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Abstract

An iterative substructuring method with Lagrange multipliers is considered for the second order elliptic problem, which is a variant of the FETI-DP method. The standard FETI-DP formulation is associated with the saddle-point problem which is induced from the minimization problem with a constraint for imposing the continuity across the interface. Starting from the slightly changed saddle-point problem by addition of a penalty term with a positive penalization parameter η, we propose a dual substructuring method which is implemented iteratively by the conjugate gradient method. In spite of the absence of any preconditioners, it is shown that the proposed method is numerically scalable in the sense that for a large value of η, the condition number of the resultant dual problem is bounded by a constant independent of both the subdomain size H and the mesh size h. We deal with computational issues and present numerical results.

Keywords: domain decomposition; iterative substructuring method; Lagrange multipliers; penalty term; elliptic equations

1 Introduction

Domain decomposition methods are widely used as fast and efficient solvers for a large sparse system of linear equations arising from the finite element discretization for boundary value problems. These are generally classified into two categories according to types of partitions of a domain into subdomains; one is an overlapping domain decomposition and the other is a nonoverlapping domain decomposition including an iterative substructuring method. We are interested in numerical solutions for a second order elliptic problem based on a dual iterative substructuring method. We begin by considering the following Poisson model problem with the homogeneous Dirichlet boundary condition

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$ and f is a given function in $L^2(\Omega)$. For simplicity, we assume that Ω is partitioned into two subdomains $\{\Omega_i\}_{i=1}^2$ such that $\overline{\Omega} = \bigcup_{i=1}^2 \overline{\Omega}_i$ and $\Omega_1 \bigcap \Omega_2 = \emptyset$. The problem (1.1) can be rewritten as

$$\min_{v\in H_0^1(\Omega)} \left(\frac{1}{2}\int_\Omega |\nabla v|^2\, dx - \int_\Omega f v\, dx \right),$$

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equivalently

$$\min_{\substack{v_i \,\in\, H^1(\Omega_i)\\ v_i \,=\, 0 \text{ on } \partial\Omega \bigcap \partial\Omega_i\\ v_1 \,=\, v_2 \text{ on } \partial\Omega_1 \bigcap \partial\Omega_2}} \quad \sum_{i=1}^{2} \left(\frac{1}{2} \int_{\Omega_i} |\nabla v_i|^2 \, dx - \int_{\Omega_i} f v_i \, dx \right). \tag{1.2}$$

Here, $H^1(\Omega)$ and $H_0^1(\Omega)$ are the usual Sobolev spaces defined as follows

$$H^1(\Omega) = \{v \in L^2(\Omega) \mid \partial^\alpha v \in L^2(\Omega),\ |\alpha| \leq 1\}, \quad H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v|_{\partial\Omega} = 0\}.$$

In the domain-decomposition approach based on the reformulated minimization problem (1.2) with a constraint, a key point is how to convert the constrained minimization problem into an unconstrained one. Most studies (e.g. [1, 11, 15]) for treatment of constrained minimizations started in the field of optimal control problem. There are three most popular methods developed for different purposes; the Lagrangian method, the method of penalty function, the augmented Lagrangian method. Such various ideas have been introduced for handling constraints as the continuity across the interface in (1.2) (see [8, 10, 14]). The FETI-DP method is one of the most advanced dual substructuring methods, which introduces Lagrange multipliers to enforce the continuity constraint by following the Lagrangian method and solves the resultant dual problem from the process of seeking a saddle-point of the relevant Lagrangian functional. The dual system is solved by the preconditioned conjugate gradient method (CGM) in company with the Dirichlet preconditioner or the lumped preconditioner.

In this paper, we propose a dual iterative substructuring algorithm which deals with the continuity constraint across the interface in view of the augmented Lagrangian method. To the Lagrangian functional, we add a penalty term which measures the jump across the interface and includes a positive penalization parameter η. In the same way as in most dual substructuring approaches, the saddle-point problem related to the augmented Lagrangian functional is reduced to the dual problem with Lagrange multipliers as unknowns. Then we solve it by the conjugate gradient method. Many studies for the augmented Lagrangian method have been done in the frame of domain-decomposition techniques which belong to families of nonoverlapping Schwarz alternating methods, variants of FETI method, etc. (cf. [4, 7, 14, 20]) Unlike FETI-DP, we do not need to introduce any preconditioners for the dual system since the proposed method is scalable in the sense that the condition number of the relevant dual system has a constant bound which is independent of the subdomain size H and the mesh size h, but depends on the chosen penalty parameter η. It is noted that without making η large, the acceleration of convergence speed on CGM can be achieved. In fact, by focusing on the iterative routine of CGM for solving the dual system, it is expected intuitively that the convergence on the modified dual system by addition of a penalty term is faster than that on the dual problem in FETI-DP. There are similar remarks for the augmented Lagrangian method in the area of constrained optimization (see [5, 12]).

This paper is organized as follows. In Section 2, we introduce a minimization problem with the continuity constraint and a penalty term, which is transformed into a saddle-point formulation with both the primal and dual variables as unknowns. We state an error estimate of the primal solution of the resultant saddle-point problem. Section 3 provides a dual iterative substructuring method and present algebraic condition number estimates. In Section 4, we mainly deal with computational issues in view of implementation of the proposed method and show the numerical results. Finally, we make the concluding remarks in Section 5.

2 Saddle-point formulation

In this section, we present a minimization problem with the pointwise matching constraint on the subdomain interfaces where the minimizer is characterized by an approximation of the solution to (1.1). The adoption of Lagrange multipliers for dealing with the constraint yields a saddle-point problem. We state the unique solvability of the resultant variational problem from the saddle-point formulation and derive the error estimate by observing the relation between the proposed constrained minimization problem and a well-organized minimization problem.

Before introducing the partitioned problem based on the domain-decomposition approach, we recall a well-known equivalence relation associated with the finite element approximation to the problem (1.1). Let $\mathcal{T}_h$ denote a family of regular triangulations on Ω where the discretization parameter h stands for the maximal mesh size of $\mathcal{T}_h$. By introducing the standard $\mathcal{P}_1$-conforming finite element space

$$X_h = \{v_h \in H_0^1(\Omega) \cap \mathcal{C}^0(\overline{\Omega}) \mid \forall \tau \in \mathcal{T}_h,\ v_h|_\tau \in \mathcal{P}_1(\tau)\},$$

we formulate a discretized variational problem for (1.1): find $u_h \in X_h$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in X_h, \tag{2.1}$$

where $a(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h \, dx$ and $(f, v_h) = \int_\Omega f v_h \, dx$. It is well-known that the solution of (2.1) is equivalent to the minimizer of the problem:

$$\min_{v \in X_h} \left(\frac{1}{2} a(v, v) - (f, v) \right). \tag{2.2}$$

Before we propose a constrained minimization problem whose minimizer has a connection with the minimizer of (2.2), we introduce some commonly-used notations. We first decompose Ω into N non-overlapping subdomains $\{\Omega_k\}_{k=1}^N$ such that

(i) Ω_k is a polygonally shaped open subset of Ω.

(ii) the decomposition $\{\Omega_k\}_{k=1}^N$ of Ω is geometrically conforming.

(iii) Γ_{kl} denotes the common interface of two adjacent subdomains Ω_k and Ω_l.

(iv) Γ is the union of the common interfaces among all subdomains, i.e., $\Gamma = \bigcup_{k<l} \Gamma_{kl}$.

Let us use $\mathcal{T}_{h_k}$ to denote a regular triangulation of each subdomain $\Omega_k, \forall k = 1, \cdots, N$, where the matching grids are taken on the boundaries of neighboring subdomains across the interfaces. On each subdomain Ω_k, we set a finite-dimensional subspace X_h^k of $H^1(\Omega_k)$:

$$X_h^k = \{v_h^k \in \mathcal{C}^0(\overline{\Omega}_k) \mid \forall \tau \in \mathcal{T}_{h_k},\ v_h^k|_\tau \in \mathcal{P}_1(\tau),\ v_h^k|_{\partial\Omega \cap \partial\Omega_k} = 0\}.$$

By enforcing the continuity at the corner points, we assemble X_h^k's into X_h^c:

$$X_h^c = \{v = (v_h^k)_k \in \prod_{k=1}^N X_h^k \mid v \text{ is continuous at each corner}\}$$

equipped with the norm

$$\|v_h\|_{X_h^c} = \left(\sum_{k=1}^{N} \|v_h^k\|^2_{H^1(\Omega_k)} \right)^{\frac{1}{2}}.$$

Next, we define a bilinear form on $X_h^c \times X_h^c$:

$$a_h(u,v) = \sum_{k=1}^{N} \int_{\Omega_k} \nabla u \cdot \nabla v \, dx.$$

Let B be a signed Boolean matrix such that for any $v \in X_h^c$, $Bv = 0$ enforces v to be continuous across the interface.

Now, we present a partitioned problem based on the domain-decomposition approach. The finite element problem (2.1) is reformulated as a minimization problem with constraints imposed by the requirement of continuity across the interface Γ:

$$\min_{v \in X_h^c} \mathcal{J}(v) \quad \text{subject to} \quad Bv = 0,$$

where an energy functional $\mathcal{J} : X_h^c \to \mathbb{R}$ is defined as

$$\mathcal{J}(v) = \frac{1}{2} a_h(v,v) - (f,v) \quad \forall v \in X_h^c.$$

Following a well-known technique for the constrained optimization, we introduce a vector μ of Lagrange multipliers in $\mathbb{R}^E$ and a Lagrangian functional $\mathcal{L} : X_h^c \times \mathbb{R}^E \to \mathbb{R}$ defined by

$$\mathcal{L}(v,\mu) = \mathcal{J}(v) + \langle Bv, \mu \rangle$$

where E represents the number of constraints used for imposing the pointwise matching condition and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^E$. Then, we have the saddle-point formulation: find a saddle point $(u_h, \lambda_h) \in X_h^c \times \mathbb{R}^E$ such that

$$\mathcal{L}(u_h, \mu_h) \le \mathcal{L}(u_h, \lambda_h) \le \mathcal{L}(v_h, \lambda_h), \quad \forall v_h \in X_h^c, \quad \forall \mu_h \in \mathbb{R}^E,$$

i.e.,

$$\mathcal{L}(u_h, \lambda_h) = \max_{\mu_h \in \mathbb{R}^E} \min_{v_h \in X_h^c} \mathcal{L}(v_h, \mu_h) = \min_{v_h \in X_h^c} \max_{\mu_h \in \mathbb{R}^E} \mathcal{L}(v_h, \mu_h). \tag{2.3}$$

Here, we shall slightly change the saddle-point formulation (2.3) by addition of a penalty term to the Lagrangian $\mathcal{L}$. Let J_η be a bilinear form on $X_h^c \times X_h^c$ defined as

$$J_\eta(u,v) = \sum_{k<l} \frac{\eta}{h} \int_{\Gamma_{kl}} (u^k - u^l)(v^k - v^l) \, ds, \quad \eta > 0,$$

where $h = \max_{k=1,\cdots,N} h_k$ with the mesh size h_k of $\mathcal{T}_{h_k}$. Given the augmented Lagrangian $\mathcal{L}_\eta$ defined by

$$\mathcal{L}_\eta(v,\mu) = \mathcal{L}(v,\mu) + J_\eta(v,v),$$

we consider the following saddle-point problem (S_h):

$$\mathcal{L}_\eta(u_h, \lambda_h) = \max_{\mu_h \in \mathbb{R}^E} \min_{v_h \in X_h^c} \mathcal{L}_\eta(v_h, \mu_h) = \min_{v_h \in X_h^c} \max_{\mu_h \in \mathbb{R}^E} \mathcal{L}_\eta(v_h, \mu_h). \tag{2.4}$$

There is a well-known characterization of a saddle-point formulation like the problem (S_h) by a variational problem. Since the analysis in [13] was done in an abstract frame, we simply review the concrete relationship related to the problem (S_h) without proof.

Proposition 2.1. *Let $a_\eta(\cdot,\cdot)$ be the bilinear form on $X_h^c \times X_h^c$ such that*

$$a_\eta(u,v) = a_h(u,v) + J_\eta(u,v).$$

Assume that $a_\eta(\cdot,\cdot)$ is symmetric and X_h^c-elliptic in the sense that there exists a constant $\alpha > 0$ such that

$$a_\eta(v,v) \geq \alpha \|v\|^2_{X_h^c}, \quad \forall v \in X_h^c.$$

Then the saddle-point of (S_h) is equivalent to the solution of the following variational problem (Q_h): find $(u_h, v_h) \in X_h^c \times \mathbb{R}^E$ such that

$$\begin{aligned} a_\eta(u_h, v_h) + \langle v_h, B^T \lambda_h \rangle &= (f, v_h), \quad \forall v_h \in X_h^c, \\ \langle B u_h, \mu_h \rangle &= 0, \quad \forall \mu_h \in \mathbb{R}^E. \end{aligned}$$

For the sake of convenience, we refer to the solutions u_h and λ_h as the primal solution and the dual solution, respectively. Moreover, the primal solution u_h of the saddle-point problem (S_h) is precisely the minimizer of the constrained minimization problem (M_c) :

$$\mathcal{J}_\eta(u_h) = \min_{\substack{v \,\in\, X_h^c \\ Bv \,=\, 0}} \mathcal{J}_\eta(v), \tag{2.5}$$

where $\mathcal{J}_\eta(v) = \frac{1}{2} a_\eta(v,v) - (f,v)$.

Since $a_h(\cdot,\cdot)$ is coercive over X_h^c due to the continuity of functions in X_h^c at corner points, it follows from the semi-definiteness of $J_\eta(\cdot,\cdot)$ that $a_\eta(\cdot,\cdot)$ is coercive over X_h^c. Hence, according to Proposition 2.1, the stationary point of (S_h) is characterized by the solution of the problem (Q_h). We now look for the necessary and sufficient conditions which ensure the unique solvability of the problem (Q_h). We set

$$V_h = \{ v \in X_h^c \mid \langle Bv, \mu \rangle = 0, \ \forall \mu \in \mathbb{R}^E \}.$$

Focusing on the fact that the problem (Q_h) is reduced to a linear system with respect to λ_h, a necessary and sufficient condition for the well-posedness of the problem (Q_h) is

$$\mathrm{Ker}(B^T) = \{0\}, \text{ i.e., } \mathrm{rank}(B) = E$$

in keeping in mind the ellipticity of $a_\eta(\cdot,\cdot)$ over V_h. Since the pointwise matching constraints over all edges nodes on Γ are independent, it is clear that $\mathrm{rank}(B) = E$. This means that the problem (Q_h) has a unique solution (u_h, λ_h). Then, with the problem (Q_h), we can associate the concerned minimization problem (2.2) by the constrained minimization (M_c). Noting that

$$V_h = X_h \subset H_0^1(\Omega),$$

we have that

$$\min_{v \in V_h} \mathcal{J}_\eta(v) = \min_{v \in X_h} \mathcal{J}_\eta(v) = \min_{v \in X_h} \mathcal{J}(v). \tag{2.6}$$

Hence, combining (2.6) with (2.5) yields that the primal solution u_h of (Q_h) is exactly equal to the solution of the variational problem (2.1). In addition, in order to check the convergence of the primal variable u_h of (Q_h), it is sufficient to observe the error estimate for the problem (2.1). By the standard error estimates for the $\mathcal{P}_1$-conforming element, we obtain the error estimate for u_h that reads in the following proposition.

Proposition 2.2. *Let u_h be the primal solution of the problem (Q_h). Assume that the problem (1.1) has a unique solution u in $H^2(\Omega)$. Then we have that*

$$\|u - u_h\|_{L^2(\Omega)} + h\|u - u_h\|_{H^1(\Omega)} \leq Ch^2\|u\|_{H^2(\Omega)}.$$

Remark 2.1. *If $\eta = 0$, the proposed method is reduced to the standard FETI-DP formulation. On the other hand, when the pointwise matching condition is removed, this approach is included in the category of penalty methods. We shall mention why we introduce both the penalty term J_η and the pointwise matching condition $Bv = 0$ simultaneously. In highlighting only the aim of treatment of the continuity constraint, it is obvious that one of them is redundant. The combination of penalization and dualization is purposed to have the best of both worlds. The presence of the $\langle Bv, \mu\rangle$ in the augmented Lagrangian functional $\mathcal{L}_\eta$ guarantee that the primal variable solution u_h converges to the solution of (2.1) without making η large. Moreover, the iterative solver for the dual system converges much faster than before augmenting the penalty term.*

3 Iterative substructuring method

The purpose of this section is to derive an iterative substructuring method in an algebraic form and to present a condition number estimate for the proposed method.

3.1 Algebraic formulation of iterative substructuring

The saddle-point formulation (Q_h) is expressed in the following algebraic form

$$\begin{bmatrix} A_\eta & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}. \tag{3.1}$$

By letting the vector u be partitioned as

$$u = \begin{bmatrix} u_i \\ u_c \\ u_e \end{bmatrix},$$

where u_i denotes the degrees of freedom in the subdomain interior, u_c those associated with the subdomain corners, and u_e those on the edge nodes on the interface except corners, we obtain the block forms of matrices and vectors in (3.1) as follows:

$$A_\eta = \begin{bmatrix} A_{ii} & A_{ic} & A_{ie} \\ A_{ci} & A_{cc} & A_{ce} \\ A_{ei} & A_{ec} & A_{ee}^\eta \end{bmatrix}, \quad B^T = \begin{bmatrix} 0 \\ 0 \\ B_e^T \end{bmatrix}, \quad F = \begin{bmatrix} f_i \\ f_c \\ f_e \end{bmatrix},$$

where

$$A_{ee}^\eta = A_{ee} + \eta J. \tag{3.2}$$

When the vectors u_i and u_c are subassembled into u_Π, (3.1) is rewritten as

$$\begin{aligned} A_{\Pi\Pi}u_\Pi + A_{\Pi e}u_e &= f_\Pi & (3.3a)\\ A_{\Pi e}^T u_\Pi + A_{ee}^\eta u_e + B_e^T\lambda &= fe & (3.3b)\\ B_e u_e &= 0 & (3.3c)\end{aligned}$$

Substituting $u_\Pi = A_{\Pi\Pi}^{-1}(f_\Pi - A_{\Pi e}u_e)$ obtained from (3.3a) into (3.3b) yields

$$u_e = S_\eta^{-1}(f_e - B_e^T\lambda - A_{\Pi e}^T A_{\Pi\Pi}^{-1} f_\Pi) \tag{3.4}$$

where

$$S_\eta = S + \eta J = (A_{ee} - A_{\Pi e}^T A_{\Pi\Pi}^{-1} A_{\Pi e}) + \eta J.$$

Here it is easily noted that S_η is symmetric positive definite because S is symmetric positive definite [19] and J is symmetric positive semidefinite. By combining (3.4) with (3.3c), we have the following system for the Lagrange multipliers:

$$F_\eta\lambda = d_\eta \tag{3.5}$$

where

$$F_\eta = B_e S_\eta^{-1} B_e^T$$

and

$$d_\eta = B_e S_\eta^{-1}(f_e - A_{\Pi e}^T A_{\Pi\Pi}^{-1} f_\Pi).$$

Based on the fact that F_η is symmetric positive definite, we solve the resulant dual system (3.5) iteratively by the conjugate gradient method.

3.2 Estimate of condition number

The key issue is to provide a sharp estimate for the condition number of F_η. Let us denote by $\langle\cdot,\cdot\rangle$ the usual inner product in the Euclidean space $\mathbb{R}^d$. The associated norm is $\|u\| = \sqrt{\langle u, u\rangle}, \forall u \in \mathbb{R}^d$. We mention the following property related to the norm induced by F_η, which can be easily checked as in [19].

Proposition 3.1. *For any $\lambda \in \mathbb{R}^E$,*

$$\lambda^T F_\eta \lambda = \max_{v_e \neq 0} \frac{|v_e^T B_e^T \lambda|^2}{\|v_e\|_{S_\eta}^2}$$

where $\|v_e\|_{S_\eta}$ is the norm induced by the symmetric positive definite matrix S_η.

Note that J in (3.2) is represented as

$$J = B_e^T D(J_B) B_e$$

with a symmetric positive definite matrix J_B induced from

$$\frac{1}{h}\int_{\Gamma_{ij}} \varphi\psi\, ds, \quad \forall\varphi,\psi \in X_h^c|_{\Gamma_{ij}},$$

where $D(J_B)$ is a block diagonal matrix such that

$$D(J_B) = \begin{bmatrix} J_B & & \\ & \ddots & \\ & & J_B \end{bmatrix}.$$

We start with defining by Λ the space of vectors of degrees of freedom associated with the Lagrange multipliers. Note that $\dim(\Lambda) = E$ where E is the number of the edge nodes on Γ. In order to give the condition number bound for F_η, based on Lemma 3.1 in [18], it is sufficient to specify a suitable norm $\|\cdot\|_\Lambda$ and to estimate the constants satisfying the relationship as follows:

$$\begin{aligned} c_1\|\lambda\|_{\Lambda'}^2 \le \langle \lambda, F_\eta \lambda\rangle \le c_2\|\lambda\|_{\Lambda'}^2 \quad \forall \lambda \in \Lambda, \\ c_3\|\mu\|_{\Lambda}^2 \le \langle \mu, \mu\rangle \le c_4\|\mu\|_{\Lambda}^2 \quad \forall \mu \in \Lambda, \end{aligned} \tag{3.6}$$

where the norm $\|\cdot\|_\Lambda$ on Λ is defined by

$$\|\mu\|_\Lambda^2 = \mu^T D(J_B)\mu, \quad \forall \mu \in \Lambda$$

and the dual norm on Λ is defined by

$$\|\lambda\|_{\Lambda'} = \max_{\mu\in\Lambda} \frac{|\langle \lambda, \mu\rangle|}{\|\mu\|_\Lambda}, \quad \forall \lambda \in \Lambda.$$

Remark 3.1. *Based on the fact that* $\Lambda = \mathrm{Range}(B_e)$*, we have*

$$\begin{aligned} \|\lambda\|_{\Lambda'} &= \max_{B_e v_e \neq 0} \frac{|(B_e v_e)^T \lambda|}{\|B_e v_e\|_\Lambda} \\ &= \max_{B_e v_e \neq 0} \frac{|v_e^T B_e^T \lambda|}{(v_e^T J v_e)^{1/2}} \\ &= \max_{\substack{v_e \perp \mathrm{Ker} B_e \\ v_e \neq 0}} \frac{|v_e^T B_e^T \lambda|}{(v_e^T J v_e)^{1/2}}. \end{aligned}$$

The third equality is obtained by

$$\frac{|v_e^T B_e^T \lambda|}{(v_e^T J v_e)^{1/2}} = \frac{|(P_\perp v_e)^T B_e^T \lambda|}{((P_\perp v_e)^T J (P_\perp v_e))^{1/2}}, \quad \forall v_e \text{ with } B_e v_e \neq 0,$$

where $P_\perp$ *is an orthogonal projection onto* $(\mathrm{Ker} B_e)^\perp$ *with respect to* $\langle \cdot, \cdot \rangle$*.*

We first mention some useful lemmas in deriving bounds on the extreme eigenvalues of F_η. It is obvious that $\dim(\mathrm{Ker} B_e) = E$.

Lemma 3.1. *Let* $\lambda_{\min}^J$ *be the nonzero smallest eigenvalue of* J*. Then,* $\lambda_{\min}^J$ *is characterized as*

$$\lambda_{\min}^J = 2\lambda_{\min}^{J_B}$$

where $\lambda_{\min}^{J_B}$ *is the smallest eigenvalue of* J_B*.*

Proof. Let λ be a nonzero eigenvalue and q be a corresponding eigenvector of J. Thanks to the relationship $B_e B_e^T = 2I$, we have

$$\begin{aligned} B_e B_e^T D(J_B) B_e q &= \lambda B_e q, \\ D(J_B)(B_e q) &= \frac{1}{2}\lambda(B_e q). \end{aligned}$$

Since $B_e q \neq 0$, it yields $\lambda_{\min}^J = 2\lambda_{\min}^{J_B}$. □

Lemma 3.2. *For $S = A_{ee} - A_{\Pi e}^T A_{\Pi\Pi}^{-1} A_{\Pi e}$, there exists a constant $C > 0$ such that*

$$v_e^T S v_e \leq C v_e^T J v_e, \quad \forall v_e \perp \mathrm{Ker} B_e.$$

Proof. Let $\lambda_{\max}^S$ denote the maximum eigenvalue of S. Note that $v_e^T S v_e \leq \lambda_{\max}^S v_e^T v_e$. Since $\mathrm{Ker} J = \mathrm{Ker} B_e$, we get that for any nonzero $v_e \perp \mathrm{Ker} B_e$,

$$\begin{aligned} \frac{v_e^T J v_e}{v_e^T S v_e} &\geq \frac{1}{\lambda_{\max}^S} \frac{v_e^T J v_e}{v_e^T v_e} \\ &\geq \frac{\lambda_{\min}^J}{\lambda_{\max}^S}. \end{aligned}$$

Hence, it follows from Lemma 3.1 that there exists a constant $C = \lambda_{\max}^S / 2\lambda_{\min}^{J_B}$ satisfying

$$v_e^T S v_e \leq C v_e^T J v_e \quad \forall v_e \perp \mathrm{Ker} B_e.$$

□

Theorem 3.1. *For any $\lambda \in \Lambda$, we have that*

$$\frac{1}{C+\eta}\|\lambda\|_{\Lambda'}^2 \leq \lambda^T F_\eta \lambda \leq \frac{1}{\eta}\|\lambda\|_{\Lambda'}^2.$$

where C is the constant estimated in Lemma 3.2.

Proof. By Proposition 3.1 and Lemma 3.2, we first get the following lower bound:

$$\begin{aligned} \lambda^T F_\eta \lambda &= \max_{v_e \neq 0} \frac{|v_e^T B_e^T \lambda|^2}{v_e^T S v_e + \eta v_e^T J v_e} \\ &\geq \max_{\substack{v_e \perp \mathrm{Ker} B_e \\ v_e \neq 0}} \frac{|v_e^T B_e^T \lambda|^2}{v_e^T S v_e + \eta v_e^T J v_e} \\ &\geq \frac{1}{C+\eta} \max_{\substack{v_e \perp \mathrm{Ker} B_e \\ v_e \neq 0}} \frac{|v_e^T B_e^T \lambda|^2}{v_e^T J v_e} \\ &= \frac{1}{C+\eta}\|\lambda\|_{\Lambda'}^2. \end{aligned}$$

The last equality holds due to Remark 3.1.

Note that $\lambda_{\min}^S v_e^T v_e \leq v_e^T S v_e$ with the minimum eigenvalue $\lambda_{\min}^S$ of S. Similarly as in the lower bound estimate, we obtain that

$$
\begin{aligned}
\lambda^T F_\eta \lambda &= \max_{v_e \neq 0} \frac{|v_e^T B_e^T \lambda|^2}{v_e^T S v_e + \eta v_e^T J v_e} \\
&\leq \max_{v_e \neq 0} \frac{|v_e^T B_e^T \lambda|^2}{\lambda_{\min}^S v_e^T v_e + \eta v_e^T J v_e} \\
&= \max_{\substack{v_e \perp \mathrm{Ker} B_e \\ v_e \neq 0}} \frac{|v_e^T B_e^T \lambda|^2}{\lambda_{\min}^S v_e^T v_e + \eta v_e^T J v_e} \\
&\leq \max_{\substack{v_e \perp \mathrm{Ker} B_e \\ v_e \neq 0}} \frac{|v_e^T B_e^T \lambda|^2}{\eta v_e^T J v_e} \\
&= \frac{1}{\eta} \|\lambda\|_{\Lambda'}^2.
\end{aligned}
$$

□

Now we give the estimate of the condition number $\kappa(F_\eta)$.

Corollary 3.1. *We have the condition number estimate of the dual system (3.5) as follows*

$$
\kappa(F_\eta) \leq \left(\frac{C}{\eta} + 1\right) \kappa(J_B), \quad C = \frac{\lambda_{\max}^S}{2\lambda_{\min}^{J_B}}.
$$

Proof. By Theorem 3.1, the constants in (3.6) are estimated as follows

$$
\begin{aligned}
\frac{1}{C+\eta} \|\lambda\|_{\Lambda'}^2 \leq \langle \lambda, F_\eta \lambda \rangle \leq \frac{1}{\eta} \|\lambda\|_{\Lambda'}^2 \quad \forall \lambda \in \Lambda, \\
\frac{1}{\lambda_{\max}^{J_B}} \|\mu\|_\Lambda^2 \leq \langle \mu, \mu \rangle \leq \frac{1}{\lambda_{\min}^{J_B}} \|\mu\|_\Lambda^2 \quad \forall \mu \in \Lambda.
\end{aligned}
$$

Thanks to Lemma 3.1 in [18], we easily find that

$$
\kappa(F_\eta) \leq \left(\frac{C}{\eta} + 1\right) \kappa(J_B) \text{ with } C = \frac{\lambda_{\max}^S}{2\lambda_{\min}^{J_B}}.
$$

□

We address the following well-known fact that informs us the eigenvalues of a special type of Toeplitz matrix.

Proposition 3.2 ([16]). *Let T be an $n \times n$ symmetric tridiagonal Toeplitz matrix*

$$
T = \begin{bmatrix}
\alpha & \beta & 0 & \cdots & 0 \\
\beta & \alpha & \beta & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \beta & \alpha & \beta \\
0 & \cdots & 0 & \beta & \alpha
\end{bmatrix}.
$$

Then, the eigenvalues of T are

$$\lambda_k = \alpha + 2\beta \cos\left(\frac{\pi k}{n+1}\right), \quad \forall k = 1, \cdots, n.$$

Corollary 3.2. *For a sufficiently large η, we have*

$$\kappa(F_\eta) \leq 3.$$

Proof. Since J_B is a tridiagonal Toeplitz matrix with $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{6}$, it is confirmed that $\kappa(J_B) < 3$ independently of h and H by Proposition 3.2. Hence Corollary 3.1 implies that $\kappa(F_\eta) \leq 3$ for a sufficiently large η. □

Remark 3.2. *To the best of our knowledge, the algorithm with such a constant bound of the condition number is unprecedented in the field of domain decomposition. Adding the penalization term J_η to the FETI-DP formulation makes a strongly scalable algorithm without any domain-decomposition-based preconditioners even if it is redundant in view of equivalence relations among the concerned minimization problems.*

4 Computational issues and numerical results

4.1 Computational issues

The formulation in the block form (3.3) is intended for the estimate of condition number. We need to reorder the relevant degrees of freedom in focusing on the implementation of the proposed algorithm. By rearranging u in order $u = [u_r, u_c]^T$ where u_i and u_e are assembled into u_r, we obtain the system in the following form

$$K_{rr}^\eta u_r + K_{rc} u_c + B_r^T \lambda = f_r \quad (4.1a)$$
$$K_{rc}^T u_r + K_{cc} u_c = f_c \quad (4.1b)$$
$$B_r u_r = 0 \quad (4.1c)$$

Note that $K_{rr}^\eta = K_{rr} + \eta \tilde{J}$ is non-singular because K_{rr} is positive definite (cf. [8]). By substituting

$$u_r = (K_{rr}^\eta)^{-1}(f_r - K_{rc} u_c - B_r^T \lambda)$$

from (4.1a) into (4.1b) and (4.1c), we have

$$\begin{bmatrix} F_{cc} & -F_{rc}^T \\ F_{rc} & F_{rr} \end{bmatrix} \begin{bmatrix} u_c \\ \lambda \end{bmatrix} = \begin{bmatrix} d_c \\ d_r \end{bmatrix} \quad (4.2)$$

where

$$F_{rr} = B_r (K_{rr}^\eta)^{-1} B_r^T, \quad F_{rc} = B_r (K_{rr}^\eta)^{-1} K_{rc}, \quad F_{cc} = K_{cc} - K_{rc}^T (K_{rr}^\eta)^{-1} K_{rc}$$

and

$$d_r = B_r^T (K_{rr}^\eta)^{-1} f_r, \quad d_c = f_c - K_{rc}^T (K_{rr}^\eta)^{-1} f_r.$$

Since A_η is invertible, so is F_{cc}, the Schur complement of K_{rr}^η in A_η. We can therefore eliminate u_c in (4.2), and get

$$F_\eta \lambda = d_\eta \quad (4.3)$$

where

$$F_\eta = F_{rr} + F_{rc}F_{cc}^{-1}F_{rc}^T, \quad d_\eta = d_r - F_{rc}F_{cc}^{-1}d_c.$$

We iteratively solve the dual problem (4.3) by the conjugate gradient method. The difference with the FETI-DP method is to invert K_{rr}^η that contains the penalization parameter η. To compare our algorithm with the FETI-DP method, we need to make more careful observation of behavior of $(K_{rr}^\eta)^{-1}$. Note that

$$K_{rr}^\eta = K_{rr} + \eta\tilde{J} = \begin{bmatrix} A_{ii} & A_{ie} \\ A_{ie}^T & A_{ee} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \eta J \end{bmatrix}$$

where $J = B_e^T D(J_B)B_e$. Thanks to the specific type of discrete Sobolev inequality proven in Lemma 3.4 of [6], we have the following Proposition without major difficulty.

Proposition 4.1. *For any v_r, there exist constants C_1 and C_2 independent of h and H such that*

$$C_1 \frac{h^2}{H^2(1+\log\frac{H}{h})}\|v_r\|^2 \le v_r^T K_{rr} v_r \le C_2 \|v_r\|^2,$$

that is,

$$\kappa(K_{rr}) \lesssim \left(\frac{H}{h}\right)^2 \left(1 + \log\frac{H}{h}\right).$$

Theorem 4.1. *For each $\eta > 0$, we have that*

$$\kappa(K_{rr}^\eta) \lesssim \left(\frac{H}{h}\right)^2 \left(1 + \log\frac{H}{h}\right)(1+\eta).$$

Proof. Since $\tilde{J}$ is positive semidefinite, it is clear that

$$\lambda_{\min}^{K_{rr}} \|v_r\|^2 \le v_r^T K_{rr}^\eta v_r \quad \forall v_r.$$

From the fact that $\eta J = \eta B_e^T D(J_B) B_e$, we obtain that for any $v_r = [v_i, v_e]^T$,

$$\begin{aligned} v_r^T K_{rr}^\eta v_r &\le \lambda_{\max}^{K_{rr}} \|v_r\|^2 + \eta \lambda_{\max}^{J_B} (B_e v_e)^T B_e v_e \\ &\le (\lambda_{\max}^{K_{rr}} + 2\lambda_{\max}^{J_B}\eta)\|v_r\|^2. \end{aligned}$$

Moreover, it is noted that $\lambda_{\max}^{J_B} < 1$ by Proposition 3.2. Hence it follows from Proposition 4.1 that

$$\kappa(K_{rr}^\eta) \lesssim \left(\frac{H}{h}\right)^2 \left(1 + \log\frac{H}{h}\right)(1+\eta).$$

$\square$

Theorem 4.1 shows how severely η deteriorates the property of K_{rr}^η as η is increased. It is expected that the large condition number of K_{rr}^η shown above may cause the computational cost relevant to K_{rr}^η to be expensive. We shall establish a good preconditioner for K_{rr}^η in order to remove a bad effect of η. We introduce the preconditioner M as follows

$$\begin{aligned} M &= \overline{K}_{rr} + \eta\tilde{J} \\ &= \begin{bmatrix} A_{ii} & 0 \\ 0 & A_{ee} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \eta J \end{bmatrix} \\ &= \begin{bmatrix} A_{ii} & 0 \\ 0 & A_{ee}^\eta \end{bmatrix}. \end{aligned}$$

Theorem 4.2. *The condition number of the preconditioned problem grows asymptotically as*

$$\kappa(M^{-1}K_{rr}^{\eta}) := \frac{\lambda_{\max}(M^{-1}K_{rr}^{\eta})}{\lambda_{\min}(M^{-1}K_{rr})} \lesssim \frac{H}{h}\left(1+\log\frac{H}{h}\right).$$

Proof. Note that, for any $v_r = [v_i, v_e]^T$,

$$\frac{v_r^T K_{rr}^{\eta} v_r}{v_r^T M v_r} = 1 + \frac{2v_i^T A_{ie} v_e}{v_i^T A_{ii} v_i + v_e^T A_{ee}^{\eta} v_e}. \tag{4.4}$$

Let

$$\gamma = \sup_{\substack{v_i \neq 0 \\ v_e \neq 0}} \frac{|v_i^T A_{ie} v_e|}{(v_i^T A_{ii} v_i \cdot v_e^T A_{ee}^{\eta} v_e)^{\frac{1}{2}}}$$

where the constant $\gamma < 1$ is referred to as the strengthened Cauchy-Schwarz-Bunyakowski constant (see [2,3,17]). Since the arithmetic-geometric inequalty gives

$$\begin{aligned} |v_i^T A_{ie} v_e| &\leq \gamma(v_i^T A_{ii} v_i \cdot v_e^T A_{ee}^{\eta} v_e)^{\frac{1}{2}} \\ &\leq \frac{1}{2}\gamma(v_i^T A_{ii} v_i + v_e^T A_{ee}^{\eta} v_e), \end{aligned}$$

it suffices to find an appropriate bound on the constant γ. Taking $v_i = A_{ii}^{-\frac{1}{2}} w_i$ and using the Cauchy inequality and the semidefiniteness of J, we have that

$$\begin{aligned} \gamma &= \sup_{\substack{w_i \neq 0 \\ v_e \neq 0}} \frac{|\langle A_{ii}^{-\frac{1}{2}} A_{ie} v_e, w_i\rangle|}{(w_i^T w_i \cdot v_e^T A_{ee}^{\eta} v_e)^{\frac{1}{2}}} \\ &\leq \sup_{v_e \neq 0} \frac{\|A_{ii}^{-\frac{1}{2}} A_{ie} v_e\|}{(v_e^T A_{ee}^{\eta} v_e)^{\frac{1}{2}}} \\ &\leq \sup_{v_e \neq 0} \frac{\|A_{ii}^{-\frac{1}{2}} A_{ie} v_e\|}{(v_e^T A_{ee} v_e)^{\frac{1}{2}}} \\ &= \left(\sup_{v_e \neq 0} \frac{v_e^T (A_{ee} - S_{ee}) v_e}{v_e^T A_{ee} v_e}\right)^{\frac{1}{2}} \\ &= (1 - C^*)^{\frac{1}{2}} \end{aligned}$$

where $S_{ee} = A_{ee} - A_{ie}^T A_{ii}^{-1} A_{ie}$ and $C^* = \inf_{v_e \neq 0} \frac{v_e^T S_{ee} v_e}{v_e^T A_{ee} v_e}$. Next, we shall estimate C^*. In a similar way as in Lemma 4.11 of [21], it is easy to show that

$$\lambda_{\min}(S_{ee}) = O\left(\frac{h}{H(1+\log\frac{H}{h})}\right) \tag{4.5}$$

based on the specific type of discrete Sobolev inequality mentioned in Lemma 3.4 of [6]. As $\lambda_{\max}(A_{ee}) = 2 - \cos((1-\frac{h}{H})\pi) < 3$, it holds from (4.5) that

$$\frac{v_e^T S_{ee} v_e}{v_e^T A_{ee} v_e} \geq \frac{\lambda_{\min}(S_{ee})}{\lambda_{\max}(A_{ee})} = O\left(\frac{h}{H(1+\log\frac{H}{h})}\right) = \hat{C}.$$

Hence,

$$\gamma \le (1-C^*)^{\frac{1}{2}} \le (1-\hat{C})^{\frac{1}{2}}. \tag{4.6}$$

Combination of (4.4) and (4.6) yields that

$$\begin{aligned}
\kappa(M^{-1}K_{rr}^{\eta}) &= \frac{\lambda_{\max}(M^{-1}K_{rr}^{\eta})}{\lambda_{\min}(M^{-1}K_{rr})} \\
&\le \frac{1+(1-\hat{C})^{\frac{1}{2}}}{1-(1-\hat{C})^{\frac{1}{2}}} \\
&\le \frac{2}{1-(1-\hat{C})^{\frac{1}{2}}} \\
&\approx C\frac{H}{h}\left(1+\log\frac{H}{h}\right) \text{ for a sufficiently large } \frac{H}{h}
\end{aligned}$$

Consequently, for a sufficiently large $\frac{H}{h}$, we have

$$\kappa(M^{-1}K_{rr}^{\eta}) \lesssim \frac{H}{h}\left(1+\log\frac{H}{h}\right).$$

□

4.2 Numerical results

In this section, we present computational results which support our theoretical arguments for the proposed method and show its efficiency in view of parallel computing. Let Ω be $[0,1]^2 \subset \mathbb{R}^2$. We consider the model problem with the exact solution

$$u(x,y) = y(1-y)\sin(\pi x)$$

as follows

$$\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned}$$

The reduced dual problem (4.3) is iteratively solved by CGM. We monitor the convergence of CGM by the following stop criterion

$$\frac{\|r_k\|}{\|r_0\|} \le \text{TOL},$$

where r_k is the residual error on the k-th CG iteration and TOL=10^{-8}. We introduce discretization parameters H, N_s, and h. These parameters stand for the subdomain size, the number of subdomains, and the mesh size, respectively. We decompose Ω into N_s square subdomains with $N_s = 1/H \times 1/H$. Each subdomain is partitioned into $2 \times H/h \times H/h$ uniform triangular elements.

Remark 4.1. *Figure 1 shows how the condition number of F_η behaves as η increases. According to Corollary 3.1, the larger η we take, the more F_η mimics J_B in terms of the condition number. But, as shown in Figure 1, we do not need to take very large values of η if we simply focus on improvement of the condition number related to computational speed. In fact, an optimal value of η can be estimated in a heuristic way, which is approximately* 2 *independently of the subdomain size H and the mesh size h.*

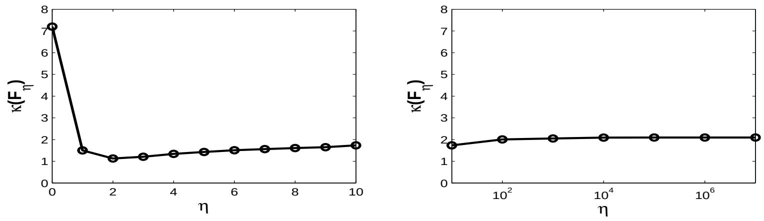


Figure 1: Condition number of F_η for various values of η where $N_s = 4 \times 4$ and $H/h = 4$.

N_s	$\frac{H}{h}$	h	$\frac{\|u-u_h\|_2}{\|u\|_2}$	ratio
4×4	4	1/16	3.2230e-3	-
	8	1/32	8.0721e-4	0.2505
	16	1/64	2.0188e-4	0.2501
	32	1/128	5.0471e-5	0.2500
	64	1/256	1.2616e-5	0.2500
8×8	4	1/32	8.0690e-4	-
	8	1/64	2.0184e-4	0.2501
	16	1/128	5.0464e-5	0.2500
	32	1/256	1.2614e-5	0.2500
16×16	4	1/64	2.0183e-4	-
	8	1/128	5.0452e-5	0.2500
	16	1/256	1.2611e-5	0.2500

Table 1: Convergence behavior.

As mentioned in Remark 4.1, an extremely large value of η is not necessary for either improvement of accuracy or speed-up of iterative solver. Just in order to avoid trouble in choosing a properly large η, we take the large values of η such as $10^4, 10^5$, etc. The choice of large η does not cause any problems in mathematically solving the system relevant to F_η. However, the situation is slightly different in practical sense. Let us look over F_η more carefully in the form of

$$F_\eta = B_e S_\eta^{-1} B_e^T$$

where

$$S_\eta = A_{ee} + \eta J - A_{\Pi e}^T {A_{\Pi\Pi}}^{-1} A_{\Pi e}.$$

An extremely large η may result in numerical instability, since it makes S_η become close to the singular matrix J. Hence, we choose a moderately large $\eta = 10^6$ throughout whenever we intend to investigate the typical properties of the proposed method.

Table 1 shows the relative errors $\frac{\|u-u_h\|_2}{\|u\|_2}$ estimated in L^2-norm while H and h change diversely. As predicted, the $O(h^2)$ convergence is observed in Table 1. Next, we make a comparison between our proposed method and the well-known FETI-DP method from the viewpoint of the conditioning of the related matrices F_η and F. Table 2 informs that the condition number $\kappa(F_\eta)$ and the CG iteration number for convergence remain almost constant when the mesh is refined and the number N_s of subdomains is increased while keeping the ratio H/h constant. It means that the designed

N_s	$\frac{H}{h}$	$\eta = 10^6$		$\eta = 0$	
		iter. no	$\kappa(F_\eta)$	iter. no	$\kappa(F)$
4×4	4	3	2.0938	14	7.2033
	8	7	2.7170	23	2.2901e+1
	16	13	2.9243	33	5.9553e+1
	32	14	2.9771	48	1.4707e+2
8×8	4	3	2.0938	18	7.9241
	8	7	2.7170	32	2.5668e+1
	16	12	2.9245	48	6.7409e+1
16×16	4	3	2.0938	19	7.9461
	8	7	2.7170	34	2.6324e+1

Table 2: Comparison between the proposed method ($\eta = 10^6$) and the FETI-DP method ($\eta = 0$).

method is a scalable algorithm in view of parallel computation. Moreover, we observe numerically that the condition number of F_η is bounded by the constant 3 independently of h and H, while the condition number in the FETI-DP method grows with a increasing H/h (cf. [8, 9]). Unlike the behavior of $\kappa(F_\eta)$, the condition number $\kappa(F)$ of the FETI-DP method increases gradually as the number of subdomains increases even if the ratio H/h is kept constant. In addition, it is shown in Table 2 that the proposed method is superior to the FETI-DP method in the number of CG iterations for convergence. In order to reduce the bad effect of η on K^η_{rr}, we proposed a preconditioner M optimal with respect to η in section 4.2. By listing the condition number of K^η_{rr} and $M^{-1}K^\eta_{rr}$, Table 3 illustrates numerically the performance of the designed preconditioner M for $(K^\eta_{rr})^{-1}$. It is confirmed that the influence of η on $\kappa(K^\eta_{rr})$ is completely removed after adopting M. Finally, in order to show the practical performance of the proposed method, we compare the proposed methods with $\eta = 2$ and 10^6 with both the FETI-DP method and the preconditioned FETI-DP by Dirichlet preconditioner in terms of the CPU time in seconds. The penalty parameter $\eta = 2$ is an optimal one chosen heuristically by focusing on efficiency in view of computational cost without introduction of the preconditioner M. All of four algorithms are implemented on a sequential machine. To highlight their efficiency as parallel solvers, the virtual wall clock time is presented in Table 4, which is measured appropriately by assuming that each algorithm is parallelized at the subdomain level. Figure 2 shows that in both cases, the proposed methods outperform than the FETI-DP methods. In details, it is noted in Table 4 that F_η with $\eta = 2$ and 10^6 spend only 24% and 15% of the virtual wall clock time for the FETI-DP method, while on average, they are 1.4 times and 2.5 times faster than the FETI-DP with Dirichlet preconditioner.

5 Conclusions

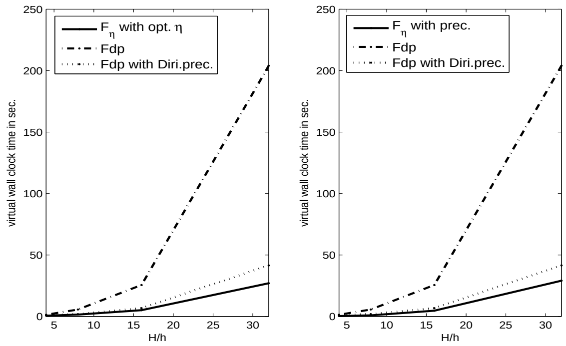
In this paper, we proposed a dual substructuring method based on an augmented Lagrangian with a penalty term. Unlike other substructuring methods, it was proven that without any precondtitioners, the designed method is scalable in the sense that for a large value of the penalty parameter η, the condition number of the relevant dual system has a constant bound independently of H and h. In addition, we dealt with some implementational issues. An optimal preconditioner with respect to η was established in order to increase the ease of use and the practical efficiency of

$N_s = 4 \times 4$	$\frac{H}{h} = 4$		$\frac{H}{h} = 8$		$\frac{H}{h} = 16$	
η	$\kappa(K_{rr}^{\eta})$	$\kappa(M^{-1}K_{rr}^{\eta})$	$\kappa(K_{rr}^{\eta})$	$\kappa(M^{-1}K_{rr}^{\eta})$	$\kappa(K_{rr}^{\eta})$	$\kappa(M^{-1}K_{rr}^{\eta})$
0	43.2794	14.8532	228.0254	40.0332	1.1070e+3	104.3459
1	34.5773	11.8232	161.1716	28.7437	7.0562e+2	68.3468
10^1	91.3072	11.4010	420.1058	28.1835	1.8390e+3	67.6093
10^2	8.5119e+2	11.3525	3.9824e+3	28.1232	1.7513e+4	67.5325
10^3	8.4538e+3	11.3475	3.9616e+4	28.1170	1.7430e+5	67.5247
10^4	8.4480e+4	11.3470	3.9596e+5	28.1164	1.7421e+6	67.5240
10^5	8.4474e+5	11.3469	3.9593e+6	28.1164	1.7420e+7	67.5239
10^6	8.4473e+6	11.3469	3.9593e+7	28.1164	1.7420e+8	67.5239
10^7	8.4473e+7	11.3469	3.9593e+8	28.1164	1.7420e+9	67.5238

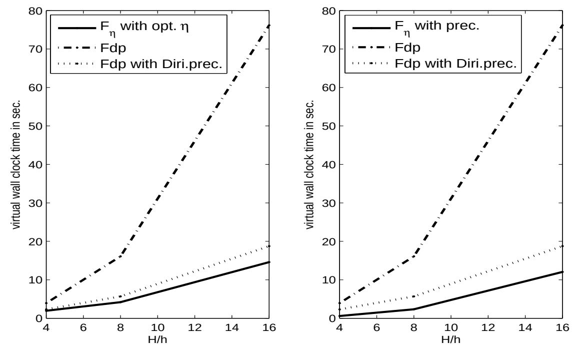
Table 3: Performance of preconditioner M for $(K_{rr}^{\eta})^{-1}$.

N_s	$\frac{H}{h}$	virtual wall clock time				$\frac{\|u-u_h\|_2}{\|u\|_2}$
		F_{η_1}	F_{η_2}	F	$\hat{F}_D^{-1}F$	
4×4	4	0.61	0.32	1.38	1.16	3.2230e-3
	8	1.50	0.88	5.65	1.86	8.0722e-4
	16	5.04	4.67	25.57	6.73	2.0188e-4
	32	27.02	29.10	203.95	41.48	5.0471e-5
8×8	4	1.95	0.60	3.95	2.31	8.0691e-4
	8	4.20	2.33	16.10	5.67	2.0184e-4
	16	14.59	12.05	76.18	18.75	5.0467e-5
16×16	4	3.42	1.24	10.24	5.06	2.0183e-4
	8	8.60	4.88	32.66	10.64	5.0457e-5

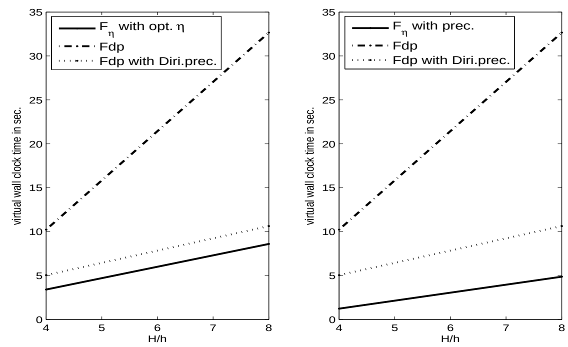
Table 4: Parallel performance in virtual wall clock time in seconds where $\eta_1 = 2$ and $\eta_2 = 10^6$.



(a) $N_s = 4 \times 4$



(b) $N_s = 8 \times 8$



(c) $N_s = 16 \times 16$

Figure 2: Comparison of four algorithms in the virtual wall clock time in seconds: $\eta = 2$ (left) and $\eta = 10^6$ (right).

the presented method. For $\eta = 0$, the proposed method is reduced to the FETI-DP method which is one of the most advanced dual substructuring methods. In this view, we compared the proposed method with the FETI-DP method from many perspectives such as the conditioning of the dual system, the CG iteration number for convergence, and the virtual wall clock time. According to the numerical results, the presented method is superior to both of the FETI-DP method and the preconditioned FETI-DP by the optimal Dirichlet preconditioner.

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