Quasi-Interpolatory Refinable Functions and Contruction of Biorthogonal Wavelet Systems

Hong Oh Kim[*], Rae Young Kim[†], Yeon Ju Lee[‡] and Jungho Yoon[‡]

Abstract

We present a new family of compactly supported and symmetric biorthogonal wavelet systems. Each refinement mask in this family has tension parameter ω. When $\omega = 0$, it becomes the minimal length biorthogonal Coifman wavelet system [22]. Choosing ω away from zero, we can get better smoothness of the refinable functions at the expense of slightly larger support. Though the construction of the new birothogonal wavelet systems, in fact, starts from a new class of quasi-interpolatory subdivision schemes, we find that the refinement masks accidently coincide with the ones by Cohen, Daubechies and Feauveau [3, §6.C] (or [8, §8.3.5]), which are designed for the purpose of generating biorthogonal wavelets close to orthonormal cases. However, the corresponding mathematical analysis is yet to be provided. In this study, we highlight the connection between the quasi-interpolatory subdivision schemes and the masks by Cohen, Daubechies and Feauveau, and then we study the fundamental properties of the new biorthogonal wavelet systems such as regularity, stability, linear independence, vanishing moments and accuracy.

1 Introduction

During the last decades, the theory of wavelets and multiresolution analysis has established itself firmly as one of the most successful methods for a broad range of signal processing applications. The construction of classical wavelets is now well-understood due to the pioneer works such as [3, 7, 8]. Many properties, such as symmetry (or antisymmetry), vanishing moments, regularity and short support, are required for a practical use in application areas. It has been well-known that orthogonality and symmetry are conflicting properties for the design of compactly supported wavelets [8]. In order to maintain the symmetric properties of wavelet systems, the orthogonality constraint has been relaxed to semi-orthogonality or biorthogonality. In particular, spline functions have been a good source for wavelet constructions. We select some of them from references [3, 5, 6]. Also, very recently, a new class of compactly supported biorthogonal wavelet systems were constructed from pseudo-splines in [13].

[*]Division of Applied Mathematics, KAIST, 373-1, Guseong-dong, Yuseong-gu, Daejeon, 305-701, South Korea (hkim@amath.kaist.ac.kr)

[†]Department of Mathematics, Yeungnam University, 214-1, Dae-dong, Gyeongsan-si, Gyeongsangbuk-do, 712-749, South Korea (rykim@ynu.ac.kr)

[‡]Department of Mathematics, Ewha W. University, Seoul, 120-750, South Korea (lee08@ewhain.net, yoon@math.ewha.ac.kr)

It is very common to introduce wavelets through the notion of multiresolution analysis [19] which is introduced as follows. First, we say that a function $\phi \in L_2(\mathbb{R})$ is a refinable function if it satisfies the so-called *refinement equation*

$$\phi(x) = \sum_{n\in\mathbb{Z}} a_n \phi(2x-n) \tag{1.1}$$

where $\mathbf{a} := \{a_n : n \in \mathbb{Z}\}$ is usually called *the refinement mask* for ϕ. The function ϕ is also termed as the basic limit function of a subdivision scheme with the mask $\mathbf{a}$ (see Definition 2.1). Let $\phi \in L_2(\mathbb{R})$ be a compactly supported refinable function and let V_j be a shift invariant space defined by

$$V_j = \overline{\operatorname{span}}\{\phi_{j,k} := 2^{j/2}\phi(2^j \cdot -k) : k \in \mathbb{Z}\}.$$

We say that a sequence of subspaces $\{V_j : j \in \mathbb{Z}\}$ forms a *multiresolution analysis* (MRA) if it satisfies the following conditions:

(1) $\{V_j : j \in \mathbb{Z}\}$ is nested, i.e., $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$.
(2) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.
(3) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$ for all $j \in \mathbb{Z}$.
(4) The set of the translates $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is a *Riesz basis* for the space V_0; see section 3.

Let $\{V_j : j \in \mathbb{Z}\}$ and $\{\tilde{V}_j : j \in \mathbb{Z}\}$ be a pair of MRAs. The concept of biorthogonal wavelets consists of finding complement space W_j and $\tilde{W}_j$ of V_j and $\tilde{V}_j$ respectively satisfying

$$\tilde{W}_j \perp V_j, \quad W_j \perp \tilde{V}_j$$

so that $W_j \perp \tilde{W}_\ell$ for $j \neq \ell$.

Our construction of biorthogonal wavelet systems starts from a new class of subdivision schemes [4] (say, S_L). The reader is refered to the paper [4] to find their intersting features in view of CAGD (computer aided geometric design). Each scheme in this class is a quasi-interpolatory scheme, which reproduces polynomials up to a certain degree, with a tension parameter ω. When $\omega = 0$, it becomes the Deslauriers-Dubuc's interpolatory scheme, whose mask is used as a key ingredient to construct the (minimal length) biorthogonal Coifman wavelets. A biorthogonal wavelet system with compact suport is called a biorthogonal Coifman wavelet system for degree L if the synthesis refinable function $\phi(x)$ and the dual wavelets $\psi(x)$ and $\tilde{\psi}(x)$ have the vanishing mements L, that is,

$$\begin{aligned} &\int_{\mathbb{R}} x^n \phi(x)dx = \delta_{0,n}, \quad \forall n = 1, \ldots, L, \\ &\int_{\mathbb{R}} x^n \psi(x)dx = \int_{\mathbb{R}} x^n \tilde{\psi}(x)dx = 0, \qquad \forall n = 0, \ldots, L. \end{aligned} \tag{1.2}$$

Thus, the main objective of this paper is to present and to analyze a new family of biorthogonal Coifman wavelet systems, which are symmetric and compactly supported. An interesting observation is that the refinement masks of the new family coincide with the ones by Cohen, Daubechies and Feauveau ([3, 8]), which are designed for the purpose of generating a biorthogonal filter close to an orthonormal cases (in fact, to some coiflets). However, the corresponding mathematical analysis has not been provided yet. In this study, we highlight the connection between the quasi-interpolatory subdivision schemes S_L and the refinement masks by [3, §6.C]) (and [8, §8.3.5]).

Furthermore, we analyze the mathematical properties of the associated refinable functions and wavelets such as stability, linear independence, regularity, vanishing monents, and accuracy. Then we will enjoy the following advantages of the suggested wavelet systems:

- Choosing ω away from zero, the corresponding refinable functions have better smoothness, at the expense of slightly larger support, than the ones of $\omega = 0$. For instance, the suggested refinable function based on the cubic polynomial can be C^3 in the sense of integer smoothness, while the one by the cubic polynomial-based Deslauriers-Dubuc scheme is C^1 and the cubic B-spline refinable function is C^2.

- One attractive property of the new wavelet systems is that some filter coefficients can be dyadic rationals, i.e., rationals of the form $(2p+1)/2^q$ for some positive integers p and $q > 0$; since division by 2 can be done very fast in a computer, this makes it very suitable for fast computation.

- If we use sample values of a smooth function as refinable function coefficients at a fine scale, the resulting biorthogonal projection (say, $P_j f(x)$) on the space V_j approximates the underlying function with optimal approxiation rate $O(2^{-jL})$ with L in (1.2).

- For some suitable values of ω, the correspoding biorthogonal wavelet systems are very close to orthonormal cases. For an algorithm to find ω and details, the reader should consult [3].

The article is organized in the following manner: In section 2, we briefly introduce the quasi-interpolatory subdivsion schemes along with the Deslauriers-Dubuc interpolatory scheme. Some analysis on their masks is also given. In section 3, we find the condition of ω which guarantees the linearly independence of the integer translates of the refinable function ϕ associated with the quasi-interpolatory scheme. The smoothess of the refinable functions are studied in Section 4. In Section 5, we construct a new class of biorthogonal wavelet systems, which are symmetric and compactly supported. Finally, we show some specific examples of biorthogonal wavelet systems based on cubic polynomial.

2 Quasi-Interpolatory Subdivision Schemes

2.1 Subdivision Scheme

Starting with the initial values $f^0 = \{f_n^0 \in \mathbb{R} : n \in \mathbb{Z}\}$, a subdivision scheme defines recursively new discrete values $f^k = \{f_n^k \in \mathbb{R} : n \in \mathbb{Z}\}$ on finer levels by linear sums of existing values as follows:

$$f_j^{k+1} = \sum_{n\in\mathbb{Z}} a_{j-2n} f_n^k, \quad k \in \mathbb{Z}_+, \tag{2.1}$$

where the sequence $\mathbf{a} = \{a_n : n \in \mathbb{Z}\}$ is termed the *mask* of the given subdivision. We denote the rule at each level by S and have the formal relation

$$f^k = S^k f^0. \tag{2.2}$$

Definition 2.1 *A subdivision scheme S is said to be C^ν if for the initial data $f^0 := \{\delta_{n,0} : n \in \mathbb{Z}\}$, there exists a limit function $\phi := S^\infty f^0 \in C^\nu(\mathbb{R})$, $\phi \not\equiv 0$, satisfying*

$$\lim_{k\to\infty} \sup_{n\in\mathbb{Z}} |f_n^k - \phi(2^{-k}n)| = 0. \tag{2.3}$$

The function ϕ is called the *basic limit function* of S and it satisfies the *refinement equation* in (1.1) [14]. It is easy to see that supp $\phi \subseteq$ [supp $\mathbf{a}$] where $[A]$ indicates the smallest closed interval containing the set A.

From the equation (2.1), we find that the mask $\{a_n : n \in \mathbb{Z}\}$ consisting of the nonzero coefficients is divided into the even and the odd masks corresponding to even and odd n respectively. To simplify the presentation of a subdivision scheme and its analysis, it is convenient to introduce the Laurent polynomial defined by $\mathbf{a} = \{a_n : n \in \mathbb{Z}\}$

$$a(z) := \sum_{n\in\mathbb{Z}} a_n z^n, \quad z \in \mathbb{C} \setminus \{0\}. \tag{2.4}$$

The Laurent polynomial $a(z)$ is also called the symbol of its corresponding refinable function ϕ; see (2.3). Next, define the Laurent polynomial $a^{[k]}(z)$, $k \in \mathbb{N}$, by

$$a^{[k]}(z) := a(z)a(z^2)\cdots a(z^{2^{k-1}}) := \sum_{n\in\mathbb{Z}} a_n^{[k]} z^n. \tag{2.5}$$

Using the coefficients $a_n^{[k]}$ in (2.5), the norm of the iterated scheme S^k in (2.2) is defined as following [14]:

$$\|S^k\|_\infty := \max\Big\{ \sum_{\beta\in\mathbb{Z}} |a^{[k]}_{\gamma+2^k\beta}| : \gamma = 0, \ldots, 2^k - 1 \Big\}. \tag{2.6}$$

2.2 Deslauriers-Dubuc Interpolatory Subdivision Scheme

Let L be an even positive integer, i.e., $L = 2N$ for some $N \in \mathbb{N}$. The L-point Deslauriers-Dubuc interpolatory subdivision scheme defines the values at the insertion points by using polynomial interpolation of degree $L - 1$ at the symmetric L-points. Let $\Pi_{<L}$ be the linear space of all polynomials on $\mathbb{R}$ of degree $< L$. The Lagrange polynomials on the set $\{-N+1, \ldots, N\}$ are defined by

$$L_n(x) = \prod_{\substack{\ell\neq n \\ \ell=-N+1}}^{N} \frac{x-\ell}{n-\ell}, \quad n = -N+1, \ldots, N, \tag{2.7}$$

so that

$$p_\ell(x) = \sum_{n=-N+1}^{N} L_n(x) p_\ell(n) \tag{2.8}$$

where $p_1, \ldots, p_L$ constitute a basis of $\Pi_{<L}$. It is obvious that $L_n(\ell) = \delta_{n,\ell}$ with $\ell = -N+1, \ldots, N$. Then, the mask of the $2N$-point Deslauriers-Dubuc interpolatory subdivision scheme is given by

$$a_{2n} := \delta_{0,n}, \quad a_{1-2n} := L_n(2^{-1}). \tag{2.9}$$

with $n = -N+1, \ldots, N$.

2.3 The Mask of Quasi-Interpolatory Subdivision Scheme

The mask of our subdivision (henceforth, denoted by S_L) can be obtained by the requirement of reproducing polynomials in $\Pi_{<L}$. As observed in (2.1), a univariate subdivision consists of two rules, which can be represented by the even and the odd masks. First, for the construction of the odd mask, we use the stencil of $L = 2N$ points to reproduce polynomials of degree $< 2N$. That is, the odd mask $\{a_{1-2n} : n = -N+1, \ldots, N\}$ is obtained by solving the linear system:

$$p_\ell(2^{-1}) = \sum_{n=-N+1}^{N} a_{1-2n} p_\ell(n), \quad \ell = 1, \ldots, L, \tag{2.10}$$

where p_ℓ, $\ell = 1, \ldots, L$, is a basis of $\Pi_{<L}$. Obviously, there is a unique solution of the linear system (2.10) which can be written in the matrix form

$$\mathbf{A}_1 = \mathbf{M}^{-1}\mathbf{R}, \tag{2.11}$$

where $\mathbf{A}_1(n) = a_{1-2n}$, $\mathbf{M}(n, \ell) = p_\ell(n)$ and $\mathbf{R}(\ell) = p_\ell(2^{-1})$, and it is exactly the same as the odd mask of the L-point Deslauriers-Dubuc scheme. Next, for the construction of the even mask, we use the stencil of $L+1 = 2N+1$ points to reproduce polynomials in $\Pi_{<L}$. That is, the even mask $\{a_{2n} : n = -N, \ldots, N\}$ is obtained by solving the linear system:

$$p_\ell(0) = \sum_{n=-N}^{N} a_{-2n} p_\ell(n), \quad p_\ell \in \Pi_{<L}. \tag{2.12}$$

This is an underdetermined system of $L+1$ unknowns a_{2n}, $n = -N, \ldots, N$, in L equations, and hence there is one degree of freedom which will be used as a tension parameter ω. Here and in the sequel, for convenience, we put

$$\omega := a_{2N}.$$

Since the odd mask is equal to the case of the L-point Deslauriers-Dubuc scheme, it is clear from (2.7) and (2.9) that

$$\begin{aligned} a_{1-2n} &= L_n(\frac{1}{2}) = \prod_{\substack{\ell \neq n \\ \ell=-N+1}}^{N} \frac{\frac{1}{2} - \ell}{n - \ell}, \quad n = -N+1, \ldots, N \\ &= \frac{(-1)^n}{1-2n} \binom{2N-2}{N-1} \binom{2N-1}{N-n} \frac{2N-1}{2^{4N-3}}. \end{aligned}$$

The following lemma treats the explicit formula of the even mask a_{2n}. Since there is one degree of freedom in the systerm (2.12),

Lemma 2.2 *Let $\{a_{2n} : n = -N, \ldots, N\}$ be the even mask of the subdivision scheme S_L obtained from* (2.12). *Then, for $n = -N+1, \ldots, N$,*

$$a_{-2n} = \delta_{n,0} - w b_{-2n}. \tag{2.13}$$

where

$$b_{-2n} = (-1)^{N-n+1} \frac{(2N)!}{(N+n)!(N-n)!}. \tag{2.14}$$

Proof. Using $a_{2N} = \omega$, the linear system (2.12) can be changed to

$$\sum_{n=-N+1}^{N} a_{-2n} p_\ell(n) = p_\ell(0) - \omega p_\ell(-N), \quad p_\ell \in \Pi_{<L}. \tag{2.15}$$

This is a $2N \times 2N$ system and it guarantees the unique solution of (2.15). Letting $\mathbf{M}$ be given as in (2.11) and $\mathbf{R}_x(\ell) = p_\ell(x)$, the solution $\mathbf{A}_2 = \{a_{-2n} : n = -N+1, \ldots, N\}$ can be expressed in the matrix form

$$\mathbf{A}_2 = \mathbf{M}^{-1}(\mathbf{R}_0 - \omega \mathbf{R}_{-N}).$$

From (2.8), it is obvious that

$$\mathbf{M}^{-1}\mathbf{R}_0 = (L_n(0) : n = -N+1, \ldots, N),$$
$$\mathbf{M}^{-1}\mathbf{R}_{-N} = (L_n(-N) : n = -N+1, \ldots, N),$$

with $L_n(x)$ the Lagrange polynomial in (2.7). Here, for any $n = -N+1, \ldots, N$, we have

$$\begin{aligned} L_n(-N) &= \prod_{\substack{\ell \neq n \\ \ell=-N+1}}^{N} \frac{-N-\ell}{n-\ell} \\ &= \frac{(-1)^{2N}(2N)!}{-N-n} \cdot \frac{(-1)^{N-n}}{(N-n)!} \cdot \frac{1}{(N+n-1)!} \\ &= (-1)^{N-n+1} \frac{(2N)!}{(N+n)!(N-n)!}. \end{aligned}$$

Denoting the last term by b_{-2n}, the proof is done. □

Remark 2.3 *The explicit forms of a_n for the case L is odd are given in section 7. However, this study is focused on the case of $L = 2N$, which is practically more useful.*

Remark 2.4 *From Lemma 2.2, it is obvious that if $\omega = 0$, $a_{2n} = \delta_{n,0}$. This implies that the $2N$-point Deslauriers-Dubuc scheme is a special case of S_L with $\omega = 0$ and $L = 2N$. For a suitable range of ω away from zero, S_L provides the same or better smoothness at the expense of slightly larger support than the case of $\omega = 0$; see [4] for the detailed (integer) smoothness corresponding to ω. In particular, the scheme S_L based on cubic polynomial (i.e., $L = 4$) can provide up to C^3, while the cubic polynomial-based Deslauriers-Dubuc scheme is C^1 and the cubic B-spline is C^2.*

The next lemma provides the explicit form of the Laurent polynomial $a(z)$ associated with the scheme S_L.

Lemma 2.5 *Let $\{a_n : n \in \mathbb{Z}\}$ be the mask of the subdivision scheme S_L with $L = 2N$ and $a(z)$ be its corresponding Laurent polynomial. If we set $a_{2N} = \omega$ and $y = \sin^2(\xi/2)$, then*

$$a(e^{i\xi}) = (1-y)^N \left[2 \sum_{n=0}^{N-1} \binom{N-1+n}{n} y^n + \omega 2^{4N} (-1)^N y^N \right]. \tag{2.16}$$

Proof. The Laurent polynomial of the $2N$-point Deslauriers-Dubuc scheme can be written as

$$a_D(z) := 1 + \sum_{n=-N+1}^{N} a_{1-2n} z^{1-2n}. \tag{2.17}$$

Using this expression and applying Lemma 2.2, we get

$$\begin{aligned} a(z) &= \omega z^{2N} + \sum_{n=-N+1}^{N} \left((\delta_{n,0} - wb_{-2n}) z^{-2n} + a_{1-2n} z^{1-2n} \right) \\ &= \omega \left(z^{2N} - \sum_{n=-N+1}^{N} b_{-2n} z^{-2n} \right) + a_D(z) \end{aligned} \tag{2.18}$$

with b_{-2n} as in (2.14). It is well known from the literatures (e.g., see [8]) that the Laurent polynomial $a_D(z)$, $z = e^{i\xi}$, has the explicit form

$$a_D(e^{i\xi}) \quad = \quad 2\cos^{2N}(\xi/2) \sum_{n=0}^{2N-1} \binom{2N-1+n}{n} \sin^{2n}(\xi/2). \tag{2.19}$$

Moreover, invoking the definition of b_{-2n} in (2.14) and using $z = e^{i\xi}$, we have

$$\begin{aligned} z^{2N} - \sum_{n=-N+1}^{N} b_{-2n} z^{-2n} &= z^{2N} - \sum_{n=-N+1}^{N} (-1)^{N-n+1} \frac{(2N)!}{(N+n)!(N-n)!} z^{-2n} \\ &= z^{2N} \left(1 + \sum_{n=1}^{2N} (-1)^{-n} \frac{(2N)!}{n!(2N-n)!} z^{-2n} \right) \\ &= z^{2N} (1 - z^{-2})^{2N} \\ &= z^{-2N} 2^{4N} i^{2N} \left(\frac{1+z}{2} \right)^{2N} \left(\frac{1-z}{2i} \right)^{2N} \\ &= 2^{4N} i^{2N} \cos^{2N}(\xi/2) \sin^{2N}(\xi/2). \end{aligned} \tag{2.20}$$

Combining (2.19) and (2.20) with (2.18),

$$\begin{aligned} a(e^{i\xi}) &= \cos^{2N}(\xi/2) \left[2 \sum_{n=0}^{N-1} \binom{N-1+n}{n} \sin^{2n}(\xi/2) + \omega 2^{4N} i^{2N} \sin^{2N}(\xi/2) \right] \\ &= (1-y)^N \left[2 \sum_{n=0}^{N-1} \binom{N-1+n}{n} y^n + \omega 2^{4N} (-1)^N y^N \right] \end{aligned}$$

where $y = \sin^2(\xi/2)$. This completes the proof. $\square$

We can find that the Laurent polynomial $a(z)$ in (2.16) coincides with the one in [8, §6.C] (see also [3, §8.3.5]), which was designed for the purpose of constructing biorthogonal filters close to orthonormal cases (or to some coiflets). However, the mathematical properties of the corresponding refinable functions are not studied. In the following sections, we will discuss the fundamental properties of $a(z)$ in (2.16) in relation to the linear independence and the smoothness of the corresponding refinable functions.

3 Linear Independance and Stability of Refinable Functions

Given a refinable function ϕ, a fundamental question is whether its integer translates are linearly independent: The integer translates of a compactly supported function $\phi \in L_2(\mathbb{R})$ are linearly independent if for any $c \in \ell(\mathbb{Z})$,

$$\sum_{j\in\mathbb{Z}} c(j)\phi(\cdot - j) = 0 \quad \text{implies} \quad c(j) = 0, \quad \forall j \in \mathbb{Z}.$$

The linear independence of the integer translates of ϕ is a necessary and sufficient condition for the existence of a compactly supported dual refinable function $\tilde{\phi} \in L_2(\mathbb{R})$ of ϕ (see [20]). Furthermore, it is well-known that the existence of a compactly supported dual refinable function of ϕ is a key step to construct a pair of biorthogonal wavelets from the the given ϕ. The studies on the linear independence of the integer translates of a compactly supported function can be found in the literatures [10, 11, 15, 21].

An issue related to the linear independence is the (somewhat weaker) notion of stability of ϕ: A function $\phi \in L_2(\mathbb{R})$ is *stable* if there exist $0 < A, B < \infty$ such that for any sequence $c \in \ell_2(\mathbb{Z})$,

$$A\|c\|_{\ell_2(\mathbb{Z})} \le \Big\| \sum_{j\in\mathbb{Z}} c(j)\phi(\cdot - j)\Big\|_{L_2(\mathbb{R})} \le B\|c\|_{\ell_2(\mathbb{Z})}. \tag{3.1}$$

In other words, the integer translates of ϕ are stable if the collection $\{\phi(\cdot - j) : j \in \mathbb{Z}\}$ is an *unconditional basis* for the subspace of $L_2(\mathbb{R})$ generated by them. The upper bound of (3.1) always exists for any compactly supported function $\phi \in L_2(\mathbb{R})$ [16, Theorem 2.1]. It is also well-known from [16, Theorem 3.5] that the lower bound is equivalent to

$$\big(\hat{\phi}(\xi + 2\pi k)\big)_{k\in\mathbb{Z}} \neq \mathbf{0}, \quad \forall \xi \in \mathbb{R}, \tag{3.2}$$

where $\mathbf{0}$ indicates the zero sequence in $\ell(\mathbb{Z})$. Thus, the stability of a compactly supported function $\phi \in L_2(\mathbb{R})$ is equivalent to (3.2).

The linear independence of the integer translates of a refinable function ϕ is characterized in terms of their masks in [17]. The main results, Theorem 1 and 2 in [17], imply directly the following lemma. Here, the notion of symmetric zeros is used: A Laurent polynomial $a(z)$ has a pair of symmetric zeros on $\mathbb{C} \setminus \{0\}$ if there is a zero $z_0 \in \mathbb{C} \setminus \{0\}$ such that $a(z_0) = a(-z_0) = 0$.

Lemma 3.1 ([17]) *Let $\phi \in L_2(\mathbb{R})$ be a compactly supported refinable function. The integer translates of ϕ are linearly independent if and only if the following two conditions are satisfied:*

(1) *The function ϕ is stable.*
(2) *The Laurent polynomial $a(z)$ does not have any symmetric zeros in $\mathbb{C} \setminus \{0\}$.*

From the above lemma, it is immediate that for a compactly supported function $\phi \in L_2(\mathbb{R})$, the linearly independence of the integer translates of ϕ implies the stability of ϕ. In what follows, we find the condition of ω which guarantees the linearly independence of the integer translates of ϕ associated with the subdivision scheme S_L. For this, two useful lemmas are introduced.

Lemma 3.2 *Let*

$$P_N(y) := \sum_{n=0}^{N-1} \binom{N-1+n}{n} y^n \tag{3.3}$$

and

$$Q_N(y) := P_N(y) - (-1)^{N+1}2^{4N-1}\omega y^N. \tag{3.4}$$

Then $Q_N(y)$ and $Q_N(1-y)$ do not vanish simultaneously for $y \in [0,1]$ if and only if

$$w \neq (-1)^{N+1}2^{-2N}.$$

Proof. First we claim that the polynomial Q_N has at most one zero for $y \in [0,1]$. For this proof, suppose that $Q_N(y_1) = 0$. Clearly, $y_1 \neq 0$, and it follows from (3.4) that

$$P_N(y_1) = (-1)^{N+1}2^{4N-1}\omega y_1^N. \tag{3.5}$$

Then, for any $y \neq y_1$, we apply (3.4) to get the relation

$$\begin{aligned} Q_N(y) &= P_N(y) - P_N(y_1)\left(\frac{y}{y_1}\right)^N \\ &= \sum_{j=0}^{N-1} \binom{N-1+j}{j} y^j \left(1 - \left(\frac{y}{y_1}\right)^{N-j}\right), \end{aligned}$$

where $Q_N(y) > 0$ if $y < y_1$ and $Q_N(y) < 0$ if $y > y_1$; hence Q_N has at most one zero in $[0,1]$. Therefore, $Q_N(y)$ and $Q_N(1-y)$ do not vanish simultaneously if and only if $Q_N(1/2) \neq 0$, that is, by (3.4), equivalent to

$$\omega \neq (-1)^{N+1}2^{-3N+1}P_N(1/2). \tag{3.6}$$

Finally, we need to evaluate $P_N(1/2)$. Since the Daubechies' polynomial P_N satisfies the equation

$$y^N P_N(1-y) + (1-y)^N P_N(y) = 1,$$

we get $P_N(1/2) = 2^{N-1}$ by setting $y = 1/2$. Combining this with (3.6) leads to the claim of the lemma. $\square$

Remark 3.3 *According to the condition* (3.6) *and its subsequent argument, we conclude that if $\omega = (-1)^{N+1}2^{-2N}$, the polynomial $Q_N(y)$ has one root at $y = 1/2$, i.e., $Q_N(1/2) = 0$.*

Lemma 3.4 *Let $\phi \in L_2(\mathbb{R})$ be a compactly supported refinable function with the symbol $a(z)$. Then ϕ is stable if and only if the following two conditions are satisfied:*

(1) *The symbol $a(z)$ does not have any symmetric zeros on the unit circle T.*
(2) *For any odd integer $m > 1$ and a primitive m-th root z_0 of unity, there is $d \in \mathbb{N}$ such that*

$$a(-z_0^{2^d}) \neq 0.$$

Proof. See [17, Theorem 1].

Based on this lemma, we prove that the refinable function ϕ associated with S_L is stable. Invoking the definition of Q_N in (3.4), it is useful for the following analysis to represent $a(e^{i\xi})$ in (2.16) as

$$a(e^{i\xi}) = 2(1-y)^N Q_N(y), \quad y = \sin^2(\xi/2). \tag{3.7}$$

Lemma 3.5 *Let ϕ be the refinable function associated with the subdivision scheme S_L with the tension parameter ω. Then ϕ is stable if and only if*

$$\omega \notin (-1)^{N+1}2^{-2N+1}\{1/2, P_N(1/4)\}.$$

Proof. We prove this lemma by using Lemma 3.4. Due to Lemma 3.2, it suffices to show that the condition (2) in Lemma 3.4 is equivalent to the condition $\omega \neq (-1)^{N+1}2^{-2N+1}P_N(1/4)$. Let z_0 be a primitive 3-rd root of unity, *i.e.*, $z_0 = e^{2\pi i/3}$ or $e^{4\pi i/3}$. We first claim that $a(-z_0^{2^d}) = 0$ for all $d \geq 0$ if and only if $\omega = (-1)^{N+1}2^{-2N+1}P_N(1/4)$. For this proof, define ζ_0 by $z_0 = e^{i\zeta_0}$. A direct calculation shows that for any integer $d \geq 0$, $\sin^2(2^{d-1}\zeta_0) = 3/4$. Therefore, we obtain from (3.4) that for all $d \geq 0$,

$$\begin{aligned} a(-z_0^{2^d}) &= 2\sin^{2N}(2^{d-1}\zeta_0)Q_N(\cos^2(2^{d-1}\zeta_0)) \\ &= 2\sin^{2N}(2^{d-1}\zeta_0)(P_N(1/4) - (-1)^{N+1}2^{4N-1}\omega 4^{-N}), \end{aligned} \tag{3.8}$$

such that $a(-z_0^{2^d}) = 0$ if and only if $\omega = (-1)^{N+1}2^{-2N+1}P_N(1/4)$. Now, assume that $\omega \neq (-1)^{N+1}2^{-2N+1}P_N(1/4)$ and that there exists $z_1 = e^{i\zeta_1} \in \mathbb{C} \setminus \{0, z_0\}$ with $\zeta_1 \in (0, 2\pi)$ such that $a(-e^{i2^d\zeta_1}) = 0$ for all integer $d \geq 0$. Let $y_1 := \sin^2(\zeta_1/2)$. Then, let us consider the cases $d = 0$ and 1. If $d = 0$,

$$a(-e^{\zeta_1 i}) = 2y_1^N Q_N(1 - y_1) = 0$$

and if $d = 1$,

$$a(-e^{2\zeta_1 i}) = 2(4y_1(1 - y_1))^N Q_N((1 - 2y_1)^2) = 0.$$

It follows that $Q_N(1 - y_1) = Q_N((1 - 2y_1)^2) = 0$. Since $Q_N(y)$ has at most one zero (as observed in the proof of Lemma 3.2), $1 - y_1$ should be equal to $(1 - 2y_1)^2$. It implies that $y_1 = 3/4$. But, due to (3.8), $Q_N(1 - y_1) = Q_N(1/4) \neq 0$ since $\omega \neq (-1)^{N+1}2^{-2N+1}P_N(1/4)$. This proves the lemma. □

Lemma 3.6 *Let $a(z)$ be the Laurent polynomial of the subdvision scheme S_L and let b_{2n} for $n = -N+1, \ldots, N$ be given as in (2.14). Assume that $z_0 \in \mathbb{C} \setminus \{0\}$ be a zero of $a(z)$. Then, z_0 is a symmetric zero of $a(z)$ if and only if*

$$\omega = \left[\sum_{n=-N+1}^{N} b_{-2n}z_0^{-2n} - z_0^{2N}\right]^{-1}. \tag{3.9}$$

Proof. Recalling the expression of $a(z)$ in (2.18), let $z_0 \in \mathbb{C} \setminus \{0\}$ be a symmetric zero of $a(z)$. Dividing $a(z)$ into even and odd degree terms and using the condition $a(z_0) = a(-z_0) = 0$, it can be easily induced that

$$a_{\text{odd}}(z_0) := \sum_{n\in\mathbb{Z}} a_{1-2n}z_0^{1-2n} = 0. \tag{3.10}$$

Since the mask $\{a_{1-2n} : n \in \mathbb{Z}\}$ is exactly the same as the case of the L-point Deslauriers-Dubuc scheme, it is clear that $a_D(z_0) = 1$ with $a_D(z)$ in (2.17). Thus, by (2.18), we obtain the required condition (3.9). □

Theorem 3.7 *Let ϕ be the refinable function generated by the subdivision scheme S_L, $L = 2N$, with a tension parameter ω. Assume that*

$$\omega \notin (-1)^{N+1}2^{-2N+1}\{1/2, P_N(1/4)\} \cup \mathcal{V}_N \tag{3.11}$$

where

$$\mathcal{V}_N = \left\{ \Big(\sum_{n=-N+1}^{N} b_{-2n} z_0^{-2n} - z_0^{2N} \Big)^{-1} : a_{\mathrm{odd}}(z_0) = 0,\ z_0 \in \mathbb{C} \setminus \{0\} \right\}$$

Then the integer translates of ϕ are linearly independent.

Proof. We check the two sufficient conditions in Lemma 3.1. The condition (1) is proved in Lemma 3.5. The condition (2) is also an immediate consequence of Lemma 3.6. □

4 Smoothness Analysis

4.1 Maximal Smoothness of Refinable Functions

Let ϕ be the refinable function associated with the subdivision scheme S_L. The detailed integer smoothness of ϕ along with the corresponding range of ω is specified in [4] for $L = 2, \ldots, 20$. An interesting observation is that as the tension parameter ω is away from zero (up to a suitable range), the smoothness of ϕ is increased. Here, we discuss the maximal Hölder smoothness of ϕ for each given L. For a given $\gamma = n + s$ with $n \in \mathbb{N}$ and $s \in [0, 1)$, the Hölder space H^γ is defined to be the space of n-times continuously differentiable functions f whose n-th derivative $f^{(n)}$ satisfies the Lipschitz condition

$$\sup_{x,h\in\mathbb{R}} \frac{|f^{(n)}(x+h) - f^{(n)}(x)|}{|h|^s} \le C.$$

In particular, it is well known that if $|\hat{f}(\xi)| \le c(1 + |\xi|)^{-1-\gamma-\epsilon}$ with $\epsilon > 0$, then f belongs to the space H^γ.

The specific maximal Hölder regularity of ϕ generated by S_L is obtained in Table 1 with the corresponding values of ω. For this computation, we used the results in [14, section 2.3]. It is remarkable to see that the refinable function ϕ based on cubic polynomial (i.e., by S_L with $L = 4$) is $H^{3.7074}$, while the cubic-based Deslauriers-Dubuc's interpolatory scheme is $H^{2-\epsilon}$ and the cubic B-spline is $H^{3-\epsilon}$

4.2 Asymptotic smoothness

For a suitable ω, we expect that the Hölder regularity of ϕ asociated with S_L, $L = 2N$, increases as N is increasing. In what follows, we estimate its asymptotic property of a regularity of ϕ for the special choice of $\omega = \omega_N$ with

$$\omega_N := 2^{-4N+1}(-1)^{N+1}\frac{N-1}{N+1}\binom{2N-1}{N-1}. \tag{4.1}$$

as N tends to ∞. Of course, it does not provide the best smoothness among all possible ranges of w for a fixed L, but at least we can compare it with the asymptotic property of the regularity

L	γ	ω	L	γ	ω
2	2.9999	$-\frac{1}{8}$	12	6.6976	$-.000078$
4	3.7074	$-.024397$	14	7.4445	$-.000019$
6	4.6783	$-.005632$	16	8.1887	$-.00000462$
8	5.4800	$-.001348$	18	8.5983	$-.00000109$
10	6.1221	$-.000332$	20	9.4260	$-.00000027$

Table 1: The maximum Hölder regularities H^γ of ϕ associated with S_L for each $L = 2, \dots, 20$ and the corresponding values of ω. These are computed by using MAPLE 8, digits = 15.

of the case $\omega = 0$, as $N \to \infty$. With the choice of $\omega = \omega_N$, the Laurent polynomial $a(z)$ becomes of the form

$$a(e^{i\xi}) = 2\sin^{2N}(\xi/2)Q_N(\sin^2(\xi/2))$$

with

$$Q_N(y) := \left[\sum_{n=0}^{N-1}\binom{N-1+n}{n}y^n - \frac{N-1}{N+1}\binom{2N-1}{N-1}y^N\right], \quad y = \sin^2\xi/2. \tag{4.2}$$

Theorem 4.1 *For a given $L = 2N$, let ω_N be given as in* (4.1)*. Let ϕ be the refinable function associated with S_L and ω_N. Then, we have the optimal decay*

$$|\hat{\phi}(\xi)| \le C(1+|\xi|)^{-2N+\kappa}, \tag{4.3}$$

where $\kappa = \log(|Q_N(3/4)|)/\log 2$ with Q_N in (4.2)*. Consequently, $\phi \in H^{2N-\kappa-1-\epsilon}$ for any $\epsilon > 0$.*

We put the proof of Theorem 4.1 in section 4.3 for the better readability of the article. The following theorem treats the special case $\omega = 0$, which is the case of the Deslauriers-Dubuc interpolatory scheme. Then with P_N in (3.3), the Laurent polynomial $a_D(z)$ can be written as

$$a_D(e^{i\xi}) = 2\sin^{2N}(\xi/2)P_N(\sin^2(\xi/2)),$$

Theorem 4.2 [1, 8] *Let ϕ be the refinable function associated with the Deslauriers-Dubuc interpolatory scheme. Then, we have the optimal decay*

$$|\hat{\phi}(\xi)| \le C(1+|\xi|)^{-2N+\tilde{\kappa}}, \tag{4.4}$$

where $\tilde{\kappa} = \log(|P_N(3/4)|)/\log 2$ with P_N in (3.3)*. Consequently, $\phi \in H^{2N-\tilde{\kappa}-1-\epsilon}$ for any $\epsilon > 0$.*

4.3 Proof of Theorem 4.1

Recall that

$$a_N(e^{i\xi}) = 2(1-y)^N Q_N(y), \quad y = \sin^2(\xi/2).$$

The next proposition characterizes the decay of $|\hat{\phi}(\xi)|$. It can also be shown by slightly adapting the technique in section 3 of [12]. However, the argument presented below is more concise.

Proposition 4.3 *[12] Let ϕ be the refinable function with the symbol $a(z)$ of the form*

$$|a(e^{i\xi})| := 2\cos^{2N}(\xi/2)|Q(\sin^2(\xi/2))|, \quad \xi \in [-\pi, \pi],$$

for some polynomial Q. Suppose that

(1) $|Q(y)| \le |Q(3/4)|$ *with* $0 \le y \le 3/4$;

(2) $|Q(y)Q(4y(1-y))| \le |Q(3/4)|^2$ *with* $3/4 \le y \le 1$.

Then, we have the optimal decay

$$|\hat{\phi}(\xi)| \le C(1+|\xi|)^{-2N+\kappa},$$

with $\kappa = \log(|Q(3/4)|)/\log 2$. *Consequently,* $\phi \in C^{2N-\kappa-1-\epsilon}$ *for any* $\epsilon > 0$.

The following lemmas will be helpful for our further anlaysis.

Lemma 4.4 *Let $\ell \le N$ be a positive integer. The following statements hold:*

(1) *For any* $j \in \mathbb{N}$, $(j+1)\binom{N+j}{j+1} = (N+j)\binom{N-1+j}{j}$.

(2) $\displaystyle\sum_{j=0}^{\ell} \binom{N-1+j}{j} j = \frac{\ell(\ell+1)}{N+1}\binom{N+\ell}{\ell+1}$.

Proof. The relation (1) is trivial, and (2) can be shown by induction on ℓ and (1). □

Lemma 4.5 *Let f be a polynomial of the form*

$$f(y) = \sum_{j=0}^{N-1} a_j y^j - b_N y^N,$$

with $a_j > 0$, $j = 0, \ldots, N-1$ *and* $b_N = \frac{1}{N}\sum_{j=1}^{N-1} j a_j$, *so that* $f'(1) = 0$. *Then f is positive and increasing on* $[0,1]$.

Proof. Note that $f(0) = a_0 > 0$. It follows from the choice of b_N that

$$f'(y) = \sum_{j=1}^{N-1} j a_j (1 - y^{N-j}) y^{j-1} \ge 0, \quad \forall y \in [0,1].$$

Therefore f is positive and increasing on $[0,1]$. □

For any $\ell = 0, 1, \ldots, N-1$, define a polynomial $Q_{N,\ell}$ by

$$Q_{N,\ell}(y) := \sum_{j=0}^{\ell} \binom{N-1+j}{j} y^j - \nu_\ell \, y^{\ell+1} \tag{4.5}$$

with

$$\nu_\ell := \frac{\ell}{N+1}\binom{N+\ell}{\ell+1} = \frac{1}{\ell+1}\sum_{j=0}^{\ell} j \binom{N-1+j}{j}. \tag{4.6}$$

Lemma 4.6 *For any $\ell = 0, \ldots, N-1$, $Q_{N,\ell}$ in (4.5) satisfies the following properites:*

$$(1)\ Q_{N,\ell+1}(y) = Q_{N,\ell}(y) + \nu_{\ell+1} y^{\ell+1} \left(\frac{\ell+2}{\ell+1} - y \right). \tag{4.7}$$

$$(2)\ Q'_{N,\ell+1}(y) = \sum_{j=0}^{\ell} (N+j) \binom{N-1+j}{j} \left(y^j - y^{l+1} \right). \tag{4.8}$$

Proof. From Lemma 4.4 (1), we rewrite $Q_{N,\ell}(y)$ as follows:

$$\begin{aligned} Q_{N,\ell}(y) &= \sum_{j=0}^{\ell} \binom{N-1+j}{j} \left(y^j - \frac{j}{\ell+1} y^{\ell+1} \right) \qquad (4.9) \\ &= \sum_{j=0}^{\ell+1} \binom{N-1+j}{j} \left(y^j - \frac{j}{\ell+1} y^{\ell+1} \right) \\ &= \sum_{j=0}^{\ell+1} \binom{N-1+j}{j} y^j - \frac{\ell+2}{\ell+1} \nu_{\ell+1} y^{\ell+1}. \end{aligned}$$

Thus the relation (4.7) is immediate from the definition of $Q_{N,\ell+1}$ in (4.5). Also, using (4.6) and Lemma 4.4 (1), we get the relation in (2). □

We now proceed to the proof of Theorem 4.1. We would like to note that the general approach for this proof is similar to the method in [13] but our analysis here is more concise. Here and in the sequel, for simplicity, we use the abbreviation:

$$\Lambda_j(y) := \Lambda_{\ell,j}(y) := y^j \left(1 - \frac{\ell+1}{\ell+2} y \right). \tag{4.10}$$

Proof of Theorem 4.1: We check the conditions (1) and (2) of Proposition 4.3. Using Lemma 4.4 and the identity $\binom{2N-1}{N-1} = \binom{2N-1}{N}$, it is easy to see that $Q_N(y)$ satisfies the hypothesis of Lemma 4.5 and hence, $Q_N(y)$ is positive and monotonically increasing on $[0,1]$. Hence the condition (1) is satisfied. Next, for the proof of the condition (2), we define

$$W_{N,\ell}(y) := Q_{N,\ell}(y) Q_{N,\ell}(4y(1-y)) - \left(Q_{N,\ell}(3/4)\right)^2 \tag{4.11}$$

and verify that for any $\ell = 0, \ldots, N-2$,

$$W_{N,\ell+1}(y) - W_{N,\ell}(y) \le 0, \quad y \in [3/4, 1]. \tag{4.12}$$

Note here that $Q_{N,0}(y) \equiv 1$, which implies $W_{N,0}(y) \equiv 0$, and that $Q_{N,N-1} = Q_N$ due to the identity $\binom{2N-1}{N-1} = \binom{2N-1}{N}$. Then Proposition 4.3 (2) follows immediately from (4.12). To this end, let us first observe from (4.7) and (4.11) that

$$\begin{aligned}
&W_{N,\ell+1}(y) - W_{N,\ell}(y)\\
&\quad=\left(\frac{\ell+2}{\ell+1} - 4y(1-y)\right)\nu_{\ell+1}(4y(1-y))^{\ell+1}Q_{N,\ell}(y)\\
&\quad\quad+\left(\frac{\ell+2}{\ell+1} - y\right)\nu_{\ell+1}y^{\ell+1}Q_{N,\ell}(4y(1-y)) - (Q_{N,\ell+1}(3/4))^2 + (Q_{N,\ell}(3/4))^2\\
&\quad=\frac{\ell+2}{\ell+1}\nu_{\ell+1}\left[\Lambda_{\ell+1}(4y(1-y))Q_{N,\ell+1}(y) + \Lambda_{\ell+1}(y)Q_{N,\ell}(4y(1-y))\right]\\
&\quad\quad-(Q_{N,\ell+1}(3/4))^2 + (Q_{N,\ell}(3/4))^2.
\end{aligned}$$

Since $W_{N,\ell+1}(3/4) - W_{N,\ell}(3/4) = 0$, in order to prove the relation (4.12), it suffices to show that $W_{N,\ell+1}(y) - W_{N,\ell}(y)$ decreases monotonically on $[3/4, 1]$. Seeing that $Q_{N,\ell+1}(y) \geq Q_{N,\ell}(y)$ for any $y \in [0, 1]$, it is equivalent to verify that

$$G(y) := \Lambda_{\ell+1}(4y(1-y))Q_{N,\ell+1}(y) + \Lambda_{\ell+1}(y)Q_{N,\ell}(4y(1-y))$$

decreases monotonically on $[3/4, 1]$, i.e., $G'(y) \leq 0$, $y \in [3/4, 1]$. Now, we compute G' as follows:

$$\begin{aligned}
G'(y) =& -(\ell+1)(8y-4)(4y(1-y))^{\ell}[1-4y(1-y)]Q_{N,\ell+1}(y) + \Lambda_{\ell+1}(4y(1-y))Q'_{N,\ell+1}(y)\\
&+(\ell+1)y^{\ell}(1-y)Q_{N,\ell}(4y(1-y)) - (8y-4)\Lambda_{\ell+1}(y)Q'_{N,\ell}(4y(1-y)).
\end{aligned}$$

From (4.7), we find the identity

$$Q'_{N,\ell}(y) = Q'_{N,\ell+1}(y) - (\ell+2)\nu_{\ell+1}y^{\ell}(1-y). \tag{4.13}$$

Also, we see from (4.7) that $Q_{N,\ell+1}(y) = Q_{N,\ell}(y) + \nu_{\ell+1}\frac{\ell+2}{\ell+1}\Lambda_{\ell+1}(y)$. This together with (4.7) and (4.13) implies that

$$\begin{aligned}
G'(y) =& -(\ell+1)(8y-4)(4y(1-y))^{\ell}[1-4y(1-y)]Q_{N,\ell}(y) + \Lambda_{\ell+1}(4y(1-y))Q'_{N,\ell+1}(y)\\
&+(\ell+1)y^{\ell}(1-y)Q_{N,\ell}(4y(1-y)) - (8y-4)\Lambda_{\ell+1}(y)Q'_{N,\ell+1}(4y(1-y)).
\end{aligned}$$

Using (4.8) and (4.9), a direct calculation shows that

$$G'(y) = \sum_{j=0}^{\ell}\binom{N-1+j}{j}y^j(4y(1-y))^j(f_{j,\ell}(y) + (N+j)g_{j,\ell}(y)), \tag{4.14}$$

where $f_{j,\ell}$ and $g_{j,\ell}$ are defined by

$$\begin{aligned}
f_{j,\ell}(y) :=& -(\ell+1)(8y-4)\,(4y(1-y))^{\ell-j}\,(1-4y(1-y))\left(1 - \frac{j}{\ell+1}y^{\ell+1-j}\right)\\
&+(\ell+1)y^{\ell-j}(1-y)\left(1 - \frac{j}{\ell+1}(4y(1-y))^{\ell+1-j}\right),\\
g_{j,\ell}(y) :=& \Lambda_{\ell+1-j}(4y(1-y))(1-y^{\ell+1-j}) - (8y-4)\Lambda_{\ell+1-j}(y)(1-(4y(1-y))^{\ell+1-j}).
\end{aligned} \tag{4.15}$$

In order to prove that $G'(y) \leq 0$, we show that for $0 \leq j \leq \ell \leq N-2$

$$f_{j,\ell}(y) + (N+j)g_{j,\ell}(y) \leq 0, \quad \forall y \in [3/4, 1]. \tag{4.16}$$

First, to estimate $f_{j,\ell}(y)$, we see that

$$\begin{aligned} f_{j,\ell}(y) &= -(\ell+1)(8y-4)\left(4y(1-y)\right)^{\ell-j}\left(1-4y(1-y)\right) + (\ell+1)y^{\ell-j}\left(1-y\right) \\ &\quad + j\left(\left(4y(1-y)\right)^{\ell-j} y^{\ell-j}\left((8y-4)y\left(1-4y(1-y)\right) - 4y(1-y)(1-y)\right)\right). \end{aligned} \tag{4.17}$$

Since $8y-4 \geq 2$, $0 \leq 4y(1-y) \leq 3/4$ for $y \in [3/4, 1]$, we have

$$(8y-4)y\left(1-4y(1-y)\right) - 4y(1-y)(1-y) \geq \frac{3}{16} > 0.$$

For $j \leq \ell$, it leads to the relation

$$\begin{aligned} f_{j,\ell}(y) \leq &- (\ell+1)(8y-4)\left(4y(1-y)\right)^{\ell-j}\left(1-4y(1-y)\right) + (\ell+1)y^{\ell-j}\left(1-y\right) \\ &+ (\ell+1)\left(\left(4y(1-y)\right)^{\ell-j} y^{\ell-j}\left((8y-4)y\left(1-4y(1-y)\right) - 4y(1-y)(1-y)\right)\right) \\ = &- (\ell+1)(8y-4)\left(4y(1-y)\right)^{\ell-j}\left(1-4y(1-y)\right)\left(1-y^{\ell+1-j}\right) \\ &+ (\ell+1)y^{\ell-j}(1-y)\left(1-\left(4y(1-y)\right)^{\ell+1-j}\right). \end{aligned} \tag{4.18}$$

Also, since $0 \leq 4y(1-y) \leq y \leq 1$ and $-16y^2+40y-20 \geq 0$ for $y \in [3/4, 1]$, we obtain

$$\begin{aligned} g_{j,\ell}(y) &\leq y^{\ell-j}\left[\Lambda_1(4y(1-y)) - (8y-4)\Lambda_1(y)\right]\left(1-\left(4y(1-y)\right)^{\ell+1-j}\right) \qquad (4.19) \\ &= y^{\ell-j+1}\left(1-\left(4y(1-y)\right)^{\ell+1-j}\right)\left(8-12y+\frac{\ell+1}{\ell+2}y\left(-16y^2+40y-20\right)\right) \\ &\leq -8y^{\ell-j+1}\left(1-\left(4y(1-y)\right)^{\ell+1-j}\right)(2y-1)(1-y)^2. \end{aligned}$$

This implies that $g_{j,\ell}(y) \leq 0$. Thus, if $N+j \geq \ell+2+j \geq \ell+2$, $(N+j)g_{j,\ell}(y) \leq (\ell+2)g_{j,\ell}(y)$. Putting this and (4.19) into (4.16), we have

$$\begin{aligned} f_{j,\ell} + (N+j)g_{\ell,j}(y) \leq &- (\ell+1)(8y-4)\left(4y(1-y)\right)^{\ell-j}\left(1-4y(1-y)\right)(1-y^{\ell+1-j}) \qquad (4.20) \\ &+ (\ell+1)y^{\ell-j}\left(1-y\right)\left(1-\left(4y(1-y)\right)^{\ell+1-j}\right) \\ &+ (\ell+2)\Lambda_{\ell+1-j}(4y(1-y))(1-y^{\ell+1-j}) \\ &- (\ell+2)(8y-4)\Lambda_{\ell+1-j}(y)(1-(4y(1-y))^{\ell+1-j}) \\ = &\left(4y(1-y)\right)^{\ell-j}(1-y^{\ell+1-j})h_1(y) + y^{\ell-j}(1-(4y(1-y))^{\ell+1-j})h_2(y) \end{aligned}$$

where h_1 and h_2 are given by

$$\begin{aligned} h_1(y) &:= -(\ell+1)(8y-4)\left(1-4y(1-y)\right) + (\ell+2)\Lambda_1(4y(1-y)) \\ h_2(y) &:= (\ell+1)\left(1-y\right) - (\ell+2)(8y-4)\Lambda_1(y). \end{aligned}$$

Since $8y-4 \geq 2$ and $4y(1-y) \leq 1$ on $[3/4, 1]$, we have

$$h_1(y) \leq -(\ell+1)2\left(1-4y(1-y)\right) + (\ell+2)\left(1-\frac{\ell+1}{\ell+2}\,4y(1-y)\right) \leq 1. \tag{4.21}$$

Moreover, since $(8y-4)y>1$ on $[3/4,1]$, we have

$$h_2(y) \le (\ell+1)(1-y) - (\ell+2)\left(1-\frac{\ell+1}{\ell+2}\,y\right) = -1. \tag{4.22}$$

Since $0 \le 4y(1-y) \le y \le 1$ for any $y \in [3/4,1]$, combining (4.21) and (4.22) with (4.20) yields

$$\begin{aligned} f_{j,\ell}(y) + (N+j)g_{\ell,j}(y) &\le (4y(1-y))^{\ell-j}\left(1-y^{\ell+1-j}\right) - y^{\ell-j}\left(1-(4y(1-y))^{\ell+1-j}\right) \\ &\le y^{\ell-j}\left(1-y^{\ell+1-j}\right) - y^{\ell-j}\left(1-y^{\ell+1-j}\right) = 0, \end{aligned}$$

which implies $G'(y) \le 0$, $y \in [3/4,1]$, with $G'(y)$ in (4.14). This completes the proof. $\square$

5 Compactly Supported Biorthogonal Wavelets

5.1 Biorthogonal Wavelet Systems

Let $\phi \in L_2(\mathbb{R})$ be a stable refinable function with the symbol $a(z)$ such that $a(-1)=0$ and $a(1)=2$. As usual, the first step for the construction of biorthogonal wavelet systems is to find a refinable function $\tilde\phi \in L_2(\mathbb{R})$ such that

$$\langle \phi, \tilde\phi(\cdot-\ell)\rangle = \delta_{0,\ell}, \quad \ell \in \mathbb{Z}. \tag{5.1}$$

If $\tilde\phi$ is stable and satisfies the condition (5.1), we call $\tilde\phi$ the dual refinable function of ϕ (or just dual of ϕ). Let $\tilde a(z)$ be the symbol of $\tilde\phi$ such that $\tilde a(0)=2$ and $\tilde a(-1)=0$. For convenience, we use the notation

$$m_0(\xi) = a(e^{-i\xi})/2, \quad \tilde m_0(\xi) = \tilde a(e^{-i\xi})/2.$$

Then, the refinement functions ϕ and $\tilde\phi$ are defined respectively by

$$\hat\phi(\xi) := \prod_{j=1}^{\infty} m_0(2^{-j}\xi), \quad \hat{\tilde\phi}(\xi) := \prod_{j=1}^{\infty} \tilde m_0(2^{-j}\xi). \tag{5.2}$$

These infinite products in (5.2) converge absolutely and uniformly on compact sets and are the Fourier transforms of compactly supported ϕ and $\tilde\phi$ with their support widths given by the filter lengths [3, 9]. A necessary condition for ϕ and $\tilde\phi$ to satisfy the duality condition (5.1) is

$$\overline{m_0(\cdot)}\tilde m_0(\cdot) + \overline{m_0(\cdot+\pi)}\tilde m_0(\cdot+\pi) = 1. \tag{5.3}$$

Given a pair of dual refinable functions ϕ and $\tilde\phi$ with their associated filters $m_0(\xi)$ and $\tilde m_0(\xi)$, the dual wavelet functions ψ and $\tilde\psi$ are defined via the relation

$$\hat\psi(\xi) = m_1(\xi/2)\hat\phi(\xi/2), \quad \hat{\tilde\psi}(\xi) = \tilde m_1(\xi/2)\hat{\tilde\phi}(\xi/2), \tag{5.4}$$

where

$$m_1(\xi) = e^{-i\xi}\overline{\tilde m_0(\xi+\pi)}, \quad \tilde m_1(\xi) = e^{-i\xi}\overline{m_0(\xi+\pi)}. \tag{5.5}$$

We usuall call $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ a biorthogonal MRA-wavelet system if the following conditions hold: (i) $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ and $\{\tilde{\psi}_{j,k} : j, k \in \mathbb{Z}\}$ are Riesz bases for $L_2(\mathbb{R})$ respectively, and they are biorthogonal in the sense that $\langle \psi_{j,k}, \tilde{\psi}_{\ell,m}\rangle = \delta_{j,\ell}\delta_{k,m}$ with $j, k, \ell, m \in \mathbb{Z}$.; (ii) the condition

$$\langle \phi, \tilde{\psi}(\cdot - \ell)\rangle = \langle \psi, \tilde{\phi}(\cdot - \ell)\rangle = 0 \tag{5.6}$$

is satisfied.

Now, we recall the results of Cohen and Daubechies [2] adapted for our purpose in the following proposition.

Proposition 5.1 [2] *Define $\phi, \tilde{\phi}, \psi, \tilde{\psi}$ as above. If ϕ and $\tilde{\phi}$ are in $L_2(\mathbb{R})$ and*

$$\langle \phi, \tilde{\phi}(\cdot - \ell)\rangle = \delta_{0,\ell},$$

then $(\phi, \tilde{\phi})$ generates a pair of biorthogonal MRAs and $(\psi, \tilde{\psi})$ generates a pair of biorthogonal wavelets associated with the biorthogonal MRA. In particular, the following conditions are also satisfied:

$$\langle \phi, \tilde{\psi}(\cdot - \ell)\rangle = \langle \psi, \tilde{\phi}(\cdot - \ell)\rangle = 0 \quad \text{and} \quad \langle \psi, \tilde{\psi}(\cdot - \ell)\rangle = \delta_{0,\ell}. \tag{5.7}$$

From now on, we will call $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ biorthogoanl MRA-wavelet system.

We will give a sufficient condition (in Theorem 5.3 below) for ψ and $\tilde{\psi}$ to be biorthognal wavelets associated with ϕ and $\tilde{\phi}$. For this, we need the following proposition, which is in fact a slight modification of Proposition 4.9 in [3].

Proposition 5.2 *Let ϕ and $\tilde{\phi}$ be refinable functions whose symbols $a(z)$ and $\tilde{a}(z)$ are respectively of the form*

$$a(z) = \Big(\frac{1+z}{2}\Big)^{\ell} b(z), \;\; \tilde{a}(z) = \Big(\frac{1+z}{2}\Big)^{\tilde{\ell}} \tilde{b}(z) \tag{5.8}$$

for some $\ell, \tilde{\ell} \in \mathbb{N}$, where $b(z)$ and $\tilde{b}(z)$ are Laurent polynomials such that $b(1) = \tilde{b}(1) = 2$. Let

$$B_k = \max_{\xi} \Big| \prod_{j=0}^{k-1} F(\xi)\Big|^{1/k}, \quad \tilde{B}_{\tilde{k}} = \max_{\xi} \Big| \prod_{j=0}^{\tilde{k}-1} \tilde{F}(\xi)\Big|^{1/k}. \tag{5.9}$$

where $F(\xi); = \frac{1}{2}b(e^{-i\xi})$ and $\tilde{F}(\xi) := \frac{1}{2}\tilde{b}(e^{-i\xi})$. Suppose that $B_k\tilde{B}_{\tilde{k}} < 2^{\ell+\tilde{\ell}-1}$ for some integers $k, \tilde{k} > 0$. Then if $\phi, \tilde{\phi} \in L_2(\mathbb{R})$, $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ in (5.2) and (5.4) is a biorthognal MRA-wavelet system.

Proof. Define u_n and $\tilde{u}_n$ by

$$u_n(w) := \left[\prod_{j=1}^{n} m_0(2^{-j}w)\right] \chi_{[-\pi,\pi]}(2^{-n}w),$$

$$\tilde{u}_n(w) := \left[\prod_{j=1}^{n} \tilde{m}_0(2^{-j}w)\right] \chi_{[-\pi,\pi]}(2^{-n}w).$$

Then u_n and $\tilde{u}_n$ converge pointwise to $\hat{\phi}$ and $\hat{\tilde{\phi}}$ respectively. Moreover, applying (5.3), it is easy to see the relation

$$\begin{aligned}
\int_{-\infty}^{\infty} u_n(w)\overline{\tilde{u}}_n(w)e^{i\ell w}dw &= \int_0^{2^{n+1}\pi}\Big(\prod_{j=1}^{n} m_0(2^{-j}w)\overline{\tilde{m}}_0(2^{-j}w)\Big)e^{i\ell w}dw \\
&= \int_0^{2^n\pi}\prod_{j=1}^{n-1} m_0(2^{-j}w)\overline{\tilde{m}}_0(2^{-j}w)e^{i\ell w} \\
&\quad \times \big[m_0(2^{-n}w)\overline{\tilde{m}}_0(2^{-n}w) + m_0(2^{-n}w+\pi)\overline{\tilde{m}}_0(2^{-n}w+\pi)\big]\,dw \\
&= \int_0^{2^n\pi}\prod_{j=1}^{n-1} m_0(2^{-j}w)\overline{\tilde{m}}_0(2^{-j}w)e^{i\ell w}dw \\
&= \int_{-\infty}^{\infty} u_{n-1}(w)\overline{\tilde{u}}_{n-1}(w)e^{i\ell w}dw.
\end{aligned}$$

Repeating this process yields the identity

$$\int_{-\infty}^{\infty} u_n(w)\overline{\tilde{u}}_n(w)e^{i\ell w}dw = \int_{-\infty}^{\infty} u_1(w)\overline{\tilde{u}}_1(w)e^{i\ell w}dw = 2\pi\delta_{0,\ell}. \tag{5.10}$$

Due to the identity

$$\int_{-\infty}^{\infty}\phi(x)\overline{\tilde{\phi}}(x-\ell)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\phi}(w)\overline{\hat{\tilde{\phi}}}(w)e^{i\ell w}dw, \quad \phi,\tilde{\phi}\in L^2(\mathbb{R})$$

it suffices to show that $u_n(\cdot)\overline{\tilde{u}}_n(\cdot)$ converges to $\hat{\phi}(\cdot)\overline{\hat{\tilde{\phi}}}(\cdot)$ in the sense of L^1-norm in order to prove $\int_{-\infty}^{\infty}\phi(x)\overline{\tilde{\phi}}(x-\ell)dx = \delta_{0\ell}$. Since

$$\prod_{j=1}^{n}\left|\frac{1+e^{-i2^{-j}w}}{2}\right| = \prod_{j=1}^{n}\left|\frac{\sin(2^{-j}w)}{2\sin(2^{-j-1}w)}\right| = \left|\frac{\sin(2^{-1}w)}{2^n\sin(2^{-n-1}w)}\right|,$$

we have

$$\begin{aligned}
|u_n(w)| &= \left|\frac{\sin(2^{-1}w)}{2^n\sin(2^{-n-1}w)}\right|^L\prod_{j=1}^{n}\left|F(2^{-j}w)\right|\chi_{[-\pi,\pi]}(2^{-n}w), \\
|\tilde{u}_n(w)| &= \left|\frac{\sin(2^{-1}w)}{2^n\sin(2^{-n-1}w)}\right|^{\tilde{L}}\prod_{j=1}^{n}\left|\tilde{F}(2^{-j}w)\right|\chi_{[-\pi,\pi]}(2^{-n}w).
\end{aligned} \tag{5.11}$$

Since $|\sin w| \geq \frac{2}{\pi}|w|$ for $|w|\leq\pi/2$, it is easy to see that

$$\left|\sin(2^{-n-1}w)\right|^{-1}\chi_{[-\pi,\pi]}(2^{-n}w) \leq \frac{\pi}{2}2^{n+1}|w|^{-1}.$$

Therefore, we arrive at the relation

$$\left|\frac{\sin(2^{-1}w)}{2^n\sin(2^{-n-1}w)}\right|\chi_{[-\pi,\pi]}(2^{-n}w) \leq \frac{\pi}{2}\left|\frac{\sin(2^{-1}w)}{2^{-1}w}\right| \leq C(1+|w|)^{-1}. \tag{5.12}$$

We now compute an upper bound for $\prod_{j=1}^{n} F(2^{-j}w)$. At $w=0$, we have $m_0(0)=1$; so $F(0)=1$. Since F is a trigonometric polynomial, there exists C_1 such that $|F(w)| \le 1 + C_1|w|$ for $|w| \le 1$. Consequently, we have

$$\Big|\prod_{j=1}^{n} F(2^{-j}w)\Big| \le \prod_{j=1}^{n}(1+C_1 2^{-j}|w|) \le \prod_{j=1}^{\infty} e^{C_1 2^{-j}|w|} \le e^{C_1}. \tag{5.13}$$

If $|w| > 1$, then, for the given k in (5.9), there exists $\ell_0 > 0$ such that $2^{k\ell_0} \le |w| < 2^{k(\ell_0+1)}$. Write $n = kn' + q$ with $0 \le q < k$, and assume without loss of generality that $\ell_0 < n'$. Note that $\sup_\zeta |F(\zeta)| \ge 1$, since $F(0)=1$. Then, letting $G(w) = F(2^{-1}w)F(2^{-2}w)\cdots F(2^{-k}w)$, we obtain

$$\Big|\prod_{j=1}^{n} F(2^{-j}w)\Big| \le \Big[\sup_\zeta |F(\zeta)|\Big]^{k-1} \prod_{j=0}^{\ell_0}\Big|G(2^{-kj}w)\Big| \prod_{j'=\ell_0+1}^{n'}\Big|G(2^{-k\ell'}w)\Big|. \tag{5.14}$$

Since $|2^{-(\ell_0+1)k}w| \le 1$, applying the same argument in (5.13) yields

$$\prod_{j=\ell_0+1}^{n'}\Big|G(2^{-kj}w)\Big| = \prod_{j=0}^{n'-\ell_0-1}\Big|G(2^{-kj}2^{-(\ell_0+1)k}w)\Big| \le e^{C_1}. \tag{5.15}$$

Moreover, invoking the definition of B_k in (5.9), we have

$$\begin{aligned}\prod_{j=0}^{\ell_0}\Big|G(2^{-kj}w)\Big| &\le B_k^{k(\ell_0+1)} \le B_k^{k+\log|w|/\log 2} \\ &\le C_2(1+|w|)^{\log B_k/\log 2}.\end{aligned} \tag{5.16}$$

Hence, combining (5.15) and (5.16) with (5.14), we arrive at the bound

$$\Big|\prod_{j=1}^{n} F(2^{-j}w)\Big| \le C_3(1+|w|)^{\log B_k/\log 2}, \tag{5.17}$$

where the constant c is independent of n. This together with (5.12) leads to the estimate of u_n in (5.11) as follows:

$$|u_n(w)| \le D(1+|w|)^{-L+\log B_k/\log 2}.$$

Next, with $\tilde{B}_{\tilde{k}}$ in (5.9), the estimate of $|\tilde{u}_n(w)|$ can be done by the same argument:

$$|\tilde{u}_n(w)| \le \tilde{D}(1+|w|)^{-L+\log \tilde{B}_{\tilde{k}}/\log 2}.$$

Thus, since $B_k\tilde{B}_{\tilde{k}} < 2^{L+\tilde{L}-1}$ by assumption, it is obvious that

$$\sup_n |u_n(w)\tilde{u}_n(w)| \le E(1+|w|)^{-L-\tilde{L}+\log(B_k\tilde{B}_{\tilde{k}})/\log 2} \in L^1(\mathbb{R}).$$

Since $u_n\tilde{u}_n \to \hat{\phi}\hat{\tilde{\phi}}$ pointwise as n tends to ∞, by applying the Lebesgue dominated convergence theorem, we have the convergence property $\|u_n\tilde{u}_n - \hat{\phi}\hat{\tilde{\phi}}\|_{L_1(\mathbb{R})} \to 0$ as $n \to \infty$. Therefore, we conclude from (5.10) that

$$\int_{-\infty}^{\infty} \phi(x)\overline{\tilde{\phi}(x-\ell)}dx = \delta_{0\ell},$$

which completes the proof by Proposition 5.1. □

This proposition is explained in view of subdivision schemes in the next theorem. For this, a reader need to remind that the norm of the iterated scheme S^k, i.e., $\|S^k\|$, is defined in (2.6).

Theorem 5.3 *Let ϕ and $\tilde{\phi}$ be refinable functions with symbols $a(z)$ and $\tilde{a}(z)$ of the form* (5.8)*. Let $b(z)$ and $\tilde{b}(z)$ be given also as in* (5.8)*. Assume that $\|(\frac{1}{2}S_b)^k\| < 1$ and $\|(\frac{1}{2}S_{\tilde{b}})^{\tilde{k}}\| < 1$ for some $k, \tilde{k} > 0$, where S_b and $S_{\tilde{b}}$ are subdivision schemes associated with $b(z)$ and $\tilde{b}(z)$ respectively. Then, if $\ell + \tilde{\ell} \geq 3$, $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ in* (5.2) *and* (5.4) *is a biorthognal MRA-wavelet system.*

Remark 5.4 *If $\|(\frac{1}{2}S_b)^k\|_\infty < 1$ for some $k > 0$, the subdivision scheme with the Laurent polynomial $a(z)$ in* (5.8) *is convergent* (see Definition 2.1)*, and further, its associated refinable function ϕ is $C^{\ell-1}$* ([14, Theorem 3.4])*,*

Proof. It is obvious from (5.8) that both ϕ and $\tilde{\phi}$ are at least C^0 and compactly supported, which implies that both ϕ and $\tilde{\phi}$ are in $L^2(\mathbb{R})$. Then, invoking the notation (2.5), (2.6) and (5.9), we get the bound

$$
\begin{aligned}
B_k &= \max_{\xi} |b^{[k]}(e^{i\xi})|^{1/k} \\
&\leq \Big(\sum_{n\in\mathbb{Z}} |b_n^{[k]}| \Big)^{1/k} = \frac{1}{2} \Big(\sum_{\gamma=0}^{2^k-1} \sum_{\beta\in\mathbb{Z}} |b^{[k]}_{\gamma+2^k\beta}| \Big)^{1/k} \\
&\leq 2 \Big(\|(\frac{1}{2}S_b)^k\|_\infty \Big)^{1/k}.
\end{aligned}
\tag{5.18}
$$

Similarly, we can obtain

$$
\tilde{B}_{\tilde{k}} \leq 2 \Big(\|(\frac{1}{2}S_{\tilde{b}})^{\tilde{k}}\|_\infty \Big)^{1/\tilde{k}}. \tag{5.19}
$$

By the hypothesis, we arrive at the conclusion $B_k \tilde{B}_{\tilde{k}} < 4$. Thus, if $\ell + \tilde{\ell} \geq 3$, the conditions of Proposition 5.2 are satisfied. Therefore, the proof is done.

5.2 Approximation Order and Vanishing Moment

The following lemma treats the relations between dual (refinable, wavelet) functions and dual symbols. Some of them are already well-known in the literature.

Lemma 5.5 *Let $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ be a biorthognal MRA-wavelet system as in the previous section and their symbols are of the form*

$$
a(z) = (1+z)^L b(z), \quad \tilde{a}(z) = (1+z)^L \tilde{b}(z),
$$

where $b(z)$ and $\tilde{b}(z)$ are Laurent polynomials such that $b(-1) \neq 0$ and $\tilde{b}(-1) \neq 0$. Then, the following conditions are equivalent:

(1) *For any $p \in \Pi_{<L}$, $\sum_{n\in\mathbb{Z}} a_{j-2n} p(n) = p(j/2)$ with $j = 0, 1$.*

(2) *For any $p \in \Pi_{<L}$, $\sum_{n\in\mathbb{Z}} \tilde{a}_{j-2n} p(n) = p(j/2)$ with $j = 0, 1$.*

(3) *The refinable function ϕ and the wavelet ψ have the vanishing moments of order L:*

$$\int_{\mathbb{R}} \phi(t)t^\ell dt = \delta_{0,\ell} \quad \textit{and} \quad \int_{\mathbb{R}} \psi(t)t^\ell dt = 0, \quad \forall \ell = 0, \dots, L-1. \tag{5.20}$$

(4) *The dual functions $\tilde{\phi}$ and $\tilde{\psi}$ have the vanishing moments of order L:*

$$\int_{\mathbb{R}} \tilde{\phi}(t)t^\ell dt = \delta_{0,\ell} \quad \textit{and} \quad \int_{\mathbb{R}} \tilde{\psi}(t)t^\ell dt = 0, \quad \forall \ell = 0, \dots, L-1. \tag{5.21}$$

Proof. The proof of this lemma is done in the following way: (1) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). (1) $\Rightarrow$ (4): It is immediate that $t^\ell = \sum_{n\in\mathbb{Z}} \phi(t-n)n^\ell$ with $\ell = 0, \dots, L-1$. It is straightforward from the duality condition (5.1), i.e., $\langle \tilde{\phi}, \phi(\cdot - n)\rangle = \delta_{n,0}$, that

$$\begin{aligned}\int_{\mathbb{R}} \tilde{\phi}(t)t^\ell dt &= \int_{\mathbb{R}} \tilde{\phi}(t) \sum_{n\in\mathbb{Z}} \phi(t-n)n^\ell dt \\ &= \sum_{n\in\mathbb{Z}} n^\ell \int_{\mathbb{R}} \tilde{\phi}(t)\phi(t-n)dt = \delta_{\ell,0}, \quad \forall \ell = 0, \dots, L-1.\end{aligned}$$

In a simlilar way, using the orthogonality condition $\langle \tilde{\psi}, \phi(\cdot - n)\rangle = 0$ in (5.6), we can show that

$$\int_{\mathbb{R}} \tilde{\psi}(t)t^\ell dt = \sum_{n\in\mathbb{Z}} n^\ell \int_{\mathbb{R}} \tilde{\psi}(t)\phi(t-n)dt = 0, \quad \forall \ell = 0, \dots, L-1.$$

(4) $\Rightarrow$ (2) Let $\tilde{a}(z) = \sum_{n\in\mathbb{Z}} \tilde{a}_n z^n$. By applying the refinement equation of $\tilde{\phi}$ in (1.1) to (5.21), we can easily obtain that $\sum_{n\in\mathbb{Z}} \tilde{a}_n n^\ell = 2\delta_{\ell,0}$ with $\ell = 0, \dots, L-1$. It implies the identity

$$\tilde{a}^{(\ell)}(1) = 2\delta_{\ell,0}, \quad \ell = 0, \dots, L-1. \tag{5.22}$$

By construction, $\tilde{a}^{(\ell)}(-1) = 0$ for any $\ell = 0, \dots, L-1$. Thus combining it with (5.22), the statement (2) is proved. (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) : The proof can be done similarly as (1) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2). $\square$

Remark 5.6 *It is easy to see that the vanishing moment of order L of ϕ (and $\tilde{\phi}$) in* (5.20) *implies the polynomial reproducing property of degree $< L$, i.e., in the sense that*

$$\int_{\mathbb{R}} \phi(x-t)p(t)dt = p(x), \quad p \in \Pi_{<L}.$$

For $f \in L^2(\mathbb{R})$, we define the biorthogonal projection $P_j f$ of f onto the space V_j by

$$P_j f = \sum_{k\in\mathbb{Z}} \langle f, \tilde{\phi}_{j,k}\rangle \phi_{j,k}.$$

For a pair of biorthogonal wavelets ψ and $\tilde{\psi}$, we also define a projection $Q_j f$ of f onto the space $W_j := \overline{\operatorname{span}}\{\psi_{j,k} : k \in \mathbb{Z}\}$ by

$$Q_j f = \sum_{k\in\mathbb{Z}} \langle f, \tilde{\psi}_{j,k}\rangle \psi_{j,k}.$$

It is obvious from the construction of the mask $\{a_n : n \in \mathbb{Z}\}$ (see (2.10) and (2.12)) and Lemma 5.5 that

$$\|P_j f - f\|_{L_2(\mathbb{R})} = O(2^{-jL}).$$

In many applications, $P_j f$ is interpreted as an approximation to f at the resolution 2^{-j}, while $Q_j f$ represents the fine detail in f. But, if we use sample values of smooth function as refinable function coefficients at a fine scale, then the resulting biorthogonal projection $P_j f(x)$ approximates the underlying function f with the (optimal) approximation rate $O(2^{-jL})$. More specifically, the value $\langle f, \tilde{\phi}_{j,k}\rangle$ can be approximated by the function value $2^{-j/2} f(2^{-j}k)$ with the error bound $O(2^{-jL})$ for $f \in C^L(\mathbb{R})$. The next theorem treats this approximation.

Theorem 5.7 *Let ϕ be the refinable function obtained from the subdivision scheme S_L with L even, and let $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ be a corresponding biorthgonal MRA-wavelet system. Assume that $f \in C^L(\mathbb{R})$. Then, for any fixed $j \in \mathbb{N}$ and $k \in \mathbb{Z}$,*

$$|f(2^{-j}k) - 2^{j/2}\langle f, \tilde{\phi}_{j,k}\rangle| = O(2^{-jL}).$$

Proof. First, by change of variables, it is clear that

$$2^{j/2}\langle f, \tilde{\phi}_{j,k}\rangle = 2^{j/2}\int_{\mathbb{R}} f(t)\tilde{\phi}_{j,k}(t)dt = \int_{\mathbb{R}} f(2^{-j}t)\tilde{\phi}(t-k)dt.$$

Using the identity $\int_{\mathbb{R}} \tilde{\phi}(t-k)dt = 1$ and taking the Taylor polynomial of $f(2^{-j}t)$ of degree $L-1$ at $2^{-j}k$, we get

$$\begin{aligned} f(2^{-j}k) - 2^{j/2}\langle f, \tilde{\phi}_{j,k}\rangle &= \int_{\mathbb{R}} (f(2^{-j}k) - f(2^{-j}t))\tilde{\phi}(t-k)dt \\ &= -\int_{\mathbb{R}} \left(\sum_{n=1}^{L-1} \frac{f^{(n)}(2^{-j}k)}{n!} 2^{-nj}(t-k)^n + R_L f(t)\right)\tilde{\phi}(t-k)dt \end{aligned}$$

where $R_L f$ is the remainder of Taylor expansion

$$R_L f(t) = f^{(L)}(\xi)2^{-jL}(t-k)^L/L!.$$

with ξ between $t2^{-j}$ and $k2^{-j}$. Due to Lemma 5.5, the first integral on the right-hand side of the above equation is identically zero. Thus, it follows that

$$|f(2^{-j}k) - 2^{j/2}\langle f, \tilde{\phi}_{j,k}\rangle| \le c\|f^{(L)}\|_\infty 2^{-jL}\int_{\mathbb{R}} |(t-k)^L\tilde{\phi}(t-k)|dt.$$

Since $\tilde{\phi}$ is compactly supported, $\displaystyle\int_{\mathbb{R}} |(t-k)^L\tilde{\phi}(t-k)|dt < \infty$, which completes the proof. $\square$

From the definition $\tilde{\psi}_{j,n} = \sum_{k\in\mathbb{Z}} \tilde{d}_{n-2k}\tilde{\phi}_{j+1,k}$, the following corollary is immediate.

Corollary 5.8 *Let ϕ be the refinable function obtained from S_L and let $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ be a corresponding biorthgonal MRA-wavelet system. Assume that $f \in C^L(\mathbb{R})$. Then, for any fixed $j \in \mathbb{N}$ and $k \in \mathbb{Z}$,*

$$|\sum_{k\in\mathbb{Z}} \tilde{d}_{n-2k} f(2^{-j-1}k) - 2^{(j+1)/2}\langle f, \tilde{\psi}_{j,k}\rangle| = O(2^{-(j+1)L}).$$

L	$\pm n$	a_n	$\tilde{a}_n$
	0	$1-2\omega$	$1/2*(4*\omega-3)/(-1+4*\omega)$
2	1	$1/2$	$1/2*(-1+5*\omega+4*\omega^2)/(-1+4*\omega)$
	2	ω	$1/4*(1+4*\omega)/(-1+4*\omega)$
	3	0	$-1/2*\omega*(1+4*\omega)/(-1+4*\omega)$
	0	$1+6\omega$	$1/64*(69+4770*\omega+24992*\omega^2+18432*\omega^3)/(16*\omega+1)/(64*\omega+1)$
	1	$9/16$	$-1/16*(-9-741*\omega-9872*\omega^2+21504*\omega^3)/(16*\omega+1)/(64*\omega+1)$
	2	-4ω	$-1/256*(15-800*\omega-109824*\omega^2+49152*\omega^3)/(16*\omega+1)/(64*\omega+1)$
4	3	$-1/16$	$1/16*(-1-113*\omega-2128*\omega^2+33792*\omega^3)/(16*\omega+1)/(64*\omega+1)$
	4	ω	$1/128*(3-10*\omega-14880*\omega^2+6144*\omega^3)/(16*\omega+1)/(64*\omega+1)$
	5	0	$-1/16*\omega*(-13-528*\omega+13312*\omega^2)/(16*\omega+1)/(64*\omega+1)$
	6	0	$1/256*(1024*\omega^2-80*\omega-1)/(16*\omega+1)/(64*\omega+1)$
	7	0	$1/16*(1024*\omega^2-80*\omega-1)*\omega/(16*\omega+1)/(64*\omega+1)$
	0	$1-20*\omega$	$[1/16384(-572427-54787440640\omega^2+1552496459776\omega^3+322878144\omega)$
	1	$150/256$	$3/128(-675-77033920\omega^2+22145925120\omega^4+3424518144\omega^3+406055\omega)$
	2	$15*\omega$	$3/32768*(66825+2837577728*\omega^2+332188876800*\omega^3-30911040*\omega)$
	3	$-25/256$	$\frac{1}{256}*(675+94789248*\omega^2-250987151360*\omega^4-4279500800*\omega^3+452090*\omega)$
	4	$-6*\omega$	$-3/8192*(7425+341966848*\omega^2+36909875200*\omega^3-3532864*\omega)$
6	5	$3/256$	$3/256*(-27-7393856*\omega^2+53016002560*\omega^4+404881408*\omega^3+26729*\omega)$
	6	ω	$1/65536*(41175+2279473152*\omega^2+204682035200*\omega^3-20958656*\omega)$
	7	0	$-3/256*\omega*(3105+129105920*\omega^2-1454144*\omega+15435038720*\omega^3)$
	8	0	$-3/32768*(675+54919168*\omega^2+3355443200*\omega^3-401600*\omega)$
	9	0	$1/256*\omega*(-690752*\omega+5771362304*\omega^3+1161+92405760*\omega^2)$
	10	0	$9/65536*(27+2490368*\omega^2-16064*\omega+134217728*\omega^3)$
	11	0	$-3/256*\omega*(27+2490368*\omega^2-16064*\omega+134217728*\omega^3)]$
			$/(64*\omega-1)/(2097152*\omega^2-14336*\omega+27)$

Table 2: The dual mask for $L = 2, 4, 6$.

6 Examples

6.1 Dual Refinable Functions

Let $\phi \in L_2(\mathbb{R})$ be a refinable function associated with S_L such that its integer translates are linear independent. The construction of its dual refinable function $\tilde{\phi}$ usually starts from finding a dual symbol $\tilde{a}(z)$ such that the relation

$$a(z)\overline{\tilde{a}(z)} + a(-z)\overline{\tilde{a}(-z)} = 4, \qquad z = e^{-i\xi}. \tag{6.1}$$

Recall that if $y = \sin^2 \xi/2$,

$$\mathcal{A}(y) := a(e^{i\xi}) = (1-y)^N \left[2\sum_{n=0}^{N-1} \binom{N-1+n}{n} y^n + \omega 2^{4N}(-1)^N y^N \right].$$

Thus, letting

$$P_1(y) = (1-y)^N \mathcal{A}(y)/2,$$

the problem to find the dual $\tilde{a}(z)$ in (6.1) is equivalent to constructing $P_2(y)$ which solves the Bezout problem

$$P_1(y)P_2(y) + P_1(1-y)P_2(1-y) = 1 \tag{6.2}$$

where the degree of $P_2(y)$ is $3N-1$; see ([8]) for the details of the Bezout problem. Since $\phi(\cdot - k)$, $k \in \mathbb{Z}$, are linear independent, it is immediate from Theorem 3.7 that there is no common zero of $P_1(y)$ and $P_2(y)$, which guarantees the existence of $P_2(y)$. For $L = 2, \ldots, 8$, the specific form of the dual mask of $\{\tilde{a}_n : n \in \mathbb{Z}\}$ are given in Table 2.

6.2 Bithogonal Wavelets for the case $L = 4$

Here, some examples of biorthogonal wavelet systems $L = 4$ are computed with respect to several different choices of ω. In this case, the refinable functions reproduce cubic polynomials and the wavelet functions have the vanishing moment of order 4. Eventually, it becomes the Coifman biorthogonal wavelet of order 4. Figure 1 indicates that the dual functions ϕ, $\tilde{\phi}$ and their associated wavelets ψ, $\tilde{\psi}$. Here, $\omega = 0.025, 0, -0.005, -0.0203$. The dual functions $(\phi, \tilde{\phi})$ has the (integer smoothness) (C^1, C^1), (C^1, C^0), (C^2, C^0), and (C^3, C^0) respectively. In particualr, if $\omega = 0$, it becomes the minimal length Coiffman biorthogonal wavelet system.

6.3 Dual Functions with Less Dissimilar Lengths

We are concerned with the biorthgonal wavelet systems of less dissimilar filter lengths. Involving the equation of Bezout problem in (6.2), let the zeros of $P_2(y)$ are $\lambda_m, \bar{\lambda}_m \in \mathbb{C}$ with $m = 1, \ldots, K$ and $y_n \in \mathbb{R}$ with $n = 1, \ldots, 3N-2K-1$. Then $P_2(y)$ is factored into the form

$$P_2(y) = M \prod_{n=1}^{3N-2K-1} (y - y_n) \prod_{m=1}^{K} (y^2 - 2y\mathrm{Re}\lambda_m + |\lambda_m|^2) \tag{6.3}$$

for some constant M. On the purpose of constructing new refinable dual functions ϕ and $\tilde{\phi}$, we regroup the factors of $P_2(y)$ in (6.3) and set

$$\begin{aligned} h(z) &:= a(z) M_1 \prod_{\ell \in I_1} (y - y_\ell) \prod_{m \in J_1} (y^2 - 2y\mathrm{Re}\lambda_m + |\lambda_m|^2) \\ g(z) &:= a(z) M_2 \prod_{\ell \in I_2} (y - y_\ell) \prod_{m \in J_2} (y^2 - 2y\mathrm{Re}\lambda_m + |\lambda_m|^2), \end{aligned}$$

where $y = (-z^{-1} + 2 - z)/4$ and

$$M = M_1 M_2, \quad I_1 \cup I_2 = \{1, \ldots, 3N-1-2K\}, \quad J_1 \cup J_2 = \{1, \ldots, K\}.$$

Then, with the new masks h and g at hand, we derive new refinable functions ϕ and $\tilde{\phi}$.

Example 6.1 (13-11 Tab biorthogonal wavelet system based on cubic) *Let $L = 4$. Then, by using symbolic computation with MAPLE 8, we obtain*

$$\begin{aligned} P_2(y) =& 2 + 4y + (12 - 256\omega)y^2 + 16y^3 + \frac{16(98304\omega^3 - 6656\omega^2 - 176\omega - 1)}{1 + 80\omega + 1024\omega^2} y^4 \\ &- \frac{1024\omega(-1 - 80\omega + 1024\omega^2)}{1 + 80\omega + 1024\omega^2} y^5. \end{aligned}$$

Let $\{y_1, \lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2\}$ be the roots of $P_2(y)$ and y_1 is the real root among them. Then we have ϕ and $\tilde{\phi}$ corresponding to

$$h(z) := (1-y)^2(y^2 - 2y\mathrm{Re}\lambda_1 + |\lambda_1|^2)(y^2 - 2y\mathrm{Re}\lambda_2 + |\lambda_2|^2)/(|\lambda_1|^2|\lambda_2|^2)$$
$$g(z) := -(1-y)^2(1+2y+256\omega y^2)(y-y_1)/y_1,$$

where $y = (-z^{-1}+2-z)/4$. Figure 6.3 indicates the dual functions ϕ, $\tilde{\phi}$ and their associated wavelets ψ, $\tilde{\psi}$. Here, $\omega = 0.002, 0, -0.005, -0.015$ and then the dual functions $(\phi, \tilde{\phi})$ has the (integer smoothness) (C^1, C^1), (C^1, C^0), (C^2, C^0), and (C^3, C^0) respectively.

7 Concluding Remarks

The scheme S_L provides a large class of refinable functions which generate multiresolution analysis. Thus, the refinable functions obtained by the new schemes can be used for the construction of wavelet systems that balance and meet various demands, such as regularity of wavelets, shapes of refinable functions and approximation power, in time-frequence analysis Our next project would be applying this biorthogonal wavelet systems for signal approximation and compression.

Our initial numerical observation verifies that by choosing ω in some suitable range, the new biorthognal wavelets systems have competitive compression ability to the Coiffman biorthogonal wavelet system as well as the 9-7 tab biorthogonal wavelet system, which is the most widely used one in the field of wavelet transform coding.

One may interested in continuing this study for the case L is odd. For the case that L is odd, i.e., $L = 2N+1$, the masks of S_L are obtained by solving the linear system

$$p_\ell(2^{-1}) = \sum_{n=-N}^{N+\tau} a_{\tau-2n} p_\ell(n), \quad \ell = 1, \ldots, L, \tag{7.1}$$

where p_ℓ, $\ell = 1, \ldots, L$, is a basis of $\Pi_{<L}$. For the even mask $\{a_{2n} : n \in \mathbb{Z}\}$, setting $a_{-2N-2} = (-1)^N\omega$, their explicit forms are

$$a_{-2n} \quad = \quad \binom{4N}{2N}\binom{2N}{N+n}\frac{(-1)^n(4N+1)}{(1-4n)4^{3N}} - \omega(-1)^{2N-n}\binom{2N+1}{N+n} \tag{7.2}$$

for $n = -N, \ldots, N$. For the odd mask $\{a_{1+2n} : n \in \mathbb{Z}\}$, setting $a_{1+2N} = (-1)^N\omega$, their explicit forms are

$$a_{1-2n} \quad = \quad \binom{4N}{2N}\binom{2N}{N-n+1}\frac{(-1)^{1-n}(4N+1)}{(4n-3)4^{3N}} - \omega(-1)^{2N+n-1}\binom{2N+1}{N-n+1} \tag{7.3}$$

for $n = -N+1, \ldots, N+1$. However, for the construction of biorthogonal wavelets, we do not pursue this direction.

References

[1] A. Cohen and J.P. Conze, Régularité des bases d'ondelettes et mesures ergodiques, Rev. Mat. Iberoamericana, **8**, no. 3, (1992), 351-365.

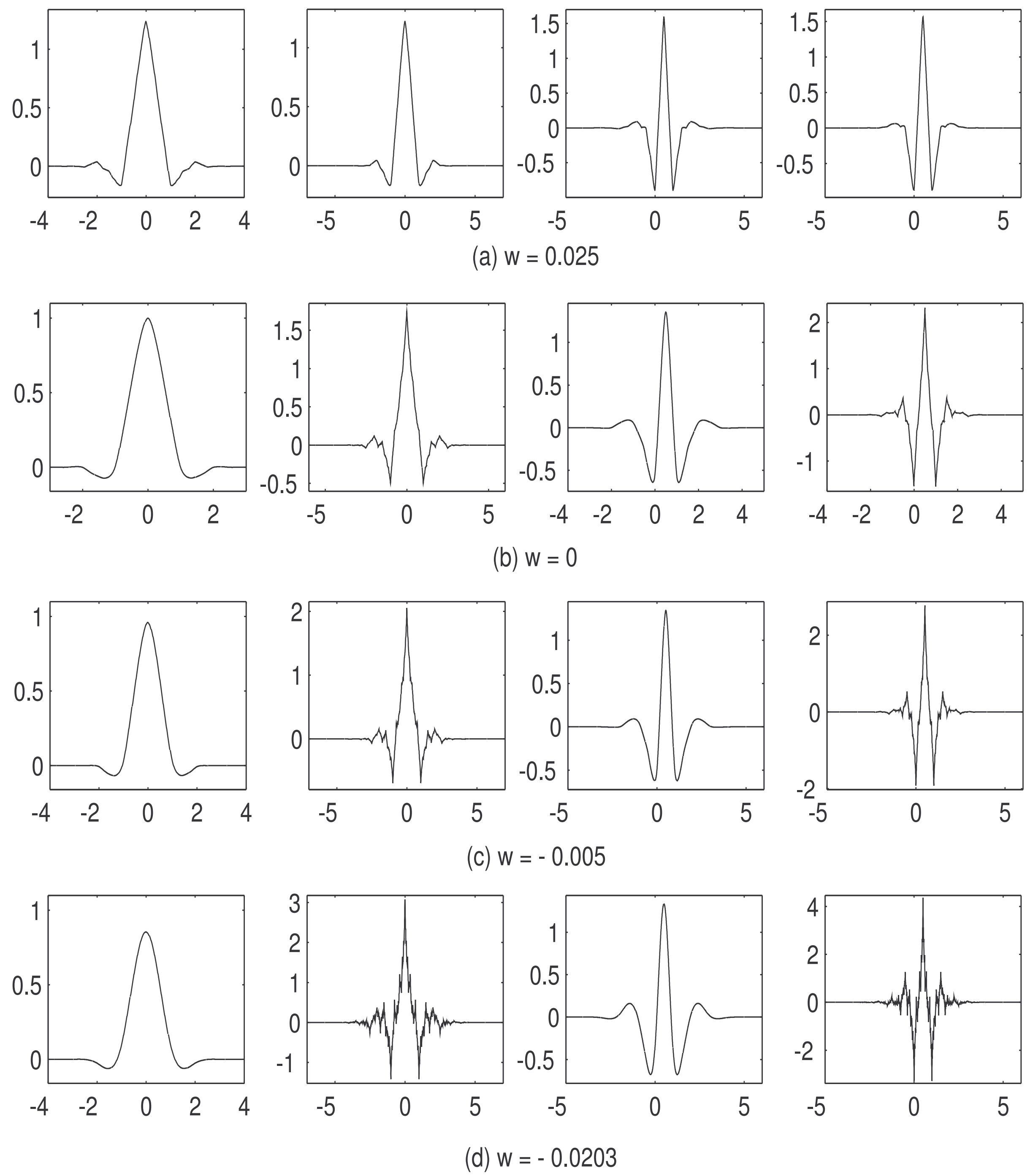

Figure 1: The biorthogonal wavelet systems based on cubic polynomials. From the left, the pictures in each column indicate the refinable functions ϕ, $\tilde{\phi}$ and wavelets ψ, $\tilde{\psi}$. From the top, ω is chosen to be $0.025,\ 0,\ -0.005,\ -0.0203$. If $\omega = 0$, it becomes the minimal length Coiffman biorthogonal wavelet system.

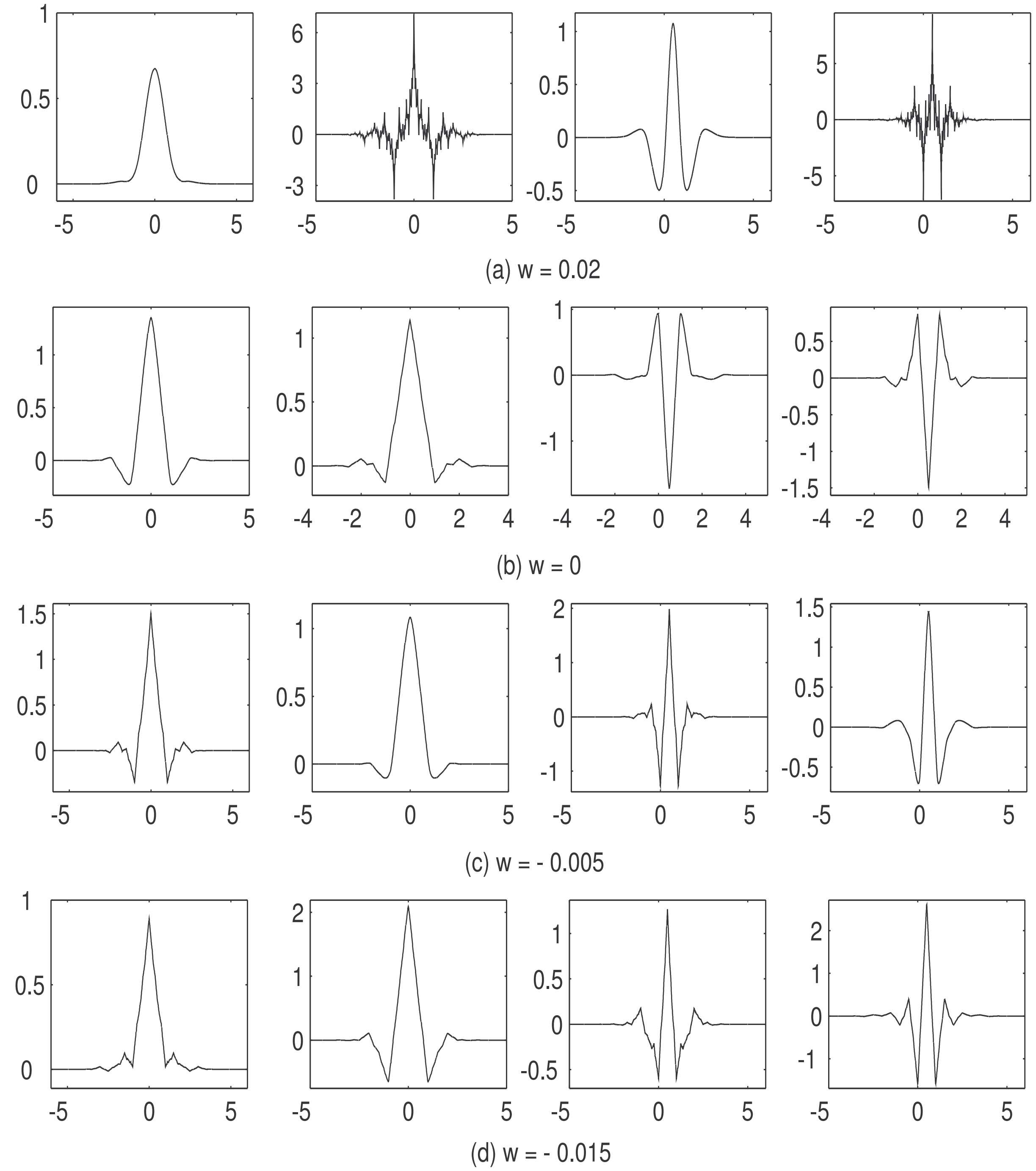

Figure 2: The 9-7 tab biorthogonal wavelet system based on cubic polynomial. From the left, the pictures in each column indicate the refinable functions ϕ, $\tilde{\phi}$ and wavelets ψ, $\tilde{\psi}$. From the top, ω is chosen to be $0.002,\ 0,\ -0.005,\ -0.015$.

[2] A. Cohen, I. Daubechies, *Stability criterion for biorthogonal wavelet bases and their related subband coding scheme*, Duke Math. J., **68**, no. 2, 1992, 313-335

[3] A. Cohen, I. Daubechies and J. C. Feauveau, *Biorthogonal bases of compactly supported wavelets*, Comm. Pure Appl. Math., **45** (1992), 485–560

[4] S. Choi, B. Lee, and Y. Lee, J. Yoon, *Stationary Subdivision Schemes Reproducing Polynomials,* Comp. Aided Geom. Design, **23**, (2006), 351-360.

[5] C.K. Chui and J. Z. Wang, *A Cardinal Spline Approach to Wavelets*, Proc. Amer. Math. Soc., 113 (1991), 785-793.

[6] C.K. Chui and J. Z. Wang, *A General Framework of Compactly Supported Splines and Wavelets*, J. Approx. Theory, 75 (1992), 903-915.

[7] I. Daubechies, *Orthogonal Bases of Compactely Supported Wavelets*, Comm. Pure Appl. Math.,**41**(1998), 909-996

[8] I. Daubechies, "*Ten Lectures on Wavelets*," CBMS-NSF series in Applied Mathematics, SIAM, Philadelphia, 1992.

[9] G. Deslauriers and S. Dubuc, *Symmetric Iterative interpolation*, Constr. Approx. **5** (1989), 49-68.

[10] W. Dahmen and C. A. Micchelli, *On the local linear independence of translates of a box spline*, Studia Math. **82** (1985), 243-262.

[11] W. Dahmen and C. A. Micchelli, *Translates of multivariate splines*, Linear Algebra Appl. **52** (1983), 217-234.

[12] B. Dong and Z. Shen, *Pseudo-splines, wavelets and framelets*, Appl. Comut. Harmon. Anal. 21 (2007), 78-104.

[13] B. Dong and Z. Shen, *Construction of biorthogonal wavelets from pseudo-splines*, J. Approx. Theory, 138 (2006), 211-231.

[14] N. Dyn, *Subdivision Schemes in Computer-Aided Geometric Design*, Advances in Numerical Analysis Vol. II: Wavelets, Subdivision Algorithms and Radial Basis Functions (W.A. Light ed.), Oxford University Press, (1992), 36-104.

[15] R. Q. Jia, *Linear independence of translates of a box spline* , J. Approx. Theorey, **40** (1984), 158-160.

[16] R. Q. Jia and C. A. Micchelli, *Using the refinement equations for the construction of prewavelets. II. Powers of two*, Curves and Surfaces, 209-246, Academic Press, Boston, MA, 1991.

[17] R. Q. Jia and J. Z. Wang, *Stability and linear independence associated with wavelet decompositions*, P. Am. Math. Soc., 117 (4) (1993), 1115-1124.

[18] H. Kim, R. Kim, Y. Lee and J. Yoon, *Quasi-Interpolatory Refinable Functions and Contruction of Biorthogonal, Wavelet Systems*, Research Report Series 06-13 of KAIST (2006); available at http://amath.kaist.ac.kr/InKorean/preprints.htm

[19] S. Mallat, *Multiresoulutional approximation and wavelets*, Trans. Amer. Math. Soc. **315** 69-88.

[20] P. G. Lemarié-Rieusset, *On the existence of compactly supported dual wavelets*, Appl. Comut. Harmon. Anal. 3 (1997), 117-118.

[21] A. Ron , *A necessary and sufficient condition for the linear independence of the integer translates of a compactly supported distribution*, Constr. Approx. **5** (1989), 2975-308.

[22] D. Wei, J. Tian, R. O. Wells, Jr., C. S. Burrus, *New Class of Biorthogonal Wavelet Systems for Image Transform Coding*, IEEE Trans. on Image Proc. 7, (1998), 1000-1013.