SINGULAR INNER FUNCTIONS WHOSE FROSTMAN SHIFTS ARE CARLESON-NEWMAN BLASCHKE PRODUCTS II

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ABSTRACT. Let $\mathcal{M}$ be the family of inner functions whose non-trivial Frostman shifts are Carleson-Newman Blaschke products. It is known that for any closed subset of the unit circle ∂D there is a discrete singular inner function S_μ with $supp\,(\mu) = E$ and $S_\mu \in \mathcal{M}$. In this paper, we are interested in continuous singular inner functions S_μ in $\mathcal{M}$ or not in $\mathcal{M}$ with nonporous $supp\,(\mu)$. For example, if E is a perfect subset of ∂D then there is a continuous singular inner function $S_\mu \in \mathcal{M}$ with $supp\,(\mu) = E$ (Theorem 2.6). We also show that if E is a perfect subset of ∂D which is not porous then there is a continuous singular inner function $S_\mu \notin \mathcal{M}$ with $supp\,(\mu) = E$ (Corollary 3.4).

1. INTRODUCTION

Let H^∞ be the Banach algebra of bounded analytic functions in the open unit disk D with the supremum norm. The pseudo-hyperbolic distance in D is given by

$$\rho(z,w) = \left|\frac{z-w}{1-\overline{w}z}\right|, \quad z,w \in D.$$

A pseudo-hyperbolic open disk with center $z \in D$ and radius $0 < r < 1$ is denoted by $D_\rho(z,r)$, that is,

$$D_\rho(z,r) = \{w \in D : \rho(z,w) < r\}.$$

We identify a function in H^∞ with its radial limit function on the unit circle ∂D. A function I in H^∞ is called an inner function if $|I(e^{i\theta})| = 1$ for almost every $e^{i\theta} \in \partial D$. For a sequence $\{z_n\}_n$ in D with

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$\sum_{n=1}^\infty (1-|z_n|)<\infty$, we have a Blaschke product defined by

$$b(z)=\prod_{n=1}^\infty \frac{-\overline{z}_n}{|z_n|}\,\frac{z-z_n}{1-\overline{z}_n z}, \quad z\in D.$$

A Blaschke product b is an inner function. Moreover, if for every bounded sequence of complex numbers $\{a_n\}_n$ there exists f in H^∞ satisfying $f(z_n)=a_n$ for every n, then both the sequence $\{z_n\}_n$ and the Blaschke product b are called interpolating. In [1], Carleson proved that $\{z_n\}_n$ is interpolating if and only if

$$\inf_n \prod_{k:k\neq n}\left|\frac{z_k-z_n}{1-\overline{z}_n z_k}\right|>0.$$

A Blaschke product b is called Carleson-Newman if b is a product of finitely many interpolating Blaschke products. In the study of H^∞, Carleson-Newman Blaschke products play an important role, see [2].

Let M_s^+ be the set of all bounded positive (nonzero) singular Borel measures on ∂D with respect to the Lebesgue measure on ∂D. We use familiar notations: for $\mu,\nu\in M_s^+$, $\mu\ll\nu$ (absolutely continuous), $\mu\sim\nu$ (equivalent, i.e., $\mu\ll\nu$ and $\nu\ll\mu$), and $\delta_{e^{i\theta}}$ (the unit point mass at $e^{i\theta}\in\partial D$), and *supp* (μ) (the closed support set). A measure μ is called continuous if $\mu(\{e^{i\theta}\})=0$ for every $e^{i\theta}\in\partial D$. A measure $\mu\in M_s^+$ is called discrete if $\mu=\sum_{n=1}^\infty a_n\delta_{e^{i\theta_n}}$. We denote by $M_{s,c}^+$ and $M_{s,d}^+$ the sets of continuous and discrete measures in M_s^+, respectively. For each $\mu\in M_s^+$, the associated singular inner function S_μ is defined by

$$S_\mu(z)=\exp\left(-\int_{\partial D}\frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(e^{i\theta})\right), \quad z\in D.$$

See [5, 6, 7] for the study of singular inner functions related to the subject of this paper.

For each $\alpha\in D$ and an inner function I, we define the Frostman shift τ_α by

$$\tau_\alpha(I)(z)=\frac{I(z)-\alpha}{1-\overline{\alpha}I(z)}, \quad z\in D.$$

Trivially, $\tau_\alpha(I)$ is inner for every $\alpha\in D$. It is known as the Frostman theorem that $\tau_\alpha(I)$ is a Blaschke product for every $\alpha\in D$ except for a set of logarithmic capacity 0. We denote by $\mathcal{M}$ the family of inner functions I for which $\tau_\alpha(I)$ is a Carleson-Newman Blaschke product for every $\alpha\in D$ with $\alpha\neq 0$. In [8], Mortini and Nicolau studied the class $\mathcal{M}$, especially singular inner functions in $\mathcal{M}$. A typical example in $\mathcal{M}$ is

$$S_{\delta_{e^{i\theta}}}(z) = \exp\Big(- \frac{e^{i\theta}+z}{e^{i\theta}-z}\Big), \quad z \in D,$$

see [3]. Also we know that $S_\mu \in \mathcal{M}$ for every $\mu = \sum_{j=1}^n a_j \delta_{e^{i\theta_j}} \in M_{s,d}^+$ with $a_j > 0$. A nonempty closed subset E of ∂D is called ε-porous, $0 < \varepsilon < 1$, if for any subarc J of ∂D with $J \cap E \neq \emptyset$, there exists a subarc $\tilde{J} \subset J$ such that $\tilde{J} \cap E = \emptyset$ and $|\tilde{J}| > \varepsilon |J|$, where $|J|$ is the arc length of J. Simply E is called porous if E is ε-porous for some $0 < \varepsilon < 1$. A union of finitely many porous sets is also porous. In [8], Mortini and Nicolau proved that for a nonempty closed subset E of ∂D, E is porous if and only if $S_\mu \in \mathcal{M}$ for every μ with $supp\,(\mu) \subset E$. Exactly, they showed that if E is not porous then there exists a discrete measure $\mu \in M_{s,d}^+$ satisfying $supp\,(\mu) \subset E$ and $S_\mu \notin \mathcal{M}$. They also showed that there exists a continuous measure $\mu \in M_{s,c}^+$ satisfying $supp\,(\mu) = \partial D$ and $S_\mu \in \mathcal{M}$. In [7], the first author proved that for every closed subset E of ∂D, there is a discrete measure $\mu \in M_{s,d}^+$ such that $supp\,(\mu) = E$ and $S_\mu \in \mathcal{M}$. More precisely, it is proved that for each $\lambda \in M_{s,d}^+$, there is $\mu \in M_{s,d}^+$ satisfying $\mu \sim \lambda$ and $S_\mu \in \mathcal{M}$.

This paper is a continuation of the paper [7]. We are interested in a singular inner function S_μ such that $supp\,(\mu)$ is not porous. In Section 2, we prove that for each perfect subset E of ∂D, there is a continuous measure $\mu \in M_{s,c}^+$ such that $supp\,(\mu) = E$ and $S_\mu \in \mathcal{M}$. In Section 3, we prove that if $S_\lambda \in \mathcal{M}$ and $supp\,(\lambda)$ is not porous, then there exists $\mu \in M_s^+$ satisfying $\mu \sim \lambda - \lambda(\{\zeta_0\})\delta_{\zeta_0}$ for some $\zeta_0 \in \partial D$ and $S_\mu \notin \mathcal{M}$. This shows that if E is perfect but not porous, then there exists $\mu \in M_{s,c}^+$ such that $supp\,(\mu) = E$ and $S_\mu \notin \mathcal{M}$.

2. Continuous singular inner functions in $\mathcal{M}$

For an inner function I, we use the following notation:

$$\{R_1 < |I| < R_2\} = \{z \in D : R_1 < |I(z)| < R_2\}, \quad 0 < R_1 < R_2 < 1.$$

The following lemma is pointed out in [8, Theorem 1] and is essentially due to Hoffman's work [4].

Lemma 2.1. *Let I be an inner function. Then $I \in \mathcal{M}$ if and only if for every pair (R_1, R_2) with $0 < R_1 < R_2 < 1$, there exists a constant $c(R_1, R_2)$ depending on (R_1, R_2) with $0 < c(R_1, R_2) < 1$ such that $D_\rho(z, r) \not\subset \{R_1 < |I| < R_2\}$ for every $z \in D$ and r with $c(R_1, R_2) \leq r < 1$.*

The following is proved in [7, Lemma 2.2].

Lemma 2.2. *Let $\{\mu_j\}_j$ be a sequence in M_s^+ satisfying supp $(\mu_i) \cap$ supp $(\mu_j) = \emptyset$ for $i \neq j$ and $\sum_{j=1}^{\infty} \mu_j(\partial D) < \infty$. Write $\mu = \sum_{j=1}^{\infty} \mu_j$ and $\widetilde{\mu}_n = \sum_{j=1}^{n} \mu_j$. Let $\{\varepsilon_j\}_j$ be a sequence of numbers satisfying $0 < \varepsilon_j < \varepsilon_{j+1} < 1$ and $\prod_{j=1}^{\infty} \varepsilon_j > 0$ (or $\sum_{j=1}^{\infty}(1-\varepsilon_j) < \infty$). Suppose that*

$$\{|S_{\widetilde{\mu}_j}| < \varepsilon_j\} \cap \{|S_{\mu_{j+1}}| < \varepsilon_{j+1}\} = \emptyset$$

for every $j \geq 1$. Then for each pair (R_1, R_2) with $0 < R_1 < R_2 < 1$, there exists a positive integer n_0 such that if $D_\rho(z,r) \subset \{R_1 < |S_\mu| < R_2\}$, then either $D_\rho(z,r) \subset \{R_1 < |S_{\widetilde{\mu}_{n_0}}| < \varepsilon_{n_0}\}$ or $D_\rho(z,r) \subset \{R_1 < |S_{\mu_j}| < (R_2+1)/2\}$ for one and only one j with $j \geq n_0 + 1$.

One easily checks the following lemma which follows from the definition of singular inner functions.

Lemma 2.3. *Let E be a closed subset of ∂D and U be an open subset of $\mathbb{C}$ with $E \subset U$. Then for each $0 < \delta < 1$, there exists $\varepsilon > 0$ such that $|S_\mu| > \delta$ on $D \backslash U$ for every $\mu \in M_s^+$ with supp $(\mu) \subset E$ and $\mu(\partial D) < \varepsilon$.*

By [8, Proof of Theorem 2], we have the following.

Lemma 2.4. *Let E be a $\frac{1}{4}$-porous subset of ∂D. For each given pair (R_1, R_2) with $0 < R_1 < R_2 < 1$, there exists a constant $c(R_1, R_2)$ depending on (R_1, R_2) with $0 < c(R_1, R_2) < 1$ such that*

$$D_\rho(z,r) \not\subset \{R_1 < |S_\mu| < R_2\}$$

for every $\mu \in M_s^+$ with supp $(\mu) \subset E$ and for every r with $c(R_1, R_2) \leq r < 1$.

Mortini and Nicolau [8, Theorem 2] proved the following.

Lemma 2.5. *If E is a porous subset of ∂D, then $S_\mu \in \mathcal{M}$ for every $\mu \in M_s^+$ with supp $(\mu) \subset E$.*

It is known that there are two types of continuous singular measures $\mu \in M_{s,c}^+$ with $S_\mu \in \mathcal{M}$. One is $\mu \in M_{s,c}^+$ such that *supp* (μ) is porous (by Lemma 2.5), and another one is given in [8, Proposition 6.1]. The following is the main theorem in this section.

Theorem 2.6. *If E is a perfect subset of ∂D, then there exists $\mu \in M_{s,c}^+$ such that supp $(\mu) = E$ and $S_\mu \in \mathcal{M}$.*

Proof. We may assume that E is not porous. We devide the proof into three steps.

Step 1. We shall prove that for every perfect subset A of ∂D, there exists a perfect $\frac{1}{4}$-porous subset B of A, First, we take a closed subarc I_0 of ∂D such that $|I_0| \le 1$, $A \cap I_0$ is perfect, and two end points of I_0 are contained in A. Next, we take an open subarc J_0 of I_0 satisfying

(2.1) $|J_0| > |I_0|/2$,

(2.2) $I_0 \setminus J_0$ consists of two closed arcs $I_{0,0}$ and $I_{0,1}$,

(2.3) $A \cap I_{0,0}$ and $A \cap I_{0,1}$ are perfect sets,

(2.4) all end points of $I_{0,0}$ and $I_{0,1}$ are contained in A.

For each $I_{0,i}, i = 0, 1$, we take an open subarc $J_{0,i}$ of $I_{0,i}$ satisfying

(2.5) $|J_{0,i}| > |I_{0,i}|/2$,

(2.6) $I_{0,i} \setminus J_{0,i}$ consists of two closed arcs $I_{0,i,0}$ and $I_{0,i,1}$,

(2.7) $A \cap I_{0,i,0}$ and $A \cap I_{0,i,1}$ are perfect sets,

(2.8) all end points of $I_{0,i,0}$ and $I_{0,i,1}$ are contained in A.

Repeating the same argument, we get a family of closed arcs $\{I_\lambda : \lambda \in \Lambda_n, n = 1, 2, \cdots\}$, where $\Lambda_n = \{(0, i_1, \cdots, i_n) : i_j = 0 \text{ or } 1\}$, and it is not difficult to see that

$$\bigcap_{n=1}^{\infty} \Big(\bigcup_{\lambda \in \Lambda_n} I_\lambda \Big) \cap A$$

is a perfect $\frac{1}{4}$-porous subset of A.

Setp 2. In this step, we show that there is a sequence of mutually disjoint perfect $\frac{1}{4}$-porous subsets $\{E_n\}_n$ of E such that $\bigcup_{n=1}^{\infty} E_n$ is dense in E.

By Step 1, there is a perfect $\frac{1}{4}$-porou subset E_1 of E. Let

$$\sigma_1 = \sup_{\xi \in E} dist(\xi, E_1).$$

Since E is not porous, $\sigma_1 > 0$. We take a perfect set E_1' with

$$E_1' \subset \big\{\zeta \in E : dist(\zeta, E_1) > \sigma_1/2\big\}.$$

By Step 1, there is a perfect $\frac{1}{4}$-porous subset E_2 of E_1'. Obviously, $E_1 \cap E_2 = \emptyset$. Let

$$\sigma_2 = \sup_{\xi \in E} dist(\xi, E_1 \cup E_2).$$

Since $E_1 \cup E_2$ is porous, $\sigma_2 > 0$. We take a perfect set E_2' with

$$E_2' \subset \big\{\zeta \in E : dist(\zeta, E_1 \cup E_2) > \sigma_2/2\big\}.$$

By Step 1, there is a perfect $\frac{1}{4}$-porous subset E_3 of E_2'. Then E_1, E_2, E_3 are mutually disjoint. Repeating the same argument, we have a sequence of mutually disjoint perfect $\frac{1}{4}$-porous subsets $\{E_n\}_n$ of E satisfying

$$E_{n+1} \subset \Big\{\zeta \in E : dist\Big(\zeta, \bigcup_{j=1}^{n} E_j\Big) > \sigma_n/2\Big\},$$

where

$$\sigma_n = \sup_{\xi \in E} dist\Big(\xi, \bigcup_{j=1}^{n} E_j\Big) > 0.$$

We shall prove that $\bigcup_{n=1}^{\infty} E_n$ is dense in E. To prove this, suppose not. Then there exists σ_0 satisfying $\sigma_n \geq \sigma_0 > 0$ for every n. Take a sequence of points $\{\zeta_n\}_n$ with $\zeta_n \in E_n$. There is a subsequence $\{\zeta_{n_j}\}_j$ of $\{\zeta_n\}_n$ such that $\zeta_{n_j} \to \zeta_0$ as $j \to \infty$ for some $\zeta_0 \in E$. By the construction of the sequence $\{E_n\}_n$,

$$dist(\zeta_{n_{j+1}}, \zeta_{n_j}) > \sigma_{n_{j+1}-1}/2 \geq \sigma_0/2 > 0$$

for every j. This is a contradiction.

Step 3. We follow the proof of Theorem 2.5 given in [7]. Let $\{\varepsilon_j\}_j$ be a sequence of numbers with $0 < \varepsilon_j < \varepsilon_{j+1} < 1$ for every $j \geq 1$ and $\prod_{j=1}^{\infty} \varepsilon_j > 0$. By Step 2, there is a sequence of mutually disjoint perfect $\frac{1}{4}$-porous subsets $\{E_n\}_n$ of E such that $\bigcup_{n=1}^{\infty} E_n$ is dense in E. For each n, take $\mu_n \in M_{s,c}^+$ with $supp\,(\mu_n) = E_n$ and $\|\mu_n\| = 1$. Applying Lemma 2.3, we can find a sequence of positive numbers $\{a_j\}_j$ with $\sum_{j=1}^{\infty} a_j < \infty$ satisfying

$$\Big\{\Big|S_{\sum_{j=1}^{n} a_j\mu_j}\Big| < \varepsilon_n\Big\} \cap \big\{|S_{a_{n+1}\mu_{n+1}}| < \varepsilon_{n+1}\big\} = \emptyset$$

for every $n \geq 1$. Write

$$\mu = \sum_{j=1}^{\infty} a_j\mu_j \quad \text{and} \quad \widetilde{\mu}_n = \sum_{j=1}^{n} a_j\mu_j.$$

To prove $S_\mu \in \mathcal{M}$ by contradiction, we assume that $S_\mu \notin \mathcal{M}$. By Lemma 2.1, there exists a pair (R_1, R_2) with $0 < R_1 < R_2 < 1$ and a sequence of pseudo-hyperbolic disks $\{D_\rho(z_k, r_k)\}_k$ such that

$$D_\rho(z_k, r_k) \subset \{R_1 < |S_\mu| < R_2\}$$

for every k and $r_k \to 1$ as $k \to \infty$. By Lemma 2.2, there exists a positive integer n_0 such that for each k, either

$$D_\rho(z_k, r_k) \subset \{R_1 < |S_{\widetilde{\mu}_{n_0}}| < \varepsilon_{n_0}\} \tag{2.9}$$

or

$$D_\rho(z_k, r_k) \subset \{R_1 < |S_{a_{j(k)}\mu_{j(k)}}| < (R_2+1)/2\} \tag{2.10}$$

for one and only one $j(k)$ with $j(k) \geq n_0 + 1$. Since $\bigcup_{j=1}^{n_0} E_j$ is a porous set, by Lemma 2.5 we have $S_{\widetilde{\mu}_{n_0}} \in \mathcal{M}$. Hence by Lemma 2.1, the number of integers k satisfying (2.9) is finite, so that (2.10) holds for all large enough k's. Since $r_k \to 1$, this contradicts the assertion of Lemma 2.4. □

3. Equivalent measures and $\mathcal{M}$

We denote by $M(\partial D)$ the space of all bounded complex Borel measures on ∂D. With the total variation norm, $M(\partial D)$ is a Banach space, and $M(\partial D) = C(\partial D)^*$, where $C(\partial D)$ is the Banach space of all continuous functions on ∂D. We may consider the weak*-topology on $M(\partial D)$.

Lemma 3.1. *Let $\lambda, \nu \in M_s^+$ with supp $(\nu) \subset$ supp (λ). Then there exists a sequence of measures $\{\lambda_n\}_n$ in M_s^+ satisfying that $\|\lambda_n\| \leq 2\|\nu\|$, $\lambda_n \sim \lambda$, and $\lambda_n \to \nu$ in the weak*-topology in $M(\partial D)$.*

Proof. Let $\{\varepsilon_n\}_n$ be a sequence of positive numbers with $\varepsilon_n\|\lambda\| \leq \|\nu\|$ and $\varepsilon_n \to 0$. For each positive integer n, let

$$J_{n,j} = \left\{e^{i\theta} : \frac{2\pi(j-1)}{n} \leq \theta < \frac{2\pi j}{n}\right\}, \quad 1 \leq j \leq n$$

and

$$\lambda_n = \varepsilon_n \lambda + \sum_j \left\{ \frac{\nu(J_{n,j}))}{\lambda(J_{n,j})} \lambda|_{J_{n,j}} : \lambda(J_{n,j}) \neq 0 \right\}.$$

It is not difficult to check that $\{\lambda_n\}_n$ has the desired properties. □

Note that if $\lambda_n \to \nu$ in the weak*-topology in $M(\partial D)$, then $S_{\lambda_n} \to S_\nu$ uniformly on each compact subset of D as $n \to \infty$.

Theorem 3.2. *Let $\lambda \in M_s^+$ with $S_\lambda \in \mathcal{M}$. If supp (λ) is not a porous set, then there exists $\mu \in M_s^+$ satisfying $\mu \sim \lambda - \lambda(\{\zeta_0\})\delta_{\zeta_0}$ for some $\zeta_0 \in \partial D$ and $S_\mu \notin \mathcal{M}$.*

Proof. By [8, Theorem 2], there exists $\nu \in M_s^+$ satisfying *supp* $(\nu) \subset$ *supp* (λ) and $S_\nu \notin \mathcal{M}$. By Lemma 2.1, there exist R_1, R_2 with $0 < R_1 < R_2 < 1$ and a sequence of pseudo-hyperbolic disks $\{D_\rho(z_j, r_j)\}_j$ with $r_j \to 1$ such that

$$D_\rho(z_j, r_j) \subset \{z \in D : R_1 < |S_\nu(z)| < R_2\} \tag{3.1}$$

for every j. Note that $|z_n| \to 1$. We may assume that $z_j \to \zeta_0$ for some $\zeta_0 \in \partial D$. Since $S_\nu = S_{\nu - \nu(\{\zeta_0\})\delta_{\zeta_0}} S_{\nu(\{\zeta_0\})\delta_{\zeta_0}}$, we may assume that

$\nu(\{\zeta_0\}) = 0$. Let $\{J_n\}_n$ be a sequence of open arcs in ∂D with $\overline{J}_{n+1} \subset J_n$ and $\bigcap_{n=1}^{\infty} J_n = \{\zeta_0\}$. Write $J_0 = \partial D$. Let α_n, β_n be the end points of J_n. We may further assume that $\lambda(\{\alpha_n, \beta_n\}) = \nu(\{\alpha_n, \beta_n\}) = 0$ for every n. For each $n \geq 1$, let

$$\nu_n = \nu|_{J_n}. \tag{3.2}$$

Since $\|\nu_n\| \to 0$,

$$|S_{\nu_n}| \to 1 \quad \text{uniformly on } D_\rho(z_j, r_j) \tag{3.3}$$

as $n \to \infty$ for each fixed j. Let $\{\varepsilon_i\}_i$ be a sequence of positive numbers with

$$0 < \varepsilon_i < 1 \quad \text{and} \quad \prod_{i=1}^{\infty} \varepsilon_i > 0. \tag{3.4}$$

By induction, we can choose a subsequence $\{J_{n_k}\}_k$ of $\{J_n\}_n$, a subsequence $\{D_\rho(z_{j_k}, r_{j_k})\}_k$ of $\{D_\rho(z_j, r_j)\}_j$, and a sequence $\{\mu_k\}_k$ in M_s^+ with $\mu_k \ll \lambda$ satisfying certain additional properties mentioned later.

Let $j_1 = 1$. By (3.1), (3.2), and (3.3), there exists a positive integer n_1 such that

$$R_1 < |S_{\nu - \nu_{n_1}}| < R_2 \quad \text{on } D_\rho(z_{j_1}, r_{j_1}) \tag{3.5}$$

and

$$|S_{\nu_{n_1}}| > \varepsilon_2 \quad \text{on } D_\rho(z_{j_1}, r_{j_1}). \tag{3.6}$$

For convenience, let $n_0 = 0$.

Since $\mathit{supp}\,(\nu) \subset \mathit{supp}\,(\lambda)$, by (3.2) $\mathit{supp}\,(\nu - \nu_{n_1}) \subset \mathit{supp}\,(\lambda) \setminus J_{n_1}$. By Lemma 3.1, there exists $\mu_1 \in M_s^+$ such that $\|\mu_1\| \leq 2\|\nu - \nu_{n_1}\|$, $\mu_1 \sim \lambda|_{J_{n_0} \setminus J_{n_1}}$, and by (3.5), $R_1 < |S_{\mu_1}| < R_2$ on $D_\rho(z_{j_1}, r_{j_1})$. Since $\zeta_0 \notin \mathit{supp}\,(\nu - \nu_{n_1})$,

$$\inf_{z \in D_\rho(z_j, r_j)} |S_{\nu - \nu_{n_1}}(z)| \to 1$$

as $j \to \infty$. Then by (3.1), there exists a positive integer j_2 with $j_2 > j_1$ such that

$$R_1 < |S_{\nu_{n_1}}| < R_2 \quad \text{on } D_\rho(z_{j_2}, r_{j_2}). \tag{3.7}$$

Since $\mu_1 \sim \lambda|_{J_{n_0} \setminus J_{n_1}}$,

$$\inf_{z \in D_\rho(z_j, r_j)} |S_{\mu_1}(z)| \to 1$$

as $j \to \infty$, so we may further assume that

$$|S_{\mu_1}| > \varepsilon_1 \quad \text{on } D_\rho(z_{j_2}, r_{j_2}).$$

By (3.3) and (3.7), there exists a positive integer n_2 with $n_2 > n_1$ such that

$$R_1 < |S_{\nu_{n_1}-\nu_{n_2}}| < R_2 \quad \text{on } D_\rho(z_{j_2}, r_{j_2}),$$

$$|S_{\nu_{n_2}}| > \varepsilon_3 \quad \text{on } D_\rho(z_{j_1}, r_{j_1}) \cup D_\rho(z_{j_2}, r_{j_2}), \tag{3.8}$$

and by (3.6)

$$|S_{\nu_{n_1}-\nu_{n_2}}| > \varepsilon_2 \quad \text{on } D_\rho(z_{j_1}, r_{j_1}).$$

By Lemma 3.1, there exists $\mu_2 \in M_s^+$ such that $\|\mu_2\| \le 2\|\nu_{n_1} - \nu_{n_2}\|$, $\mu_2 \sim \lambda|_{J_{n_1}\setminus J_{n_2}}$,

$$R_1 < |S_{\mu_2}| < R_2 \quad \text{on } D_\rho(z_{j_2}, r_{j_2}),$$

and

$$|S_{\mu_2}| > \varepsilon_2 \quad \text{on } D_\rho(z_{j_1}, r_{j_1}).$$

There exists a positive integer j_3 with $j_3 > j_2$ such that

$$R_1 < |S_{\nu_{n_2}}| < R_2 \quad \text{on } D_\rho(z_{j_3}, r_{j_3})$$

and

$$|S_{\mu_i}| > \varepsilon_i \quad \text{on } D_\rho(z_{j_3}, r_{j_3}) \text{ for } i = 1, 2.$$

Take a positive integer n_3 with $n_3 > n_2$ satisfying

$$R_1 < |S_{\nu_{n_2}-\nu_{n_3}}| < R_2 \quad \text{on } D_\rho(z_{j_3}, r_{j_3}),$$

$$|S_{\nu_{n_3}}| > \varepsilon_4 \quad \text{on } \bigcup_{k=1}^{3} D_\rho(z_{j_k}, r_{j_k}),$$

and by (3.8)

$$|S_{\nu_{n_2}-\nu_{n_3}}| > \varepsilon_3 \quad \text{on } D_\rho(z_{j_1}, r_{j_1}) \cup D_\rho(z_{j_2}, r_{j_2}).$$

Then there exists $\mu_3 \in M_s^+$ such that $\|\mu_3\| \le 2\|\nu_{n_2} - \nu_{n_3}\|$, $\mu_3 \sim \lambda|_{J_{n_2}\setminus J_{n_3}}$,

$$R_1 < |S_{\mu_3}| < R_2 \quad \text{on } D_\rho(z_{j_3}, r_{j_3}),$$

and

$$|S_{\mu_3}| > \varepsilon_3 \quad \text{on } D_\rho(z_{j_1}, r_{j_1}) \cup D_\rho(z_{j_2}, r_{j_2}).$$

Inductively, we can get sequences of positive integers $\{n_k\}_k$ and $\{j_k\}_k$, and a sequence $\{\mu_k\}_k$ in M_s^+ such that

$$\|\mu_k\| \le 2\|\nu_{n_{k-1}} - \nu_{n_k}\|, \tag{3.9}$$

$$\mu_k \sim \lambda|_{J_{n_{k-1}}\setminus J_{n_k}}, \tag{3.10}$$

$$R_1 < |S_{\mu_k}| < R_2 \quad \text{on } D_\rho(z_{j_k}, r_{j_k}), \tag{3.11}$$

$$|S_{\mu_k}| > \varepsilon_k \quad \text{on } \bigcup_{i=1}^{k-1} D_\rho(z_{j_i}, r_{j_i}), \tag{3.12}$$

and

$$|S_{\mu_i}| > \varepsilon_i \quad \text{on} \;\; D_\rho(z_{j_k}, r_{j_k}) \quad \text{for } 1 \leq i < k. \tag{3.13}$$

Let

$$\mu = \sum_{k=1}^{\infty} \mu_k. \tag{3.14}$$

Then

$$\begin{aligned}
\|\mu\| &\leq \sum_{k=1}^{\infty} \|\mu_k\| \\
&\leq 2\|\nu - \nu_{n_1}\| + 2\sum_{k=2}^{\infty} \|\nu_{n_{k-1}} - \nu_{n_k}\| \qquad \text{by (3.9)} \\
&= 2\Big(\|\nu\| - \|\nu_{n_1}\| + \sum_{k=2}^{\infty}(\|\nu_{n_{k-1}}\| - \|\nu_{n_k}\|)\Big) \\
&\leq 2\|\nu\| < \infty.
\end{aligned}$$

Hence $\mu \in M_s^+$, and by (3.10) $\mu \sim \lambda - \lambda(\{\zeta_0\})\delta_{\zeta_0}$. Also by (3.11),

$$\sup_{z \in D_\rho(z_{j_k}, r_{j_k})} |S_\mu(z)| \leq \sup_{z \in D_\rho(z_{j_k}, r_{j_k})} |S_{\mu_k}(z)| \leq R_2,$$

and by (3.11), (3.12), and (3.13), we have

$$\begin{aligned}
\inf_{z \in D_\rho(z_{j_k}, r_{j_k})} |S_\mu(z)| &= \inf_{z \in D_\rho(z_{j_k}, r_{j_k})} \prod_{i=1}^{\infty} |S_{\mu_i}(z)| \\
&\geq \Big(\prod_{i=1}^{k-1} \varepsilon_i\Big) R_1 \Big(\prod_{i=k+1}^{\infty} \varepsilon_i\Big) \\
&= R_1 \prod_{i=1}^{\infty} \varepsilon_i \\
&> 0 \qquad \text{by (3.4)}.
\end{aligned}$$

Therefore, we have

$$D_\rho(z_{j_k}, r_{j_k}) \subset \Big\{ z \in D : R_1 \prod_{i=1}^{\infty} \varepsilon_i \leq |S_\mu(z)| \leq R_2 \Big\}$$

for every k. By Lemma 2.1, $S_\mu \notin \mathcal{M}$. □

Corollary 3.3. *If $\lambda \in M_{s,c}^+$ and supp(λ) is not porous, then there exists $\mu \in M_{s,c}^+$ such that $\mu \sim \lambda$ and $S_\mu \notin \mathcal{M}$.*

Corollary 3.4. *If E be a perfect subset of ∂D which E is not porous, then there exists $\mu \in M^+_{s,c}$ satisfying supp $(\mu) = E$ and $S_\mu \notin \mathcal{M}$.*

We end the paper with the following two problems.

Problem 3.5. *Is there $\lambda \in M^+_{s,c}$ satisfying $S_\mu \notin \mathcal{M}$ for every $\mu \in M^+_{s,c}$ with $\mu \sim \lambda$?*

Problem 3.6. *Is there $\lambda \in M^+_{s,d}$ such that $S_\mu \in \mathcal{M}$ for every $\mu \in M^+_{s,d}$ with $\mu \sim \lambda$ and supp (λ) is not porous?*

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