SAMPLING THEORY IN ABSTRACT REPRODUCING KERNEL HILBERT SPACE

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Abstract. Let H be a separable Hilbert space and $k(t)$ an H-valued function on a subset Ω of the real line $\mathbb{R}$ such that $\{k(t) \mid t \in \Omega\}$ is total in H. Then

$$\{f_x(t) := \langle x, k(t)\rangle_H \mid x \in H\}$$

becomes a reproducing kernel Hilbert space (RKHS) in a natural way. Here, we develop a sampling formula for functions in this RKHS, which generalizes the well-known celebrated Whittaker-Shannon-Kotel'nikov sampling formula in the Paley-Wiener space of band-limited signals. To be more precise, we develop a multi-channel sampling formula in which each channel is given rather arbitrary sampling rate. We also discuss the stability and oversampling.

1. Introduction

Let $f(t)$ be a band-limited signal with band region $[-\pi, \pi]$, that is, a square-integrable function on $\mathbb{R}$ of which the Fourier transform $\hat{f}$ vanishes outside $[-\pi, \pi]$. Then f can be recovered by its uniformly spaced discrete values as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)},$$

which converges absolutely and uniformly over $\mathbb{R}$. This series is called the cardinal series or the Whittaker-Shannon-Kotel'nikov (WSK) sampling series. This formula tells us that once we know the values of a band-limited signal f at certain discrete points, we can recover f completely. In 1941, Hardy [4] recognized that this cardinal series is actually an orthogonal expansion.

WSK sampling series was generalized by Kramer [8] in 1957 as follows: Let $k(\xi, t)$ be a kernel on $I \times \Omega$, where I is a bounded interval and Ω is a subset of $\mathbb{R}$. Assume that $k(\cdot, t) \in L^2(I)$ for each t in Ω and there are points $\{t_n\}_{n\in\mathbb{Z}}$ in Ω such that $\{k(\xi, t_n)\}$ is an orthonormal basis of $L^2(I)$. Then any $f(t) = \int_I F(\xi)k(\xi, t)d\xi$ with $F(\xi) \in L^2(I)$ can be expressed as a sampling series

$$f(t) = \sum_n f(t_n) \int_I k(\xi, t)\, \overline{k(\xi, t_n)}\, d\xi,$$

which converges absolutely and uniformly over the subset D on which $\|k(\cdot, t)\|_{L^2(I)}$ is bounded. While WSK sampling series treats sample values taken at uniformly spaced points, Kramer's series may take sample values at nonuniformly spaced points.

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Recently, A. G. Garcia and A. Portal [3] extended the WSK and Kramer sampling formulas further to a more general setting using a suitable abstract Hilbert space valued kernel.

On the other hand, Papoulis [10] (see also [7]) introduced a multi-channel sampling formula for band-limited signals, in which a signal is recovered from discrete sample values of several transformed versions of the signal.

In this work, following the setting introduced by Garcia and Portal [3], we first extends and modify Theorem 1 in [3] into a single channel sampling formula (see Theorem 3.2 below), which is more transparent. It is then easy to extend it to a multi-channel sampling formula, in which each channel can be given rather arbitrary sampling rate. Comparing two-channel sampling formula, Theorem 3 in [3] and our multi-channel sampling formula, Theorem 3.3, reveals the advantage of modification made in Theorem 3.2. Finally, we also discuss the oversampling and recovery of missing samples in the single-channel sampling formula.

2. Preliminaries

For $f(t) \in L^2(\mathbb{R})$, we let

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\xi t}\, dt$$

be the Fourier transform of $f(t)$ and

$$f(t) = \mathcal{F}^{-1}(\hat{f})(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{i\xi t}\, d\xi$$

the inverse Fourier transform $\hat{f}(\xi)$.

Definition 2.1. For any $w > 0$, the Paley-Wiener space $PW_{\pi w}$ is defined to be

$$PW_{\pi w} := \{f \mid f \in L^2(\mathbb{R}),\ supp\, \widehat{f} \in [-\pi w, \pi w]\}.$$

Note that $PW_{\pi w}$ is isometrically isomorphic onto $L^2[-\pi w, \pi w]$ under the Fourier transform.

We call a basis $\{\varphi_n\}$ of a separable Hilbert space H to be an unconditional basis of H if for every $f \in H$, the expansion $f = \sum c_n(f)\varphi_n$ still converges to f after any permutation of its terms. We also call a basis $\{\varphi_n\}$ to be a Riesz basis of H if there is a linear isomorphism T from H onto H such that $T(e_n) = \varphi_n$ where $\{e_n\}$ is an orthonormal basis for H. Then any Riesz basis of H is an unconditional basis of H but not conversely in general.

Definition 2.2. [12] A Hilbert space H consisting of complex-valued functions defined on a set $D(\neq \emptyset)$ is called a reproducing kernel Hilbert space (RKHS in short) if there exists a function $k(s,t)$ on $D \times D$ satisfying

(1) $k(\cdot, t) \in H$ for each $t \in D$;
(2) $\langle f(s), k(s,t)\rangle_H = f(t)$ for all $f \in H$ and all $t \in D$.

Such a function $k(s,t)$ is called a reproducing kernel of H.

We need some properties of RKHS's.

Proposition 2.3. [5] *Let H be a Hilbert space as in Definition 2.2. Then we have:*

(a) *H is an RKHS if and only if the point evaluation map $l_t(f) := f(t)$ is a bounded linear functional on H for each $t \in D$;*

(b) *an RKHS H has a unique reproducing kernel;*
(c) *the convergence of a sequence in an RKHS H implies its uniform convergence over any subset of D on which $k(t,t)$ is bounded.*

For example, the Paley-Wiener space $PW_{\pi w}$ is an RKHS with the reproducing kernel $k(s,t) = w\,\dfrac{\sin \pi w(s-t)}{\pi w(s-t)}$.

3. Multi-channel sampling

Let H be a separable Hilbert space and $k : \Omega \longrightarrow H$ be an H-valued function on a subset Ω of the real line $\mathbb{R}$. Define a linear operator T on H by

$$T(x)(t) = f_x(t) := \langle x, k(t)\rangle_H,\ t \in \Omega.$$

We call $k(t)$ the kernel of the linear operator T.

Lemma 3.1. *([3])*

(a) *T is one-to-one if and only if $\{k(t) \mid t \in \Omega\}$ is total in H.*

Assume $\{k(t) \mid t \in \Omega\}$ is total in H so that $T : H \longrightarrow T(H)$ is a bijection. Then

(b) *$\langle T(x), T(y)\rangle_{T(H)} := \langle x, y\rangle_H$ defines an inner product on $T(H)$, with which $T(H)$ is a Hilbert space and $T : H \longrightarrow T(H)$ is unitary. Moreover, $T(H)$ becomes an RKHS with the reproducing kernel $k(s,t) := \langle k(t), k(s)\rangle_H$.*

Proof. (a) T is one-to-one if and only if $\{k(t) \mid t \in \Omega\}^{\perp} = \{0\}$ if and only if $\overline{span}\{k(t) \,|\, t \in \Omega\} = H$, that is, $\{k(t) \mid t \in \Omega\}$ is total in H.

(b) It is trivial that $\langle T(x), T(y)\rangle_{T(H)} := \langle x, y\rangle_H$ defines an inner product on $T(H)$ with which $T : H \longrightarrow T(H)$ is unitary. Now for any $f(\cdot) = \langle x, k(\cdot)\rangle_H$ in $T(H)$ and $t \in \Omega$,

$$|f(t)| = |\langle x, k(t)\rangle_H| \le \|x\|_H \|k(t)\|_H = \|f\|_{T(H)} \|k(t)\|_H$$

so that $l_t(f) = f(t)$ is a bounded linear functional on $T(H)$. Hence $T(H)$ is an RKHS by Proposition 2.3. Since

$$f(t) = \langle x, k(t)\rangle_H = \langle T(x)(s), T(k(t))(s)\rangle_{T(H)} = \langle f(s), \langle k(t), k(s)\rangle_H\rangle_{T(H)},$$

the reproducing kernel $k(s,t)$ of $T(H)$ is $\langle k(t), k(s)\rangle_H$. $\square$

First, we develop a single-channel sampling formula. Let $\tilde{k} : \Omega \longrightarrow H$ be another H-valued function on Ω and $\widetilde{T}$ the linear operator on H defined by

$$\widetilde{T}(x)(t) = \tilde{f}_x(t) = \langle x, \tilde{k}(t)\rangle_H.$$

Theorem 3.2. *If $KerT \subseteq Ker\widetilde{T}$ and there exists a sequence $\{t_n\}$ in Ω such that $\{\tilde{k}(t_n)\}_n$ is a basis of H, then T is one-to-one so that $T(H)$ becomes an RKHS under the inner product $\langle T(x), T(y)\rangle_{T(H)} := \langle x, y\rangle_H$. Moreover, there is a basis $\{S_n(t)\}_n$ of $T(H)$ with which we have the sampling expansion:*

$$f_x(t) = \sum_n \tilde{f}_x(t_n)\, S_n(t), \qquad f_x(t) \in T(H) \tag{3.1}$$

which converges not only in $T(H)$ but also uniformly over any subset on which $\|k(t)\|_H$ is bounded.

Proof. Assume $\widetilde{T}(x)(t) = \langle x, \tilde{k}(t)\rangle = 0$ on Ω. Then $\langle x, \tilde{k}(t_n)\rangle = 0$ for any n so that $x = 0$ since $\{\tilde{k}(t_n)\}_n$ is a basis of H. Hence $KerT = Ker\widetilde{T} = \{0\}$ and $T(H)$ becomes an RKHS as in Lemma 3.1 (b).

Let $\{x_n\}_n = \{\tilde{k}(t_n)\}_n$ and $\{x_n^*\}_n$ be its dual. Then $\{T(x_n)(t)\}$ and $\{T(x_n^*)(t)\}$ are bases of $T(H)$, which are dual each other since T is unitary.

Expanding any $f_x(t) = T(x)(t)$ in $T(H)$ via the basis $\{S_n(t)\}_n = \{T(x_n^*)(t)\}$ gives

$$\begin{aligned} f_x(t) &= \sum_n \langle T(x),\, T(x_n)\rangle_{T(H)} S_n(t) = \sum_n \langle x,\, x_n\rangle_H \; S_n(t) \\ &= \sum_n \langle x,\, \tilde{k}(t_n)\rangle_H \; S_n(t) = \sum_n \tilde{f}_x(t_n)\, S_n(t). \end{aligned}$$

Uniform convergence of the series (3.1) follows from Proposition 2.3 (c). $\square$

The single channel sampling expansion (3.1) may not converge absolutely unless $\{x_n\}_n$ is an unconditional basis and may not be stable. However, if $\{x_n\}_n$ is an unconditional basis and $\sup_n \|x_n^*\| < \infty$, then (3.1) is a stable sampling expansion, which converges absolutely on Ω. In fact, if then, $\{S_n(t)\}_n$ becomes an unconditional basis of $T(H)$ and $\sup_n \|S_n(t)\| = \sup_n \|x_n^*\| < \infty$. Since $\{\frac{1}{\|S_n(t)\|} S_n(t)\}_n$ is a Riesz basis of $T(H)$ by the Köthe-Toeplitz Theorem [9], there is a constant $B > 0$ such that

$$\|f_x(t)\|^2_{T(H)} \le B \sum_n |\tilde{f}_x(t_n)|^2 \|S_n(t)\|^2 \le (\sup_n \|S_n(t)\|)^2 B \sum_n |\tilde{f}_x(t_n)|^2, \quad f_x(t) \in T(H).$$

Furthermore, the sampling series expansion (3.1) remains valid when $\{\tilde{k}(t_n)\}_n$ is not a basis but a frame of H. When $\tilde{k}(t) = k(t)$ on Ω so that $T = \tilde{T}$, Theorem 3.2 is essentially Theorem 1 in [3]. However, Theorem 3.2 might have some advantage over Theorem 1 in [3]. While Theorem 1 in [3] requires first the expansion of the kernel $k(t)$ in terms of a given basis of H and then the interpolatory condition for the expansion coefficients at some points in Ω, Theorem 3.2 simply requires points in Ω, of which values under $k(\cdot)$ form a basis of H.

Now, we can extend Theorem 3.2 naturally to a multi-channel setting. Let $\{k_i\}_{i=1}^N$ be N H-valued functions on Ω and $\{T_i\}_{i=1}^N$ linear operators on H defined by

$$T_i(x)(t) = f_x^i(t) := \langle x, k_i(t)\rangle_H, \quad x \in H.$$

Theorem 3.3. *(Asymmetric nonuniform multi-channel sampling formula) If $KerT \subseteq \cap_{i=1}^N KerT_i$ and there exist points $\{t_{i,n} \mid 1 \le i \le N, n \in \mathbb{Z}\} \subset \Omega$ and constants $\{\alpha_{i,n}^j \mid 1 \le i \le N, 1 \le j \le M \text{ and } n \in \mathbb{Z}\}$ for some integer $M \ge 1$ such that $\{\sum_{i=1}^N \alpha_{i,n}^j k_i(t_{i,n}) \mid 1 \le j \le M \text{ and } n \in \mathbb{Z}\}$ is an unconditional basis of H, then there is a basis $\{S_{j,n}(t) \mid 1 \le j \le M \text{ and } n \in \mathbb{Z}\}$ of $T(H)$ such that for any $f_x(t) = T(x)(t) \in T(H)$,*

$$f_x(t) = \sum_{n\in Z}\sum_{j=1}^{M} \{\overline{\alpha_{1,n}^j}\, f_x^1(t_{1,n}) + \overline{\alpha_{2,n}^j}\, f_x^2(t_{2,n}) + \cdots + \overline{\alpha_{N,n}^j}\, f_x^N(t_{N,n})\} S_{j,n}(t) \quad (3.2)$$

which converges in $T(H)$. Moreover, the series (3.2) converges absolutely and uniformly on any subset of Ω over which $\|k(t)\|_H$ is bounded.

Proof. First, we prove that T is one-to-one. Suppose $T(x)(t) = \langle x, k(t)\rangle = 0$ for all $t \in \Omega$. Then, $\langle x, k_i(t)\rangle = 0$, $1 \le i \le N$ on Ω since $KerT \subseteq \cap_{i=1}^N KerT_i$. In particular, $\langle x, \sum_{i=1}^N \alpha_{i,n}^j k_i(t_{i,n})\rangle = 0$ for all $1 \le j \le M$ and $n \in \mathbb{Z}$ so that $x = 0$ since $\{\sum_{i=1}^N \alpha_{i,n}^j k_i(t_{i,n}) \,|\, 1 \le j \le M \,\text{and}\, n \in \mathbb{Z}\}$ is a basis of H. Therefore, $T : H \longrightarrow T(H)$ is a bijection and $T(H)$ becomes an RKHS under the inner product $\langle T(x), T(y)\rangle_{T(H)} := \langle x, y\rangle_H$ by Lemma 3.1.

Let $x_n^j := \sum_{i=1}^N \alpha_{i,n}^j k_i(t_{i,n})$ for $1 \le j \le M$ and $n \in \mathbb{Z}$ and $\{x_n^{j*}\}_{j=1,n\in\mathbb{Z}}^M$ be the dual of $\{x_n^j\}$. Then, $\{T(x_n^j)\}_{j=1,n\in\mathbb{Z}}^M$ becomes an unconditional basis of $T(H)$ with the dual basis $\{T(x_n^{j*})\}_{j=1,n\in\mathbb{Z}}^M := \{S_{j,n}(t)\}_{j=1,n\in\mathbb{Z}}^M$, which is also unconditional.

Expanding $f_x(t) = T(x)(t)$ in $T(H)$ with respect to $\{S_{j,n}(t)\}_{j=1,n\in\mathbb{Z}}^M$, we have

$$\begin{aligned} f(t) &= \sum_{n\in\mathbb{Z}} \sum_{j=1}^{M} \langle T(x), T(x_n^j)\rangle_{T(H)} S_{j,n}(t) \\ &= \sum_{n\in\mathbb{Z}} \sum_{j=1}^{M} \langle x, x_n^j\rangle_H S_{j,n}(t) \\ &= \sum_{n\in\mathbb{Z}} \sum_{j=1}^{M} \{\overline{\alpha_{1,n}^j}\, f_x^1(t_{1,n}) + \cdots + \overline{\alpha_{N,n}^j}\, f_x^N(t_{N,n})\}\, S_{j,n}(t). \end{aligned}$$

Uniform convergence of the series (3.2) follows from Proposition 2.3 (c). Finally, the series (3.2) converges also absolutely since it is an unconditional basis expansion. □

If either $KerT = \{0\}$ or $k_i(t) = A_i(k(t)), 1 \le i \le N$, where A_i's are automorphisms of H, then the first assumption $KerT \subseteq \cap_{i=1}^N KerT_i$ of Theorem 3.3 is trivially satisfied. For example, it is so when $H = L^2[-\pi, \pi]$, $\Omega = \mathbb{R}$ and $k(t) = \dfrac{e^{-it\xi}}{\sqrt{2\pi}}$ so that $T = \mathcal{F}^{-1}$ is the inverse Fourier transform. In particular, if $N = M = 2$, $k_1(t) = k(t), t_{1,n} = t_{2,n} = t_n$, and $\{\alpha_{1,n}^1 k(t_n) + \alpha_{2,n}^1 k_2(t_n)\} \cup \{\alpha_{1,n}^2 k(t_n) + \alpha_{2,n}^2 k_2(t_n)\}$ is a Riesz basis of H, then Theorem 3.3 is essentially the same as Theorem 3 in [3].

When $H = L^2[-\pi w, \pi w](w > 0), \Omega = \mathbb{R}$ and

$$k(t) = \frac{1}{\sqrt{2\pi}}\, e^{-it\xi},\; k_i(t) = \frac{1}{\sqrt{2\pi}}\, \overline{A_i(\xi)}\, e^{-it\xi} \quad (1 \le i \le N)$$

for suitable bounded measurable functions $A_i(\xi)(1 \le i \le N)$ on $[-\pi w, \pi w]$, we have

$$\begin{aligned} T(\phi)(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi w}^{\pi w} \phi(\xi)\, e^{it\xi}\, d\xi = \mathcal{F}^{-1}(\phi)(t) \\ T_i(\phi)(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi w}^{\pi w} A_i(\xi)\, \phi(\xi)\, e^{it\xi}\, d\xi = \mathcal{F}^{-1}(A_i\phi)(t) \quad (1 \le i \le N). \end{aligned}$$

Hence, $T(H)$ becomes the Paley-Wiener space $PW_{\pi w}$ and then Theorem 3.3 reduces to an asymmetric multi-channel sampling handled in [7].

If $\{\sum_{i=1}^N \alpha_{i,n}^j k_i(t_{i,n}) | \, 1 \le j \le M \,\text{and}\, n \in \mathbb{Z}\}$ is a frame of H in Theorem 3.3, then the sampling series expansion (3.2) still holds.

As in the single channel case, if $\sup\limits_{i,j,n} \|\alpha_{i,n}^j x_n^{j*}\| < \infty$, then the multi-channel sampling expansion (3.2) is also stable in the following sense.

Definition 3.4. (cf. Rawn [11] and Yao and Thomas [13]) We say that $\{t_{i,n}\,|\,1 \le i \le N\,\text{and}\,n \in \mathbb{Z}\}$ is a set of stable sampling for $T(H)$ if there exists $A > 0$ which is independent of $f \in T(H)$ such that

$$\|f(t)\|^2_{T(H)} \le A \sum_{n=-\infty}^{\infty} \sum_{i=1}^{N} |f^i_x(t_{i.n})|^2 \quad f \in T(H).$$

Let $B > 0$ be the upper Riesz bound for the Riesz basis $\{\frac{1}{\|S_{j,n}\|} S_{j,n}(t)\}$ of $T(H)$. Then

$$\|f_x\|^2_{T(H)} \le B \sum_n \sum_{j=1}^{M} \sum_{i=1}^{N} |\overline{\alpha^j_{i,n}} f^i_x(t_{i,n})|^2 \|S_{j,n}\|^2 \le (\sup_{i,j,n} \|\alpha^j_{i,n} S_{j,n}\|)^2 BM \sum_n \sum_{i=1}^{N} |f^i_x(t_{i,n})|^2$$

so that (3.2) is a stable sampling expansion with respect to $\{t_{i,n}\}$ when $\sup_{i,j,n} \|\alpha^j_{i,n} x^{j*}_n\| < \infty$.

We now discuss several examples in which we always take $H = L^2[-\pi, \pi], \Omega = \mathbb{R}$ and $k(t) = \frac{1}{\sqrt{2\pi}} e^{-it\xi}$ so that $T = \mathcal{F}^{-1}$ is the inverse Fourier transform and $T(H) = PW_\pi$.

Example 3.5 (Sampling with Hilbert transform)**.**
Take $\tilde{k}(t) = i\,\mathrm{sgn}(\xi)\,k(t)$ so that $\widetilde{T}(f)(t) = \tilde{f}(t)$ is the Hilbert transform of $f(t)$ in PW_π. Choosing $\{t_n\}_{n\in\mathbb{Z}} = \{n\}_{n\in\mathbb{Z}}$, $\{x_n\}_{n\in\mathbb{Z}} = \{i\,\mathrm{sgn}\xi\, \frac{e^{-in\xi}}{\sqrt{2\pi}}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2[-\pi, \pi]$ so that $\{x^*_n\}_{n\in\mathbb{Z}} = \{x_n\}_{n\in\mathbb{Z}}$. We then have

$$S_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} i\,\mathrm{sgn}(\xi)\, \frac{e^{-in\xi}}{\sqrt{2\pi}}\, e^{it\xi}\, d\xi = -\mathrm{sinc}\frac{1}{2}\,(t-n) \sin\frac{\pi}{2}\,(t-n)$$

where $\mathrm{sinc}t := \frac{\sin \pi t}{\pi t}$. Hence, we have

$$f(t) = -\sum_{n\in\mathbb{Z}} \tilde{f}\,(n)\ \mathrm{sinc}\frac{1}{2}\,(t-n)\ \sin\frac{\pi}{2}\,(t-n)\,, \quad f(t) \in PW_\pi.$$

Using the operational relation $\tilde{\tilde{f}} = -f$([5, Appendix B]) and the fact that if $f \in PW_\pi$, then so does $\tilde{f}$, we also have

$$\tilde{f}(t) = \sum_{n\in\mathbb{Z}} f\,(n)\ \mathrm{sinc}\frac{1}{2}\,(t-n)\ \sin\frac{\pi}{2}\,(t-n)\,, \quad f(t) \in PW_\pi.$$

Example 3.6. Here, we derive asymmetric derivative sampling formula on PW_π, in which we take samples from $f(t)$ and $f'(t)$ with ratio 2:1.

Take $k_1(t) = k(t) = \frac{1}{\sqrt{2\pi}} e^{-it\xi}$ and $k_2(t) = -i\xi\,k(t) = k'(t)$ so that $f^1(t) = f(t)$ and $f^2(t) = f'(t)$ for $f(t) \in PW_\pi$. Now, take the set of sampling points $\{t_{1,n} = \frac{3n}{2}\}_{n\in\mathbb{Z}}$ for $f^1_x(t)$ and $\{t_{2,n} = 3n\}_{n\in\mathbb{Z}}$ for $f^2_x(t)$. With $\alpha^1_{1,n} = \sqrt{\frac{3}{2}}$, $\alpha^1_{2,n} = \alpha^2_{1,n} = 0$ and $\alpha^2_{2,n} = -\sqrt{3}$, $\{\alpha^1_{1,n}\,k_1(t_{1,n}) + \alpha^1_{2,n}\,k_2(t_{2,n})\}_{n\in\mathbb{Z}} \cup \{\alpha^2_{1,n}\,k_1(t_{1,n}) + \alpha^2_{2,n}\,k_2(t_{2,n})\}_{n\in\mathbb{Z}} = \{\sqrt{\frac{3}{4\pi}}\, e^{-i3n\xi/2}\}_{n\in\mathbb{Z}} \cup \{\sqrt{\frac{3}{2\pi}}\, i\xi\, e^{-i3n\xi}\}_{n\in\mathbb{Z}}$ is a Riesz basis of

$L^2[-\pi,\pi]$, of which the dual (cf. [6]) is

$$\left\{\sqrt{\frac{3}{4\pi}}\,\mu_{1,n}(\xi)\,e^{-i3n\xi/2}\right\}\cup\left\{\sqrt{\frac{3}{2\pi}}\,\mu_2(\xi)\,e^{-i3n\xi}\right\},\quad n\in\mathbb{Z} \tag{3.3}$$

where

$$\mu_{1,n}(\xi)=\begin{cases}3\,(\xi+\pi)/2\pi, & \xi\in[-\pi,-\pi/3);\\ 1, & \xi\in[-\pi/3,\pi/3);\\ -3\,(\xi-\pi)/2\pi, & \xi\in[\pi/3,\pi]\end{cases}$$

if n is even,

$$\mu_{1,n}(\xi)=\begin{cases}1/2, & \xi\in[-\pi,-\pi/3);\\ 1, & \xi\in[-\pi/3,\pi/3);\\ 1/2, & \xi\in[\pi/3,\pi]\end{cases}$$

if n is odd, and

$$\mu_2(\xi)=\begin{cases}-3i/4\pi, & \xi\in[-\pi,-\pi/3);\\ 0, & \xi\in[-\pi/3,\pi/3);\\ 3i/4\pi, & \xi\in[\pi/3,\pi].\end{cases}$$

Taking inverse Fourier transform on (3.3), we have a Riesz basis of PW_π:

$$S_{1,n}(t) = \begin{cases}\sqrt{\frac{2}{3}}\,\mathrm{sinc}\frac{1}{3}\left(t-\frac{3n}{2}\right)\,\mathrm{sinc}\frac{2}{3}\left(t-\frac{3n}{2}\right) & \text{if } n \text{ is even,}\\[2ex] \sqrt{\frac{3}{8}}\left\{\frac{1}{3}\,\mathrm{sinc}\frac{1}{3}\left(t-\frac{3n}{2}\right)+\mathrm{sinc}\left(t-\frac{3n}{2}\right)\right\} & \text{if } n \text{ is odd,}\end{cases}$$

$$S_{2,n}(t) = -\frac{\sqrt{3}}{2\pi}\,\mathrm{sinc}\frac{1}{3}\,(t-3n)\,\sin\frac{2\pi}{3}\,(t-3n)\,.$$

With these setting we have the nonsymmetric derivative sampling formula:

$$f(t)=\sum_{n\in\mathbb{Z}}\sqrt{\frac{3}{2}}f(\frac{3n}{2})S_{1,n}(t)-\sqrt{3}f'(3n)S_{2,n}(t),\quad f(t)\in PW_\pi.$$

Example 3.7. We now take $k_1(t)=k(t)=\dfrac{1}{\sqrt{2\pi}}e^{-it\xi}$ and $k_2(t)=e^{i\xi}\,k(t)$ so that $f^1(t)=f(t)$ and $f^2(t)=f(t-1)$. We want to express $f\in PW_\pi$ via samples from $f(t)$ and $f(t-1)$ with ratio $3:2$. Note that $\{\sqrt{\frac{5}{6\pi}}\,e^{-i5n\xi/3}\}_{n\in\mathbb{Z}}\cup\{\sqrt{\frac{5}{4\pi}}\,e^{i\xi}\,e^{-i5n\xi/2}\}_{n\in\mathbb{Z}}$ forms a Riesz basis of $L^2[-\pi,\pi]$ with the dual

$$\begin{aligned}&\left\{\sqrt{\frac{5}{6\pi}}\,\frac{1}{e^{-i4\pi/5}-e^{-i2\pi/5}}\,\mu_{1,n}(\xi)\,e^{-i5n\xi/3}\right\}_{n\in\mathbb{Z}}\\ \cup&\left\{\sqrt{\frac{5}{4\pi}}\,\frac{1}{e^{-i4\pi/5}-e^{-i2\pi/5}}\,\mu_{2,n}(\xi)\,e^{i\xi}\,e^{-i5n\xi/2}\right\}_{n\in\mathbb{Z}}\end{aligned} \tag{3.4}$$

where

$$\mu_{1,n}(\xi)=\begin{cases}e^{-i\frac{4}{5}\pi}+e^{-i2n\pi/3}+e^{-i\frac{6}{5}\pi}\,e^{i2n\pi/3}, & -\pi\le\xi<-\pi/5;\\ e^{-i\frac{4}{5}\pi}-e^{-i\frac{2}{5}\pi}, & -\pi/5\le\xi<\pi/5;\\ -e^{-i\frac{2}{5}\pi}-e^{-i2n\pi/3}-e^{-i\frac{6}{5}\pi}\,e^{i2n\pi/3}, & \pi/5\le\xi\le\pi\end{cases}$$

and

$$\mu_{2,n}(\xi) = \begin{cases} -(-1)^n e^{-i\frac{8}{5}\pi} - e^{-i\frac{2}{5}\pi}, & -\pi \le \xi < -\pi/5; \\ 0, & -\pi/5 \le \xi < \pi/5; \\ e^{-i\frac{4}{5}\pi} + (-1)^n e^{-i\frac{8}{5}\pi}, & \pi/5 \le \xi \le \pi. \end{cases}$$

Then, we can obtain the sampling series

$$f(t) = \sum_n \sqrt{\frac{5}{3}} f(\frac{5n}{3}) S_{1,n}(t) + \sqrt{\frac{5}{2}} f(\frac{5n}{2} - 1) S_{2,n}(t), \quad f(t) \in PW_\pi,$$

where $\{S_{1,n}(t)\} \cup \{S_{2,n}(t)\}$ are the inverse Fourier transforms of functions in (3.4).

4. Oversampling and reconstruction of missing samples

We now develop the oversampling expansion, which extends the one in Kramer [2]. We extend an oversampling expansion in Kramer's setting [2] to a more general case. Again, let k and $\tilde{k} : \Omega \longrightarrow H$ be H-valued functions. Assume that there exists $\{t_n\} \subset \Omega$ such that $\{x_n := \tilde{k}(t_n)\}_n$ is a basis of H with the dual basis $\{x_n^*\}_n$. Define linear operators T and $\widetilde{T}$ on H by $T(x)(t) = \langle x, k(t)\rangle_H := f_x(t)$ and $\widetilde{T}(x)(t) = \langle x, \tilde{k}(t)\rangle_H := \tilde{f}_x(t)$, respectively and assume $KerT \subseteq Ker\widetilde{T}$. Then, both T and $\widetilde{T}$ are one-to-one and so $T(H)$ and $\widetilde{T}(H)$ become RKHS's.

Now, let G be a proper closed subspace of H and $P : H \longrightarrow G$ the orthogonal projection onto G. Then, for any $x \in G$ we have

$$x = \sum_n \langle x, \tilde{k}(t_n)\rangle_H x_n^*$$

so that

$$x = P(x) = \sum_n \langle x, \tilde{k}(t_n)\rangle_H P(x_n^*) = \sum_n \tilde{f}_x(t_n) P(x_n^*). \tag{4.1}$$

Theorem 4.1. *Under the above setting, there is a sequence of sampling functions $\{T_n(t)\}$ in $T(G)$ such that for any $x \in G$*

$$f_x(t) = \sum_n \tilde{f}_x(t_n) T_n(t) \tag{4.2}$$

which converges in $T(H)$ and uniformly on any subset of Ω over which $\|k(t)\|_H$ is bounded. Moreover, if $\{x_n\}$ is a Riesz basis of H, then $\{T_n(t)\}$ is a frame of $T(G)$.

Proof. Applying T on both sides of (4.1) gives

$$\begin{aligned} f_x(t) = T(x)(t) &= \sum_n \tilde{f}_x(t_n) T(P(x_n^*))(t) \\ &= \sum_n \tilde{f}_x(t_n) T_n(t), \end{aligned}$$

where $T_n(t) = T(P(x_n^*))(t)$. Since

$$\begin{aligned} |f_x(t) - \sum_{|n|\le N} \tilde{f}_x(t_n) T_n(t)| &= |T(x) - \sum_{|n|\le N} \tilde{f}_x(t_n) T(P(x_n^*))| \\ &= |\langle x - \sum_{|n|\le N} \tilde{f}_x(t_n) P(x_n^*), k(t)\rangle_H| \\ &\le \|x - \sum_{|n|\le N} \tilde{f}_x(t_n) P(x_n^*)\|_H \|k(t)\|_H, \end{aligned}$$

the series (4.2) converges uniformly on any subset over which $\|k(t)\|_H$ is bounded. Finally, if $\{x_n\}$ is a Riesz basis of H, then $\{x_n^*\}$ is also a Riesz basis of H so that $\{P(x_n^*)\}$ is a frame of G since G is a closed subspace of H [1, Proposition 5.3.5]. Hence $\{T_n(t) = T(P(x_n^*))(t)\}$ is a frame of $T(G)$. $\square$

We may call (4.2) an oversampling expansion of $f_x(t)$ for $x \in G$.

Now assume that finitely many sample values $\{\tilde{f}_x(t_n) \,|\, n \in X = \{n_1, n_2, \cdots, n_N\}\}$ are missing. Applying $\widetilde{T}$ on both sides of (4.1) gives

$$\tilde{f}_x(t) = \widetilde{T}(x)(t) = \sum_n \tilde{f}_x(t_n)\widetilde{T}(P(x_n^*))(t) \tag{4.3}$$

which converges not only in $\widetilde{T}(H)$ but also pointwisely in Ω since $\widetilde{T}(H)$ is an RKHS. Setting $t = t_{n_j}$ in (4.3), we have

$$\begin{aligned} \tilde{f}_x(t_{n_j}) &= \sum_n \tilde{f}_x(t_n)\widetilde{T}(P(x_n^*))(t_{n_j}) \quad \text{for } 1 \le j \le N \\ &= \sum_{k=1}^{N} \tilde{f}_x(t_{n_k})\widetilde{T}(P(x_{n_k}^*))(t_{n_j}) + \sum_{n \notin X} \tilde{f}_x(t_n)\widetilde{T}(P(x_n^*))(t_{n_j}),\ 1 \le j \le N, \end{aligned}$$

which can be rewritten in the matrix form as

$$(\mathbf{I} - \mathbf{T})\,\mathbf{f} = \mathbf{h}$$

where $\mathbf{f} = (\tilde{f}_x(t_{n_1}), \cdots, \tilde{f}_x(t_{n_N}))^T$ is the column vector consisting of missing samples, $\mathbf{h} = (h_1, \cdots, h_n)^T$, where

$$h_j = \sum_{n \notin X} \tilde{f}(t_n)\widetilde{T}(P(x_n^*))(t_{n_j})$$

and $\mathbf{T}$ is the $N \times N$ matrix with entries

$$T_{ij} = \widetilde{T}(P(x_{n_j}^*))(t_{n_i}) = \langle P(x_{n_j}^*), x_{n_i}\rangle_H = \langle P(x_{n_j}^*), P(x_{n_i})\rangle_H.$$

Note that if $\mathbf{I} - \mathbf{T}$ is invertible, the missing samples $\mathbf{f}$ can be recovered uniquely. In particular, if $\langle \mathbf{T}\mathbf{v}, \mathbf{v}\rangle \neq \|\mathbf{v}\|^2$ for any $\mathbf{v} \in \mathbb{C}^N \setminus \{0\}$, then $\mathbf{I} - \mathbf{T}$ is invertible. We have:

Theorem 4.2. *Under the same hypotheses as in Theorem 4.1, we assume further that $\{x_n\}_n$ is a Riesz basis of H such that $x_n = U(e_n)$ where $\{e_n\}_n$ is an orthonormal basis of H and U is an automorphism of H. Then any finitely many missing samples $\{\tilde{f}_x(t_{n_i}) \,|\, 1 \le i \le N\}$ in the oversampling expansion (4.2) can be uniquely recovered if $PU = UP$ and*

$$span\{e_{n_i} \,|\, 1 \le i \le N\} \cap G = \{0\}. \tag{4.4}$$

Proof. Note first that $x_n^* = (U^*)^{-1}(e_n)$ where $\{x_n^*\}_n$ is the dual of $\{x_n\}_n$. Hence we have for any $\mathbf{v} = (v_1, \cdots, v_N)^T \in \mathbb{C}^N \setminus \{0\}$,

$$\begin{aligned}
\langle \mathbf{Tv}, \mathbf{v} \rangle &= \sum_{i,j=1}^{N} \langle P(x_{n_j}^*), P(x_{n_i}) \rangle_H v_j \overline{v_i} \\
&= \langle P(U^*)^{-1}(\sum_{j=1}^{N} v_j e_{n_j}), PU(\sum_{i=1}^{N} v_i e_{n_i}) \rangle_H \\
&= \langle \sum_{j=1}^{N} v_j e_{n_j}, U^{-1}PU(\sum_{i=1}^{N} v_i e_{n_i}) \rangle_H \\
&= \langle \sum_{j=1}^{N} v_j e_{n_j}, P(\sum_{i=1}^{N} v_i e_{n_i}) \rangle_H \\
&= \|P(\sum_{j=1}^{N} v_j e_{n_j})\|_H^2 \\
&< \|\sum_{j=1}^{N} v_j e_{n_j}\|_H^2 = \sum_{j=1}^{N} |v_j|^2 = \|\mathbf{v}\|^2
\end{aligned}$$

since $\sum_{j=1}^{N} v_j e_{n_j} \notin G$ and $\{e_n\}_n$ is an orthonormal basis of H. Hence $\mathbf{I} - \mathbf{T}$ is invertible. $\square$

If moreover, $\{x_n\}_n$ is an orthonormal basis of H in Theorem 4.2, then any finitely many missing samples $\{\tilde{f}_x(t_{n_i}) \,|\, 1 \leq i \leq N\}$ can be uniquely recovered when the condition (4.4) holds.

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