)\right\} = P_N(y) + y^N \left\{S(y) + P(y)F(y)\right\}. \tag{2.2}$$

Lemma 2.1 *Define P, S, P_N as in* (1.8) *and* (2.1)*. If* $P(0) \neq 0$, *then the following statements are equivalent:*

(a) *There exists an odd polynomial R such that $P_N(y) + y^N R(y - 1/2)$ can be divisible by P.*

(b) *There exists a polynomial F such that $S(y) + P(y)F(y)$ is antisymmetric about 1/2 .*

In this case, we can choose

$$R(y) = S(y + 1/2) + P(y + 1/2)F(y + 1/2). \tag{2.3}$$

Proof. (a) $\Leftarrow$ (b): It is trivial by the choice of R as in (2.3) and by the use of (2.2) .
(a) $\Rightarrow$ (b): Suppose that there exists an odd polynomial R such that $P_N(y)+y^N R(y-1/2)$ is divisible by P, *i.e.*, there is a polynomial $\tilde{P}$ satisfying Condition (1.7). The difference of Equations (1.7) and (2.1) leads to

$$P(y)\{\tilde{P}(y) - Q(y)\} = y^N\{R(y - 1/2) - S(y)\}. \tag{2.4}$$

Since $P(0) \neq 0$, there exists a polynomial F such that

$$\tilde{P}(y) - Q(y) = y^N F(y).$$

Substituting this equation into (2.4) leads

$$R(y - 1/2) = S(y) + P(y)F(y).$$

The oddness of R implies that

$$S(y) + P(y)F(y) + S(1 - y) + P(1 - y)F(1 - y) = 0,$$

which shows that $S(y) + P(y)F(y)$ is antisymmetric about 1/2 . $\square$

By Lemma 2.1, we are going to seek for the polynomial F so that R, defined as in (2.3), be an odd polynomial. Then the polynomial $\tilde{P}$, defined by $\tilde{P}(y) := Q(y) + y^N F(y)$, will satisfy Condition (1.7). Let N_A denote the degree of a polynomial A. Expanding F, P and S as the Taylor polynomials at $y = 1/2$, we write

$$F(y) = \sum_{n=0}^{N_F} f_n (y - 1/2)^n;$$

$$P(y) = \sum_{n=0}^{N_P} p_n (y - 1/2)^n;$$

$$S(y) = \sum_{n=0}^{N_S} s_n (y - 1/2)^n.$$

Then

$$R(y) = \sum_{n=0}^{N_S} s_n y^n + \sum_{k=0}^{N_F+N_P} (p * f)_k y^k,$$

where $f := (f_k)_{k=0}^{N_F}, p := (p_k)_{k=0}^{N_P}$. In order for R to be an odd polynomial, its even coefficients must vanish, *i.e.*,

$$(p * f)(2k) = \begin{cases} -s_{2k} & 0 \le 2k \le N_S \\ 0 & N_S < 2k < N_P + N_F. \end{cases} \tag{2.5}$$

We note that $N_P + N_F$ is odd. By taking $N_F = 0$ if N_p is odd; $N_F = 1$ otherwise, we obtain the filter $\tilde{m}_0$ of shortest length. Equation (2.5) can be written in the matrix form

$$\boldsymbol{P f} = -\boldsymbol{s}, \tag{2.6}$$

where

$$\boldsymbol{f} := \begin{pmatrix} f_0 & f_1 & f_2 & \cdots & f_{N_F} \end{pmatrix}^T;$$
$$\boldsymbol{s} := \begin{pmatrix} s_0 & s_2 & s_4 & \cdots & s_{(N_P+N_F-1)} \end{pmatrix}^T;$$
$$\boldsymbol{P} := \big(p_{2i-j-1}\big)_{1 \le i \le (N_P+N_F+1)/2,\ 1 \le j \le N_F+1}.$$

Note that the size of $\boldsymbol{P}$ is $(N_P + N_F + 1)/2 \times (N_2 + 1)$. If $N_F \le N_P - 1$, then this system is not overdetermined. Hence, for $\boldsymbol{s} \in \operatorname{ran} \boldsymbol{P}$, we have a solution $\boldsymbol{f}$ of Equation (2.6) and so a solution F of (2.3) producing an odd polynomial in (2.3).

We summarize the above discussion as an algorithm for constructing a dual filter $\tilde{m}_0$ for a given m_0 as follows:

Algorithm 2.2 1. Determine $P(y)$ from m_0 in (1.6);

2. Choose the regularity parameter $\tilde{\ell}$ of the dual filter $\tilde{m}_0$, which determine N in P_N ;
3. Determine Q and S from P and P_N in (2.1);
4. If $S(y-1/2)$ is an odd polynomial, then we set $\tilde{P} := Q$;
5. Otherwise choose N_F with $N_F \le N_P - 1$ and solve the matrix equation (2.6);
6. Set $\tilde{P}(y) := Q(y) + y^N F(y)$. Then a dual filter $\tilde{m}_0$ is determined by the equation (1.6).

We now illustrate our algorithm by examples. The examples below recover the biorthogonal dual filters in [10].

Example 2.3 Consider the quasi-interpolatory filter $m_0(\xi)$ of order 1 defined by

$$m_0(\xi) = (1-y)(1-8\omega y), \qquad y = \sin^2(\xi/2),$$

which yields the scaling function reproducing polynomials. Here ω is a tension parameter. See [3, 10]. In this case, $P(y) = (1-8\omega y)$. Fix $\tilde{\ell} = 1$. Then $N = 2$ and $P_2 = 1+2y$. From (2.1), we have $Q(y) = 1 + (2+8\omega)y, S(y) = -8\omega(2+8\omega)$. Since $S(y-1/2)$ is not odd, we choose $N_F = 0$. By solving the matrix equation (2.6), we obtain $F(y) = \frac{8\omega(2+8\omega)}{1-4\omega}$. Hence

$$\tilde{m}_0(\xi) = (1-y)\tilde{P}(y) = (1-y)\left(1 + (2+8\omega)y + y^2 \frac{8\omega(2+8\omega)}{1-4\omega}\right).$$

□

Example 2.4 Let $m_0(\xi) = (1-y)^2(1+2y+128\omega y^2)$, $y = \sin^2(\xi/2)$, which is the quasi-interpolatory filter of order 2. Fix $\tilde{\ell} = 2$. Then we have

$$P(y) = 1 + 2y + 128\omega y^2 = 2 + 32\omega + (2+128\omega)(y-1/2) + 128\omega(y-1/2)^2;$$
$$Q(y) = 1 + 2y + (6-128\omega)y^2 + 8y^3;$$
$$S(y) = 128\omega(6-128\omega) + 16 + 1024\omega y = 16 + 1280\omega - 16384\omega^2 + 1024\omega(y-1/2).$$

Choose $N_F = 0$ if $\omega = 0$; $N_F = 1$ if $\omega \neq 0$. Then

$$F(y) = \begin{cases} -8, & \text{if } \omega = 0; \\ -\dfrac{8(96\omega+1)(1+80\omega-1024\omega^2)}{(1+16\omega)(64\omega+1)} + \dfrac{512\omega(1+80\omega-1024\omega^2)}{(1+16\omega)(64\omega+1)}y, & \text{if } \omega \neq 0. \end{cases}$$

Hence

$$\tilde{m}_0(\xi) = \begin{cases} (1-y)^2(1+2y+6y^2+8y^3-8y^4), & \text{if } \omega = 0; \\ (1-y)^2\Bigg(1+2y+(6-128\omega)y^2+8y^3 - \dfrac{8(96\omega+1)(1+80\omega-1024\omega^2)}{(1+16\omega)(64\omega+1)}y^4 & \\ \quad + \dfrac{512\omega(1+80\omega-1024\omega^2)}{(1+16\omega)(64\omega+1)}y^5\Bigg), & \text{if } \omega \neq 0. \end{cases}$$

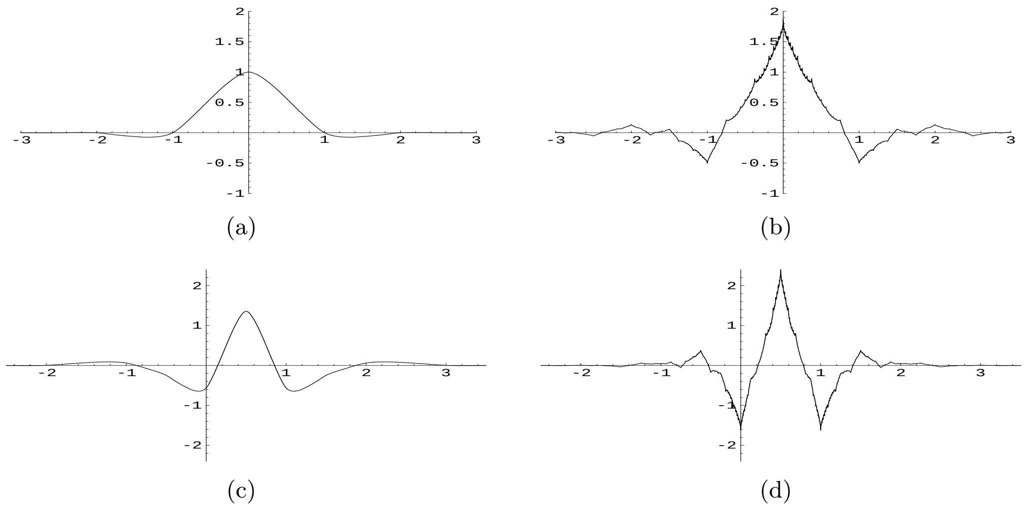

Figure 1: The functions φ [Figure (a)], $\tilde{\varphi}$ [Figure (b)], ψ [Figure (c)] and $\tilde{\psi}$ [Figure (d)] for $w = 0$ in Example 2.4.

Figures 1 and 2 indicate the scaling functions $\varphi, \tilde{\varphi}$ and their associated biorthogonal wavelets $\psi, \tilde{\psi}$ for $w = 0$ and 0.025, respectively. □

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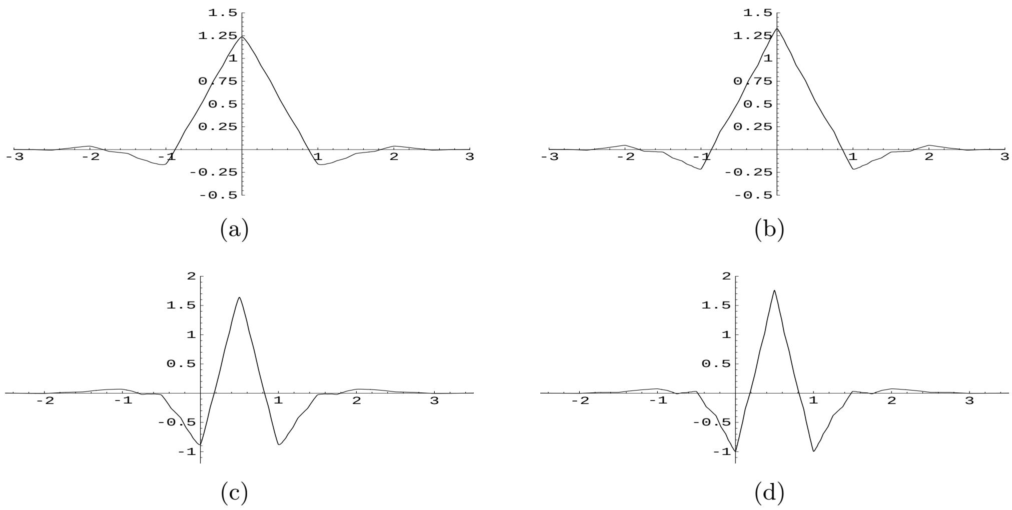

Figure 2: The functions φ [Figure (a)], $\tilde{\varphi}$ [Figure (b)], ψ [Figure (c)] and $\tilde{\psi}$ [Figure (d)] for $w = 0.025$ in Example 2.4.

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