A Pair of Orthogonal Frames *

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Abstract

We start with a characterization of a pair of frames to be orthogonal in a shift-invariant space and then give a simple construction of a pair of orthogonal shift-invariant frames. This is applied to obtain a construction of a pair of Gabor orthogonal frames as an example. We also give a construction of a pair of orthogonal wavelet frames.

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1 Introduction

Let X be a (countable) system for a separable Hilbert space $\mathcal{H}$ over the complex field $\mathbb{C}$. The *synthesis operator* $T_X : \ell_2(X) \to \mathcal{H}$ is defined by

$$T_X a := \sum_{h \in X} a_h h$$

for $a = (a_h)_{h \in X}$. The adjoint operator T_X^* of T_X, called the *analysis operator*, is

$$T_X^* : \mathcal{H} \to \ell_2(X); \quad T_X^* f := (\langle f, h \rangle)_{h \in X}.$$

Recall that X is a frame for $\mathcal{H}$ if and only $S_X := T_X T_X^* : \mathcal{H} \to \mathcal{H}$, the *frame operator or dual Gramian*, is bounded and has a bounded inverse [3, 6] and it is a tight frame (with frame bound 1) if and only if S_X is the identity operator. The system X is a Riesz system (might be a subspace of $\mathcal{H}$) if and only if its *Gramian* $G_X := T_X^* T_X$ is bounded and has a bounded inverse and it is an orthonormal system of $\mathcal{H}$ if and only if G_X is the identity operator.

Definition 1.1 *Let X and $Y = RX$, where $R : h \to Rh$ is an association between X and Y, be two frames of $\mathcal{H}$. We call that X and Y are a pair of orthogonal frames if $T_Y T_X^* = 0$, i.e., $\sum_{h \in X} \langle f, h \rangle Rh = 0$, for all $f \in \mathcal{H}$.*

Notice that the definition is symmetric with respect to X and Y. The orthogonal frames have been studied in [11] and [12]. The various applications of orthogonal frames are also discussed in both papers.

For a pair of frames X and $Y = RX$ in $\mathcal{H}$, we have the following simple characterization of orthogonal frames via its Gramians.

Proposition 1.2 *Let X and $Y = RX$ be frames for $\mathcal{H}$ with synthesis operators T_X and T_Y, respectively. Then, X and Y are a pair of orthogonal frames if and only if $G_Y G_X = 0$.*

Proof. Suppose that $T_Y T_X^* = 0$. Then $T_Y^* T_Y T_X^* T_X = T_Y^* 0 T_X = 0$. Suppose, on the other hand, that $T_Y^* T_Y T_X^* T_X = 0$. Then

$$0 = (T_Y T_Y^*)(T_Y T_X^*)(T_X T_X^*) = S_Y (T_Y T_X^*) S_X.$$

Since $S_Y, T_Y T_X^*$ and S_X are bounded operators from $\mathcal{H}$ to $\mathcal{H}$ and since S_X and S_Y are invertible, $0 = T_Y T_X^*$. $\square$

The paper is organized as follows: in Section 2, we discuss the orthogonal frames in a general shift-invariant subspace of $L_2(\mathbb{R}^d)$, and apply the results to construct Gabor orthogonal frames. Section 3 provides construction of wavelet orthogonal frames.

2 Orthogonal frames in shift invariant space

In this section, we consider orthogonal frames in a shift-invariant subspace of $L_2(\mathbb{R}^d)$. Let Φ be a countable subset of $L_2(\mathbb{R}^d)$, and $E(\Phi) := \{\phi(\cdot - k) : k \in \mathbb{Z}^d\}$. Define

$$\mathcal{S}(\Phi) := \overline{\operatorname{span}}\, E(\Phi),$$

the smallest closed subspace that contains $E(\Phi)$. The space $\mathcal{S}(\Phi)$ is called the *shift-invariant* space generated by Φ and Φ is called a *generating set* for $\mathcal{S}(\Phi)$. The shift invariant space has been studied in the literature, e.g., [1, 2, 5, 7].

For $\omega \in \mathbb{T}^d$ we define the *pre-Gramian* via

$$J_\Phi(\omega) = \left(\widehat{\phi}(\omega + \alpha)\right)_{\alpha \in 2\pi\mathbb{Z}^d, \phi \in \Phi},$$

where $\widehat{\phi}$ is the Fourier transform of ϕ. Note that the domain of the pre-Gramian matrix as an operator is $\ell_2(\Phi)$ and its co-domain is $\ell_2(\mathbb{Z}^d)$. The pre-Gramian can be regarded as the synthesis operator represented in Fourier domain as it was extensively studied in [7]. In particular, we have (see, e.g., [7, 2])

Proposition 2.1 *The shift invariant system $E(\Phi)$ is a frame of $\mathcal{S}(\Phi)$ if and only if $J_\Phi(\omega) J_\Phi^*(\omega)$ is uniformly bounded with uniformly bounded inverse on the range of*

$J_\Phi(\omega)$ *for a.e.* ω *such that* $\operatorname{ran} J_\Phi(\omega) \neq \{0\}$. *In particularly, when* $\mathcal{S}(\Phi) = L_2(\mathbb{R}^d)$, $E(\Phi)$ *is a frame of* $L_2(\mathbb{R}^d)$ *if and only if there are* $0 < A \leq B < \infty$, *such that* $AI_{\ell_2(\mathbb{Z}^d)} \leq J_\Phi(\omega) J_\Phi^*(\omega) \leq BI_{\ell_2(\mathbb{Z}^d)}$, *a.e.* $\omega \in \mathbb{R}^d$; *and it is a tight frame of* $L_2(\mathbb{R}^d)$ *if and only if* $J_\Phi(\omega) J_\Phi^*(\omega) = I_{\ell_2(\mathbb{Z}^d)}$, *for a.e.* $\omega \in \mathbb{R}^d$.

Let Φ and $\Psi = R\Phi$ where R is an association satisfying $R\phi(\cdot - k) = (R\phi)(\cdot - k)$ be countable subsets of $L_2(\mathbb{R}^d)$. Suppose that $\mathcal{S}(\Phi) = \mathcal{S}(\Psi)$ and both $E(\Phi)$ and $E(\Psi)$ are frames of $\mathcal{S}(\Phi)$. Then, $E(\Phi)$ and $E(\Psi)$ are orthogonal frames in $\mathcal{S}(\Phi)$ if and only if for all $f \in \mathcal{S}(\Phi)$,

$$Sf := T_{E(\Psi)} T^*_{E(\Phi)} f = 0.$$

We define the *mixed dual Gramian* as in [9] as

$$\widetilde{G}(\omega) = J_\Psi(\omega) J_\Phi^*(\omega),$$

and *Gramians* as

$$G_\Phi(\omega) = J_\Phi^*(\omega) J_\Phi(\omega), \quad G_\Psi(\omega) = J_\Psi^*(\omega) J_\Psi(\omega).$$

Then, it was proven in [9] that for an arbitrary $f \in L_2(\mathbb{R}^d)$

$$\widehat{Sf}_{|\omega+\alpha} = \widetilde{G}(\omega) \hat{f}_{|\omega+\alpha},$$

where $\hat{g}_{|\omega+\alpha}$ is the column vector $(\hat{g}(\omega+\gamma)_{\gamma \in 2\pi\mathbb{Z}^d})^T$. With this, one can prove easily that $Sf = 0$ for all $f \in L_2(\mathbb{R}^d)$ if and only if $\widetilde{G}(\omega) = 0$ for a.e. $\omega \in \mathbb{R}^d$. When $f \in \mathcal{S}(\Phi)$, then

$$\hat{f} = \sum_{\phi \in \Phi} \widehat{a}_\phi \hat{\phi},$$

where $\widehat{a}_\phi$ is defined on $\mathbb{T}^d$. Further the column vector

$$\hat{f}_{|\omega+\alpha} = J_\Phi(\omega) A(\omega)$$

where column vector $A(\omega) = (\widehat{a}_\phi(\omega))^T_{\phi \in \Phi}$. Putting everything together, we have:

Theorem 2.2 *Let* Φ *and* $\Psi = R\Phi$ *be defined as above. Suppose that* $\mathcal{S}(\Phi) = \mathcal{S}(\Psi)$ *and that* $E(\Phi)$ *and* $E(\Psi)$ *are frames for* $\mathcal{S}(\Phi)$. *Then, the following are equivalent:*

(1) *Systems* $E(\Phi)$ *and* $E(\Psi)$ *are orthogonal frames for* $\mathcal{S}(\Phi)$*;*

(2) $J_\Psi(\omega)J^*_\Phi(\omega)J_\Phi(\omega) = 0$ *a.e.* $\omega \in \mathbb{R}^d$*;*

(3) $G_\Psi(\omega)G_\Phi(\omega) = 0$ *a.e.* $\omega \in \mathbb{R}^d$*.*

In particular, when $\mathcal{S}(\Phi) = L_2(\mathbb{R}^d)$*,* $E(\Phi)$ *and* $E(\Psi)$ *are a pair of orthogonal frames if and only if* $J_\Psi(\omega)J^*_\Phi(\omega) = 0$ *a.e.* $\omega \in \mathbb{R}^d$*.*

Item (3) follows from (2) and the fact that $J^*_\Psi(\omega)$ has bounded inverse on the range of $J_\Psi(\omega)$ a.e. $\omega \in \mathbb{R}^d$, whenever $E(\Psi)$ is a frame of $S(\Psi)$ (see [7]).

Suppose that $\Phi := \{\phi_1, \phi_2, \cdots, \phi_r\} \subset L_2(\mathbb{R}^d)$ where r can be ∞, and that $E(\Phi)$ is a frame for $\mathcal{S}(\Phi)$. We now give a construction of a pair of orthogonal frames in $\mathcal{S}(\Phi)$. Let $U := (U_1; U_2)$ be a $2r \times 2r$ matrix with each entry being a 2π periodic function of $L_2(\mathbb{T}^d)$ satisfying $U^*(\omega)U(\omega) = I_{2r}$ a.e. $\omega \in \mathbb{R}^d$, where U_1 is the submatrix of the first r columns and U_2 the last r columns. Define $\widehat{\Phi}_1 := U_1\widehat{\Phi}$, and $\widehat{\Phi}_2 := U_2\widehat{\Phi}$. It is easy to check by the Bessel property of $E(\Phi)$ that $\mathcal{S}(\Phi) = \mathcal{S}(\Phi_1) = \mathcal{S}(\Phi_2)$ with each of Φ_1 and Φ_2 consists of $2r$ elements of $L_2(\mathbb{R}^d)$. Furthermore, it is direct to check that

$$J_{\Phi_1}(\omega) = J_\Phi(\omega)U_1^T(\omega); \quad \text{and} \quad J_{\Phi_2}(\omega) = J_\Phi(\omega)U_2^T(\omega).$$

It is easy to see that $\operatorname{ran} J_{\Phi_1}(\omega) = \operatorname{ran} J_\Phi(\omega)$ a.e., since $U_1^T(\omega) : \ell^2(\Phi_1) \to \ell^2(\Phi)$ is onto by $U^T(\omega)(U^T(\omega))^* = I_{2r}$ a.e. $\omega \in \mathbb{T}^d$. Moreover,

$$\begin{aligned} J_{\Phi_1}(\omega)J_{\Phi_1}(\omega)^* &= J_\Phi(\omega)U_1^T(\omega)(J_\Phi(\omega)U_1^T(\omega))^* = J_\Phi(\omega)(U_1^*(\omega)U_1(\omega))^T J_\Phi(\omega)^* \\ &= J_\Phi(\omega)I_r J_\Phi(\omega)^* = J_\Phi(\omega)J_\Phi(\omega)^*. \end{aligned}$$

Hence, $E(\Phi_1)$ is a frame on $\mathcal{S}(\Phi_1) = \mathcal{S}(\Phi)$ by Proposition 2.1. Similarly, we have that $E(\Phi_2)$ forms a frame of $\mathcal{S}(\Phi_2) = \mathcal{S}(\Phi)$ as well. Furthermore, $E(\Phi_1)$ and $E(\Phi_2)$ form a pair of orthogonal frames of $\mathcal{S}(\Phi)$. Indeed, this follows from the fact for a.e. $\omega \in \mathbb{R}^d$

$$\begin{aligned} G_{\Phi_1}(\omega)G_{\Phi_2}(\omega) &= J_{\Phi_1}(\omega)^* J_{\Phi_1}(\omega) J_{\Phi_2}(\omega)^* J_{\Phi_2}(\omega) \\ &= J_{\Phi_1}(\omega)^* J_{\Phi}(\omega) U_1^T(\omega)(U_2^T(\omega))^* J_{\Phi}(\omega)^* J_{\Phi_2}(\omega) \\ &= J_{\Phi_1}(\omega)^* J_{\Phi}(\omega) U_1^T(\omega)(U_2^*(\omega))^T J_{\Phi}(\omega)^* J_{\Phi_2}(\omega) \\ &= J_{\Phi_1}(\omega)^* J_{\Phi}(\omega)(U_2^*(\omega)U_1(\omega))^T J_{\Phi}(\omega)^* J_{\Phi_2}(\omega) \\ &= J_{\Phi_1}(\omega)^* J_{\Phi}(\omega) 0 J_{\Phi}(\omega)^* J_{\Phi_2}(\omega) = 0 \end{aligned}$$

and Theorem 2.2. Finally, we note that the matrix U can be chosen to be a constant $2r \times 2r$ unitary matrix.

Since the Gabor system is shift-invariant, we next apply the above construction to the Gabor system to give an example. Let $G := \{g_1, g_2, \cdots, g_\gamma\} \subset L_2(\mathbb{R}^d)$, where γ is a positive integer, and

$$\Phi := \{M^l g_j : l \in \mathbb{Z}^d, 1 \le j \le \gamma\},$$

where $M^t f(x) := e^{it\cdot x} f(x)$ is the modulation operator for $t \in \mathbb{R}^d$. Then $E(\Phi)$ is equivalent to a Gabor system generated by G [10]. Note, in general, the shift operator and modulation operator can be chosen to be any d-dimensional lattice instead of $\mathbb{Z}^d$. For the simplicity, we assume that both the shift lattice and the modulation lattice are $\mathbb{Z}^d$, however, the discussion here can be carried out similarly for the more general shift and modulation lattices.

Suppose that $E(\Phi)$ is a frame for its closed linear span. Let $V := (V_1; V_2)$ be a $2\gamma \times 2\gamma$ constant unitary matrix, where V_1 is the submatrix formed by the first γ columns of V and V_2 is the submatrix formed by the last γ columns of V. We show that the Gabor systems generated by $G_1 := V_1 G$ and $G_2 := V_2 G$ are orthogonal frames by using the above result.

Let U_1 be the block diagonal (infinite) matrix of size $(\mathbb{Z}^d \times \{1, 2 \cdots, 2\gamma\}) \times (\mathbb{Z}^d \times \{1, 2, \cdots, \gamma\})$ such that

$$\text{the } (l,j)(l',j')\text{-th entry of } U_1 = \begin{cases} 0 & \text{if } l \neq l', \\ (V_1)_{j,j'} & \text{if } l = l'. \end{cases}$$

Similarly, one can define block diagonal matrix U_2 by V_2. Then, the matrix $U := (U_1; U_2)$ is unitary. Furthermore, the Gabor system generated by $V_1 G$ is $E(\Phi_1)$ satisfying $\Phi_1 := U_1 \Phi$ and the system generated by $V_2 G$ is $E(\Phi_2)$ satisfying $\Phi_2 := U_2 \Phi$. Hence $E(\Phi_1)$ and $E(\Phi_2)$ are a pair of orthogonal frames.

3 Orthogonal wavelet frames

Let $\Psi := \{\psi_1, \psi_2, \cdots, \psi_r\} \subset L_2(\mathbb{R}^d)$, where r is a positive integer, and s be an integer-valued invertible $d \times d$ matrix such that s^{-1} is contractive. Define a unitary dilation operator D on $L_2(\mathbb{R}^d)$ via

$$D : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d) : f \mapsto |\det s|^{1/2} f(s\cdot).$$

Then, the following collection is called a *wavelet (or affine)* system generated by $\Psi = \{\psi_1, \ldots, \psi_r\}$:

$$X(\Psi) := \{D^j E^k \psi_l : j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \le l \le r\}, \tag{3.1}$$

where $E^k f := f(\cdot - k)$.

The wavelet system is not shift-invariant. To apply the theorem in the previous section, one needs to use quasi-affine system $X^q(\Psi)$, i.e. the smallest shift invariant system containing $X(\Psi)$. Then, applying the similar approach of [9], one can obtain that two wavelet frame systems $X(\Psi_1)$ and $X(\Psi_2)$ are a pair of orthogonal frames if and only if the mixed dual Gramian of the corresponding quasi-affine systems $X^q(\Psi)$ and $X^q(\Psi_2)$ are zero almost everywhere. This is exactly what has been obtained by Weber in [11], with a different approach, as given below:

Proposition 3.1 ([11]) *Let $\Psi_1 := \{\psi_1^1, \psi_2^1, \ldots, \psi_r^1\}$ and $\Psi_2 := \{\psi_1^2, \psi_2^2, \ldots, \psi_r^2\}$. Suppose that $X(\Psi_1)$ and $X(\Psi_2)$ are frames in $L_2(\mathbb{R}^d)$. $X(\Psi_1)$ and $X(\Psi_2)$ generate a pair of orthogonal frames for $L_2(\mathbb{R}^d)$ if and only if the following two equations are satisfied* *a.e.:*

$$\sum_{i=1}^{r}\sum_{j\geq 0}\overline{\hat{\psi}_i^2}(s^{*j}\omega)\hat{\psi}_i^1\left(s^{*j}(\omega+q)\right)=0,\quad q\in 2\pi\mathbb{Z}^d\setminus 2\pi s^*\mathbb{Z}^d; \tag{3.2}$$

$$\sum_{i=1}^{r}\sum_{j\in\mathbb{Z}^d}\overline{\hat{\psi}_i^2}(s^{*j}\omega)\hat{\psi}_i^1(s^{*j}\omega)=0. \tag{3.3}$$

We remark here that the double sums in (3.2) and (3.3) are the entries of the 'mixed dual Gramian' of the affine systems generated by Ψ_1 and Ψ_2 [9].

Applying above result of Weber, one can construct a pair of orthogonal wavelet frames easily. Suppose that $X(\Psi)$ is a frame for $L_2(\mathbb{R}^d)$. Let $V := (V_1; V_2) = (v_{i,j})$ be a $2r\times 2r$ constant unitary matrix, where V_1 denotes the submatrix formed by the first r columns of V and V_2 denotes the submatrix formed by the last r columns of V. Also let $\Psi_1 := V_1\Psi$ and $\Psi_2 := V_2\Psi$. It is easy to show that $X(\Psi_1)$ and $X(\Psi_2)$ are frames by using dual Gramian characterization of frames given by Corollary 5.7 in [8] by a simple computing of the dual Gramian of $X^q(\Psi_1)$ and $X^q(\Psi_2)$. Or one can compute each entry of the dual Gramian of $X^q(\Psi_1)$ and $X^q(\Psi_2)$ similar to what are we doing next.

We show that the wavelet systems generated by Ψ_1 and Ψ_2 are a pair of orthogonal frames for $L_2(\mathbb{R}^d)$. Since $X(\Psi)$ is assumed to be a frame, the double sums converge absolutely a.e. Now, we apply Theorem 3.1 to $\Psi_1 := \{\psi_1^1, \psi_2^1, \cdots, \psi_{2r}^1\}$ and $\Psi_2 := \{\psi_1^2, \psi_2^2, \cdots, \psi_{2r}^2\}$. For a fixed $q\in 2\pi\mathbb{Z}^d\setminus 2\pi s^*\mathbb{Z}^d$, we then have

$$\begin{aligned}\sum_{i=1}^{2r}\sum_{j\geq 0}\overline{\widehat{\psi_i^1}}(s^{*j}\omega)\widehat{\psi_i^2}\left(s^{*j}(\omega+q)\right) &= \sum_{i=1}^{2r}\sum_{j\geq 0}\sum_{l=1}^{r}\overline{v_{i,l}}\overline{\widehat{\psi_l}}(s^{*j}\omega)\sum_{l'=1}^{r}v_{i,r+l'}\widehat{\psi_{l'}}\left(s^{*j}(\omega+q)\right)\\ &= \sum_{j\geq 0}\sum_{l=1}^{r}\overline{\widehat{\psi_l}}(s^{*j}\omega)\sum_{l'=1}^{r}\widehat{\psi_{l'}}\left(s^{*j}(\omega+q)\right)\sum_{i=1}^{2r}\overline{v_{i,l}}v_{i,r+l'}\\ &= \sum_{j\geq 0}\sum_{l=1}^{r}\overline{\widehat{\psi_l}}(s^{*j}\omega)\sum_{l'=1}^{r}\widehat{\psi_{l'}}\left(s^{*j}(\omega+q)\right)0=0,\end{aligned} \tag{3.4}$$

where we used the orthogonality of the columns of V. A similar calculation shows that (3.3) holds also. This completes the proof that Ψ_1 and Ψ_2 generate a pair of orthogonal frames by Theorem 3.1.

When the wavelet tight frame system $X(\Psi)$ is constructed from a multiresolution analysis based on the *unitary extension principle* (UEP) of [8], one can construct a pair of orthogonal tight frames from the same multiresolution analysis as we describe below.

We first give a brief discussion here on the UEP for the one variable case with trigonometric polynomial masks, while the more general version and comprehensive discussions of the UEP can be found in [4] and [8].

Let $\phi \in L_2(\mathbb{R})$ be a refinable function, i.e., $\widehat{\phi}(2\xi) = \widehat{a}_0(\xi)\widehat{\phi}(\xi)$, where $\widehat{a}_0$ is a trigonometric polynomial called the *refinement mask* of $\phi \in L_2(\mathbb{R})$ satisfying $\widehat{a}_0(0) = 1$ and let $\widehat{a}_j$, $j = 1, 2, \ldots, r$, be a set of trigonometric polynomials called the *wavelet masks.* The column vector $\vec{\widehat{a}} = (\widehat{a}_0, \widehat{a}_1, \ldots, \widehat{a}_r)^T$ is called the *refinement-wavelet mask.* Let

$$A(\omega) = \begin{pmatrix} \widehat{a}_0(\omega) & \widehat{a}_0(\omega+\pi) \\ \widehat{a}_1(\omega) & \widehat{a}_1(\omega+\pi) \\ \vdots & \vdots \\ \widehat{a}_r\omega) & \widehat{a}_r(\omega+\pi) \end{pmatrix} = (\vec{\widehat{a}}(\omega), \vec{\widehat{a}}(\omega+\pi)).$$

Assuming

$$A^*(\omega)A(\omega) = I.$$

for a.e. $\omega \in [-\pi, \pi]$. Define $\Psi := \{\psi_1, \psi_2, \ldots, \psi_r\} \subset L_2(\mathbb{R})$ by

$$\widehat{\psi_l}(2\xi) := \widehat{a}_j(\xi)\widehat{\phi}(\xi), \quad l = 1, 2, \ldots, r,$$

then the UEP asserts that $X(\Psi)$ is a tight frame for $L_2(\mathbb{R})$.

By using the UEP the construction of compactly supported tight frame becomes painless. For example, it is easy to obtain the compactly supported symmetric spline tight wavelet frames as shown in [8] and [4]

Next, we briefly describe how to obtain a pair of compactly supported orthogonal tight frames from a given compactly supported tight frame system $X(\Psi)$ constructed via the UEP. The main idea of this construction is from a paper by Bhatt, Johnson and Weber [12] where orthogonal tight frames are constructed from orthogonal wavelets.

Let $V(\omega) := (V_1(\omega); V_2(\omega)) = (v_{i,j}(\omega))$ be a $2r \times 2r$ unitary matrix with each entry being a π periodic trigonometric polynomial, where V_1 denotes the submatrix formed by the first r columns of V and V_2 denotes the submatrix formed by the last r columns of V. Let

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & V_1 \end{pmatrix}; \quad U_2 = \begin{pmatrix} 1 & 0 \\ 0 & V_2 \end{pmatrix}.$$

Define two new sets of the refinement-wavelet mask from $\vec{\widetilde{a}}$ by

$$\vec{\widetilde{a}}_1 = U_1 \vec{\widetilde{a}}; \quad \vec{\widetilde{a}}_2 = U_2 \vec{\widetilde{a}}.$$

The corresponding wavelets are defined via its Fourier transform as: $\widehat{\Psi}_1 := V_1 \widehat{\Psi}$ and $\widehat{\Psi}_2 := V_2 \widehat{\Psi}$ with their wavelet masks given above. It is easy to check that both entries in the column vectors Ψ_1 and Ψ_2 are compactly supported. Let

$$A_1(\omega) = (\vec{\widetilde{a}}_1(\omega); \vec{\widetilde{a}}_1(\omega + \pi)); \quad A_2(\omega) = (\vec{\widetilde{a}}_2(\omega); \vec{\widetilde{a}}_2(\omega + \pi)).$$

Then, it is easy to see

$$A_1 = U_1 A; \quad A_2 = U_2 A,$$

since each entry of U_1 and U_2 are π periodic. This leads to

$$A_1^*(\omega) A_1(\omega) = I; \quad A_2^*(\omega) A_2(\omega) = I,$$

for all $\omega \in [-\pi, \pi]$. Hence, both $X(\Psi_1)$ and $X(\Psi_2)$ are tight frames by the UEP.

Let B_1 and B_2 be the matrices generated by A_1 and A_2 respectively by removing the first rows of them. Then, it is clear that

$$B_1^*(\omega) B_2(\omega) = 0,$$

for all $\omega \in [-\pi, \pi]$. This asserts that $X(\Psi_1)$ and $X(\Psi_2)$ are a pair of orthogonal frames by Theorem 2.1.1 of [12] whose proof was obtained by a computation similar to (3.4). In fact, Theorem 2.1.1 of [12] can also be proved via a method similar to the proof of the mixed unitary extension principle in [9]. Finally, we remark that this construction can be modified to more general cases, e.g., one may start with two different tight frames instead of starting with one tight frame $X(\Psi)$.

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