ASYMMETRIC MULTI-CHANNEL SAMPLING AND ITS ALIASING ERROR

J. M. KIM, K. H. KWON, AND E. H. LEE

Abstract. We present a general asymmetric(multi-rate) multi-channel sampling formula in the Paley-Wiener space. It is well-known that a function in Paley-Wiener space, i.e., a band-limited signal can be recovered by its equidistant samples. Its frequency bound determines the minimum rate, called Nyquist rate at which the reconstruction process is stable. In the multi-channel sampling it is not necessary to distribute the sampling rates equally among the channels. In this paper, we modify the sampling series so that the sampling densities are weighed in favor of some channels at the expense of other channels. We give a general sampling formula with asymmetric sampling rate by the Riesz basis method. In case of 2-channel derivative sampling, we find condition on the ratio of sampling rates, under which the sampling formula is possible. We also give the aliasing error bound for asymmetric multi-channel sampling formula, when it is applied to non-band limited signals.
Key words : Asymmetric multi-channel sampling, aliasing error, sampling theory

1. Introduction

The sampling theory is one of most important mathematical tools used in communication engineering since it enables us to reconstruct signals from their discrete sampled data. The most fundamental result in sampling theory is the Shannon-Whittaker-Kotel'nikov(WSK) sampling theorem, which states that any band-limited signal $f(t)$ with bandwidth $\pi\omega$, i.e.,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi\omega}^{\pi\omega} F(\xi) e^{it\xi} d\xi,$$

for some $F(\xi) \in L^2(-\pi\omega, \pi\omega)$, can be reconstructed from its sampled data at uniformly distributed points $t_k = \frac{k}{\omega}$, $k \in \mathbb{Z}$.

Reconstructing a band-limited function f from samples which are taken from several transformed versions of f is called the multi-channel sampling. The multi-channel sampling method goes back to the work of Shannon[8], where the reconstruction of a band-limited signal from samples of the signal and of its derivatives was suggested. General method for multi-channel sampling was carried by Papoulis[7]. In ordinary WSK sampling formula, we need to take samples at a rate ω samples per second, which is called the Nyquist sampling rate. However, in the multi-channel sampling by Papoulis, we distribute sampling densities equally among N channels, so that we take $\frac{\omega}{N}$ samples per second from each channel.

In Papoulis' result, the sampling densities assigned in each channel are equal. But it is not necessary to distribute the sampling densities equally to each channel. In section 3, we will modify the sampling series of Papoulis so that the sampling densities are weighed in favor of some channels at the expense of others. Higgins[5] presented the derivative sampling formula which has densities $\frac{2}{3}$ and $\frac{1}{3}$ for the signal itself and its derivative respectively. We will give a general asymmetric multi-channel sampling formula which has different sampling rate in each channel using the Riesz basis method. The sampling expansion by the Riesz basis guarantees the stability of the sampling expansion.

In general sampling series of a bandlimited function converges to the original function uniformly. However, the series of the sampling functions with coefficients obtained from sampled values of the non-bandlimited function may not converge to the original function. This discrepancy between the original function and the series is called the *aliasing error*. A classical result on the aliasing error bound was presented originally by P.Weiss[9] and later proved by Brown[1]. Later, extensive researches were done and Higgins[4] presented the result for single channel sampling in his monograph and Splettstösser has obtained some related results for multidimensional non-bandlimited functions. We can also get a similar result on the asymmetric multi-channel sampling. In Section 4, we will give an upper bound for aliasing error in asymmetric sampling formula.

2. Asymmetric multi-channel sampling

For $\omega > 0$, we denote by $PW_{\pi\omega}$ the Paley-Wiener space of band-limited signals, which consists of functions of the form

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi\omega}^{\pi\omega} F(\xi) e^{it\xi} d\xi,\ t \in \mathbb{R}$$

for some $F(\xi) \in L^2[-\pi\omega, \pi\omega]$ so that $F(\xi) = \mathcal{F}(f)(xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-it\xi} dt$ is the Fourier transform of f. Let $A_j(\xi)$, $1 \le j \le N$ be N channels which are bounded and measurable on $[-\pi\omega, \pi\omega]$.

For $f(t) \in PW_{\pi\omega}$, let

$$c_j(f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi\omega}^{\pi\omega} A_j(\xi) \hat{f}(\xi) e^{it\xi} d\xi,\ 1 \le j \le N$$

be channeled signals of $f(t)$.

In this section, we give a multi-channel sampling formula of recovering the signal $f(t)$ via discrete samples taken from channeled signals $c_j(f)(t)$, $1 \le j \le N$ with arbitrary sampling rate on each channeled signal. We take samples from each channeled signal $c_j(f)(t)$ with ratio $m_1 : m_2 : \cdots : m_N$ where m_j's are positive integers with $gcd(m_1, m_2, \cdots, m_N) = 1$. In other words, we take $\frac{m_j\omega}{M}$ samples per second from each channeled signal $c_j(f)(t)$ for $j = 1, 2, \cdots, N$ where $M = \sum_{j=1}^{N} m_j$, i.e, we take samples

$$c_j(f)\Big(\frac{nM}{m_j\omega}\Big),\ n \in \mathbb{Z} \text{ and } 1 \le j \le N$$

or equivalently

$$c_j(f)\Big(\frac{M}{m_j\omega}(m_j n + k)\Big),\ n \in \mathbb{Z}, 0 \le k \le m_j - 1 \text{ and } 1 \le j \le N.$$

Then, in total, we take $\frac{m_1\omega}{M} + \frac{m_2\omega}{M} + \cdots + \frac{m_N\omega}{M} = \omega$ samples per second, which is the Nyquist rate for signals in $PW_{\pi\omega}$.

We divide the interval $[-\pi\omega, \pi\omega]$ into M subintervals of length $d = \frac{2\pi\omega}{M}$ as

$$I_1 = [-\pi\omega, -\pi\omega + d], I_2 = [-\pi\omega + d, -\pi\omega + 2d], \cdots, I_M = [-\pi\omega + (M-1)d, -\pi\omega + Md]$$

and set $I = I_1$. Then

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi\omega}^{\pi\omega} \hat{f}(\xi) e^{it\xi} d\xi = \frac{1}{\sqrt{2\pi}} \sum_{l=1}^{M} \int_I \hat{f}(\xi + (l-1)d) e^{it(\xi+(l-1)d)} d\xi, \tag{2.1}$$

where $\hat{f}(\xi) = \mathcal{F}(f)(\xi)$.

Define an operator D from $L^2[-\pi\omega, \pi\omega]$ into $L^2(I)^M$ by $D(\phi)(\xi) = (\phi_1(\xi), \cdots, \phi_M(\xi))^T$ for $\phi \in L^2[-\pi\omega, \pi\omega]$, where $\phi_l(\xi) = \phi(\xi + (l-1)d)$, $l = 1, 2, \cdots, M$. Then D is a unitary operator from $L^2[-\pi\omega, \pi\omega]$ onto $L^2(I)^M$ with the inverse $D^{-1}((\phi_j(\xi))_{j=1}^M) = \phi(\xi)$, where $\phi(\xi) = \phi_l(\xi - (l-1)d)$ on I_l, $1 \le l \le M$. The we can rewrite (2.1) as

$$f(t) = \frac{1}{\sqrt{2\pi}} < D(\hat{f})(\xi), D(e^{-it\xi}) >_{L^2(I)^M} . \tag{2.2}$$

Similarly, we have

$$c_j(f)(t) = \frac{1}{\sqrt{2\pi}} \sum_{l=1}^{M} \int_I A_j(\xi + (l-1)d) \hat{f}(\xi + (l-1)d) e^{it(\xi+(l-1)d)} d\xi$$

so that

$$\begin{aligned} c_j(f)\Big(\frac{M}{m_j\omega}(m_j n + k)\Big) &= \frac{1}{\sqrt{2\pi}} \sum_{l=1}^{M} \int_I A_j(\xi + (l-1)d) \hat{f}(\xi + (l-1)d) e^{i\frac{M}{m_j\omega}(m_j n+k)(\xi+(l-1)d)} d\xi \\ &= \frac{1}{\sqrt{2\pi}} \sum_{l=1}^{M} \int_I A_j(\xi + (l-1)d) \hat{f}(\xi + (l-1)d) e^{i\frac{Mk}{m_j\omega}[\xi+(l-1)d]} e^{i\frac{nM}{\omega}\xi} d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_I G_{j,k}(\xi) e^{i\frac{nM}{\omega}\xi} d\xi \end{aligned}$$

where

$$G_{j,k}(\xi) = \sum_{l=1}^{M} A_j(\xi + (l-1)d) \hat{f}(\xi + (l-1)d) e^{i\frac{Mk}{m_j\omega}(\xi+(l-1)d)}$$

for $1 \le j \le N$, $0 \le k \le m_j - 1$, and $\xi \in I$.

Let $\mathbf{G}(\xi) = (G_{1,0}(\xi), G_{1,1}(\xi), \cdots, G_{1,m_1-1}(\xi), \cdots, G_{N,0}(\xi), G_{N,1}(\xi), \cdots, G_{N,m_N-1}(\xi))^T$. Then

$$\mathbf{G}(\xi) = \mathbf{A}(\xi) D(\hat{f})(\xi), \tag{2.3}$$

where $A(\xi) := [B_1(\xi), \cdots, B_N(\xi)]^T$ is an $M \times M$ matrix where

$$B_j(\xi) := [A_j(\xi + (l-1)d) e^{i\frac{Mk}{m_j\omega}(\xi+(l-1)d)}]_{k=0,\ l=1}^{m_j-1\ M}$$

is an $m_j \times M$ matrix. We call $A(\xi)$ the transfer matrix. We are now ready to state and prove our main result.

Theorem 2.1. *With the notations as above, assume that there is a constant $\alpha > 0$ such that*

$$|\det A(\xi)| \ge \alpha,\ \ \xi \in I. \tag{2.4}$$

Then for any band-limited signal $f(t)$ in $PW_{\pi\omega}$, we have the following multi-channel sampling expansion formula

$$f(t) = \frac{M}{\sqrt{2\pi}\omega} \sum_{j=1}^{N} \sum_{k=0}^{m_j-1} \sum_n c_j(f) \left(\frac{M}{m_j\omega}(m_j n + k)\right) y_{j,k} \left(t - \frac{nM}{\omega}\right),\ \ t \in \mathbb{R} \tag{2.5}$$

where $y_{j,k}(t) = \frac{1}{\sqrt{2\pi}} \int_I Y_{j,k}(\xi, t) e^{it\xi} d\xi$ *and* $Y_{j,k}(\xi, t)$ *is the* $\lambda(j,k)$*-th component of the vector*

$$Y(\xi, t) = [A(\xi)^{-1}]^T E(t).$$

Here, $\lambda_{(j,k)} := m_1 + m_2 + \cdots + m_{j-1} + k + 1$ $(m_0 = 0)$ *and* $E(t) := (1, e^{idt}, \cdots, e^{i(M-1)dt})^T$ *is the carrier vector.*

Proof. By (2.3), we can rewrite (2.2) as

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{2\pi}} \langle D(\hat{f})(\xi), D(e^{-it\xi}) \rangle_{L^2(I)^M} = \frac{1}{\sqrt{2\pi}} \langle \mathbf{A}(\xi)^{-1} \mathbf{G}(\xi), D(e^{-it\xi}) \rangle_{L^2(I)^M} \\ &= \frac{1}{\sqrt{2\pi}} \langle \mathbf{G}(\xi), [\mathbf{A}(\xi)^{-1}]^* D(e^{-it\xi}) \rangle_{L^2(I)^M} \\ &= \frac{1}{\sqrt{2\pi}} \langle \mathbf{G}(\xi), \overline{[\mathbf{A}(\xi)^{-1}]^T D(e^{it\xi})} \rangle_{L^2(I)^M} \\ &= \frac{1}{\sqrt{2\pi}} \langle \mathbf{G}(\xi), \overline{Y(\xi, t) e^{it\xi}} \rangle_{L^2(I)^M} \end{aligned}$$

since $D(e^{it\xi}) = (e^{it\xi}, e^{it(\xi+d)}, \cdots, e^{it(\xi+(M-1)d)})^T = E(t)e^{it\xi}$.
By setting

$$Y(\xi, t) = (Y_{1,0}(\xi, t), \cdots, Y_{1,m_1-1}(\xi, t), \cdots, Y_{N,0}(\xi, t), \cdots, Y_{N,m_N-1}(\xi, t))^T,$$

we obtain

$$f(t) = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{N} \sum_{k=0}^{m_j-1} \langle G_{j,k}(\xi), \overline{Y_{j,k}(\xi, t) e^{it\xi}} \rangle_{L^2(I)}, \ t \in \mathbb{R}. \tag{2.6}$$

Now, by expanding $G_{j,k}(\xi)$ and $\overline{Y_{j,k}(\xi, t) e^{it\xi}}$ via an orthonormal basis $\{\phi_n(\xi)\}_n = \{\frac{1}{\sqrt{d}} e^{-i\frac{nM}{\omega}\xi}\}_n$ of $L^2(I)$, we obtain

$$\begin{aligned} G_{j,k}(\xi) &= \sum_n \langle G_{j,k}(\xi), \frac{1}{\sqrt{d}} e^{-i\frac{nM}{\omega}\xi} \rangle_{L^2(I)} \phi_n(\xi) \\ &= \frac{1}{\sqrt{d}} \sum_n \Big(\int_I G_{i,k}(\xi) e^{i\frac{nM}{\omega}\xi} d\xi \Big) \phi_n(\xi) \\ &= \sqrt{\frac{M}{\omega}} \sum_n c_j(f) \Big(\frac{M}{m_j \omega} (m_j n + k) \Big) \phi_n(\xi), \end{aligned} \tag{2.7}$$

and

$$
\begin{aligned}
\overline{Y_{j,k}(\xi,t)e^{it\xi}} &= \sum_n \Big\langle \overline{Y_{j,k}(\xi,t)e^{it\xi}}, \frac{1}{\sqrt{d}} e^{-i\frac{nM}{\omega}\xi} \Big\rangle_{L^2(I)} \phi_n(\xi) \\
&= \frac{1}{\sqrt{d}} \sum_n \Big(\int_I \overline{Y_{j,k}(\xi,t)e^{it\xi}} e^{i\frac{nM}{\omega}\xi} d\xi \Big) \phi_n(\xi) \\
&= \sqrt{\frac{M}{\omega}} \sum_n \frac{1}{\sqrt{2\pi}} \Big(\int_I \overline{Y_{j,k}(\xi,t) e^{i(t-\frac{nM}{\omega})\xi}} \Big) \phi_n(\xi) \qquad (2.8) \\
&= \sqrt{\frac{M}{\omega}} \sum_n \frac{1}{\sqrt{2\pi}} \Big(\int_I \overline{Y_{j,k}\Big(\xi, t-\frac{nM}{\omega}\Big) e^{i(t-\frac{nM}{\omega})\xi}} \Big) \phi_n(\xi) \\
&= \sqrt{\frac{M}{\omega}} \sum_n \overline{y_{j,k}\Big(t-\frac{nM}{\omega}\Big)} \phi_n(\xi)
\end{aligned}
$$

since

$$
\mathbf{Y}(\xi, t-\tfrac{nM}{\omega}) = [\mathbf{A}(\xi)^{-1}]^T E(t-\tfrac{nM}{\omega}) = [\mathbf{A}(\xi)^{-1}]^T E(t) = \mathbf{Y}(\xi,t).
$$

Hence, by Parseval's identity, we have (2.5) from (2.6), (2.7), and (2.8). □

The pointwise convergence of the sampling series (2.5) in Theorem 2.1 can be strengthened as:

Theorem 2.2. *The set of functions* $\{y_{j,k}(t-\frac{nM}{\omega}) | n \in \mathbb{Z}, 1 \le j \le N, 0 \le k \le m_j - 1\}$ *constitutes a Riesz basis of* $PW_{\pi\omega}$. *Hence, the sampling series (2.5) converges absolutely and uniformly on* $\mathbb{R}$. *Moreover, the dual* $\{y^*_{j,k,n}(t)\}$ *of* $\{y_{j,k}(t-\frac{nM}{\omega})\}$ *is*

$$
y^*_{j,k,n}(t) = \frac{1}{d} y^*_j \left(t - \frac{Mk}{m_j\omega} - \frac{nM}{\omega} \right), \qquad (2.9)
$$

where

$$
y^*_j(t) = \mathcal{F}^{-1}(\overline{A_j(\xi)}) = \frac{1}{\sqrt{2\pi}} \int_{-\pi\omega}^{\pi\omega} \overline{A_j(\xi)} e^{it\xi} d\xi.
$$

Proof. We first show that $\{y_{j,k}(t-\frac{nM}{\omega}) | n \in \mathbb{Z}, 1 \le j \le N, 0 \le k \le m_j - 1\}$ is a Riesz basis of $PW_{\pi\omega}$. With the notations as in Theorem 2.1 and $\mathbf{A}(\xi)^{-1} = [q_{l,\lambda}(\xi)]_{l,\lambda=1}^M$, we have

$$
Y_{j,k}(\xi,t) = \sum_{l=1}^{M} q_{l,\lambda(j,k)}(\xi) e^{i(l-1)dt}
$$

where $\lambda(j,k) = m_1 + m_2 + \cdots + m_{j-1} + k + 1$ $(m_0 = 1)$. Hence

$$
\begin{aligned}
y_{j,k}(t) =& \frac{1}{\sqrt{2\pi}} \int_I Y_{j,k}(\xi, t) e^{it\xi} d\xi \\
=& \frac{1}{\sqrt{2\pi}} \int_I \sum_{l=1}^{M} q_{l,\lambda(j,k)}(\xi) e^{i(l-1)td} e^{it\xi} d\xi = \frac{1}{\sqrt{2\pi}} \langle \mathbf{Q}_{\lambda(j,k)}(\xi), \overline{E(t)e^{it\xi}} \rangle_{L^2(I)^M} \\
=& \frac{1}{\sqrt{2\pi}} \langle D^{-1}(\mathbf{Q}_{\lambda(j,k)}), \overline{D^{-1}(E(t)e^{it\xi})} \rangle_{L^2[-\pi\omega,\pi\omega]} \\
=& \frac{1}{\sqrt{2\pi}} \langle D^{-1}(\mathbf{Q}_{\lambda(j,k)}), \overline{e^{it\xi}} \rangle_{L^2[-\pi\omega,\pi\omega]} \\
=& \frac{1}{\sqrt{2\pi}} \int_{-\pi\omega}^{\pi\omega} D^{-1}(\mathbf{Q}_{\lambda(j,k)}) e^{it\xi} d\xi = \mathcal{F}^{-1}(D^{-1}(\mathbf{Q}_{\lambda(j,k)}))(t),
\end{aligned}
$$

where $\mathbf{Q}_{\lambda(j,k)}(\xi)$ is the $\lambda(j,k)$-th column of $A(\xi)^{-1}$. Therefore,

$$
\begin{aligned}
\mathcal{F}\Big(y_{j,k}\Big(t - \frac{nM}{\omega}\Big)\Big)(\xi) &= e^{-i\frac{nM}{\omega}\xi} \mathcal{F}(y_{j,k}(t))(\xi) \\
&= \sqrt{d}\phi_n(\xi) D^{-1}(\mathbf{Q}_{\lambda(j,k)})(\xi), \ \xi \in [-\pi\omega, \pi\omega].
\end{aligned}
$$

Hence

$$
\begin{aligned}
D\Big[\mathcal{F}\Big(y_{j,k}\Big(t - \frac{nM}{\omega}\Big)\Big)\Big](\xi) =& \sqrt{d} D(\phi_n D^{-1}(\mathbf{Q}_{\lambda(j,k)}))(\xi)) \\
=& \sqrt{d}\phi_n(\xi) \mathbf{Q}_{\lambda(j,k)}(\xi), \ \xi \in I
\end{aligned}
$$

since $\phi_n(\xi)$ is periodic with period d. Hence

$$
D\Big[\mathcal{F}\Big(y_{j,k}\Big(t - \frac{mn}{\omega}\Big)\Big)\Big](\xi) = \sqrt{d}\mathbf{A}(\xi)^{-1} e_{\lambda(j,k)} \phi_n(\xi)
$$

where $e_{\lambda(j,k)} = (0, 0, \cdots, 0, 1, 0, \cdots, 0)^T$ is the $\lambda(j,k)$-th unit vector in $\mathbb{C}^M$ where 1 appears only in the $\lambda(j,k)$-th position. Hence

$$
y_{j,k}\Big(t - \frac{Mn}{\omega}\Big) = \sqrt{d}\mathcal{F}^{-1} D^{-1} \mathbf{A}(\xi)^{-1}(e_{\lambda(j,k)}\phi_n(\xi)).
$$

Therefore $\{y_{j,k}(t - \frac{nM}{\omega})\}$ is a Riesz basis of $PW_{\pi\omega}$ since $\{e_\lambda \psi_n(\xi) | n \in \mathbb{Z}$ and $1 \leq \lambda \leq M\}$ is an orthonormal basis of $L^2(I)^M$, $\mathbf{A}(\xi)$ is an isomorphism from $L^2(I)^M$ onto $L^2(I)^M$ and D^{-1} and $\mathcal{F}^{-1}$ are unitary. This completes the first part of Theorem 2.2.

For the second part of Theorem 2.2, let $y_{j,k,n}(t) = y_{j,k}(t - \frac{nM}{\omega})$ and $\{y^*_{j,k,n}(t)\}$ be the dual of $\{y_{j,k,n}(t)\}$. From the biorthogonality, we have

$$
\begin{aligned}
\delta_{mn} =& \langle y_{j,k,m}(t), y^*_{j,k,n}(t) \rangle_{L^2(\mathbb{R})} \\
=& \Big\langle \sqrt{d}\mathcal{F}^{-1} D^{-1} \mathbf{A}(\xi)^{-1}(e_{\lambda(j,k)}\phi_m(\xi)), y^*_{j,k,n}(t) \Big\rangle_{L^2(\mathbb{R})} \\
=& \Big\langle e_{\lambda(j,k)}\phi_m(\xi), \sqrt{d}[\mathbf{A}(\xi)^{-1}]^* D\mathcal{F}(y^*_{j,k,n}(t)) \Big\rangle_{L^2(I)^M} \text{ for } m \text{ and } n \in \mathbb{Z}.
\end{aligned}
$$

Thus we obtain

$$
\sqrt{d}[\mathbf{A}(\xi)^{-1}]^* D\mathcal{F}(\tilde{y}^*_{j,k,n}(t)) \quad = \quad e_{\lambda(j,k)}\phi_n(\xi)
$$

so that

$$
y^*_{j,k,n}(t) \quad = \quad \frac{1}{\sqrt{d}} \mathcal{F}^{-1} D^{-1} \mathbf{A}(\xi)^*(e_{\lambda(j,k)}\phi_n(\xi)).
$$

Since the $\lambda(j,k)$-th row of $\mathbf{A}(\xi)$ is the $(k+1)$-th row of $\mathbf{B}_j(\xi)$, which is

$$[A_j(\xi+(l-1)d)e^{i\frac{Mk}{m_j\omega}(\xi+(l-1)d)}]_{l=1}^{M}$$

and $D(\phi_n)(\xi) = (\phi_n(\xi), \cdots, \phi_n(\xi))^T$,

$$D^{-1}\mathbf{A}(\xi)^*(e_{\lambda(j,k)}\phi_n(\xi)) = \overline{A_j(\xi)e^{i\frac{Mk}{m_j\omega}\xi}}\phi_n(\xi).$$

Hence

$$\begin{aligned} y^*_{j,k,n}(t) =& \frac{1}{\sqrt{d}}\mathcal{F}^{-1}(\overline{A_j(\xi)e^{i\frac{Mk}{m_j\omega}\xi}}\phi_n(\xi))(t) \\ =& \frac{1}{d\sqrt{2\pi}}\int_{-\pi\omega}^{\pi\omega}\overline{A_j(\xi)}e^{i(t-\frac{Mk}{m_j\omega}-\frac{nM}{\omega})\xi}d\xi, \end{aligned}$$

from which (2.9) follows. Then, it is easy to show that the sampling series (2.5) is also a Riesz basis expansion of $f(t)$ with respect to the Riesz basis $\{y_{j,k}(t-\frac{nM}{\omega})\}$ of $PW_{\pi\omega}$ so that the series (2.5) converges also in the sense of $L^2(\mathbb{R})$. Then finally since the Paley-Wiener space $PW_{\pi\omega}$ is a reproducing kernel Hilbert space with the reproducing kernel

$$k(s,t) = \omega\frac{sin\pi\omega(s-t)}{\pi\omega(s-t)}$$

and $\|k(\cdot,t)\|_{L^2(\mathbb{R})} = \sqrt{\omega}$ is bounded on $\mathbb{R}$, the series (2.5) converges also absolutely and uniformly on $\mathbb{R}$ (See [4, p. 59]). □

When $m_1 = m_2 = \cdots = m_N = 1$ so that $M = N$ and the transfer matrix $A(\xi) = [A_j(\xi+(k-1)d)]_{j,k=1}^N$, we obtain the Papoulis multi-channel sampling formula([7]) with uniform sampling rates on all channels as a special case of Theorem 2.1. However, A. Papoulis[7] did not mention the determinant condition (2.4) explicitly.

Example 2.1. Take $A_1(\xi) = 1$, $A_2(\xi) = -isgn\xi$ and $m_1 = 2$, $m_2 = 1$ so that $c_1(f)(t) = f(t)$ and $c_2(f)(t) = \tilde{f}(t)$ is the Hilbert transform of $f(t)$. Then

$$A(\xi) = \begin{bmatrix} 1 & 1 & 1 \\ e^{i\frac{3}{2\omega}\xi} & -e^{i\frac{3}{2\omega}\xi} & e^{i\frac{3}{2\omega}\xi} \\ i & -isgn(\xi+\frac{2}{3}\pi\omega) & -i \end{bmatrix}$$

and

$$A(\xi)^{-1} = \begin{bmatrix} \frac{1}{4}(1+sgn(\xi+\frac{2}{3}\pi\omega)) & \frac{1}{4}e^{-i\frac{3}{2\omega}\xi}(1-sgn(\xi+\frac{2}{3}\pi\omega)) & -\frac{i}{4} \\ \frac{1}{2} & -\frac{1}{2}e^{-i\frac{3}{2\omega}\xi} & 0 \\ \frac{1}{4}(1-sgn(\xi+\frac{2}{3}\pi\omega)) & \frac{1}{4}e^{-i\frac{3}{2\omega}\xi}(1+sgn(\xi+\frac{2}{3}\pi\omega)) & \frac{i}{4} \end{bmatrix}.$$

Now we have for any $f(t) \in PW_{\pi\omega}$,

$$f(t) = \frac{3}{\sqrt{2\pi}\omega}\sum_n\Big[f\Big(\frac{3n}{\omega}\Big)y_{1,0}\Big(t-\frac{3n}{\omega}\Big) + f\Big(\frac{3n}{\omega}+\frac{3}{2\omega}\Big)y_{1,1}\Big(t-\frac{3n}{\omega}\Big) + \tilde{f}\Big(\frac{3n}{\omega}\Big)y_{2,0}\Big(t-\frac{3n}{\omega}\Big)\Big].$$

where $\text{sinc}t = \frac{\sin \pi t}{\pi t}$ and

$$\begin{aligned} y_{1,0}(t) =& \frac{\omega\sqrt{2\pi}}{3}\text{sinc}\Big(\frac{2\omega}{3}t\Big), \\ y_{1,1}(t) =& \frac{\omega\sqrt{2\pi}}{6}\Big[\text{sinc}\Big(\frac{\omega}{3}(t-\frac{3}{2\omega})\Big)+\text{sinc}\Big(\frac{1}{6}\omega(t-\frac{3}{2\omega})\Big)\cos\Big(\frac{5\pi\omega}{6}(t-\frac{3}{2\omega})\Big)\Big], \\ y_{2,0}(t) =& -\frac{\omega\sqrt{2\pi}}{6}\text{sinc}\Big(\frac{\omega}{3}t\Big)\sin\Big(\frac{2\pi\omega}{3}t\Big). \end{aligned}$$

In general, the determinant condition (2.4) on the transfer matrix $A(\xi)$ is not easy to check. Note first that $|\det A(\xi)| = |\det \tilde{A}(\xi)|$ where $\tilde{\mathbf{A}}(\xi) = [\tilde{\mathbf{B}}_1(\xi)\cdots\tilde{\mathbf{B}}_N(\xi)]^T$ is an $M\times M$ matrix and

$$\tilde{\mathbf{B}}_j(\xi) = \Big[A_j(\xi+(l-1)d)e^{i\frac{k(l-1)2\pi}{m_j}}\Big]_{k=0,l=1}^{m_j-1,M}$$

is an $m_j\times M$ matrix. Let $S = diag[S_1, S_2, \cdots, S_N]$ be the block matrix, where $S_i (1\le i\le N)$ is an $m_i\times m_i$ constant matrix whose (j,k) entry is

$$[S_i]_{jk} = \frac{1}{m_i}e^{-i\frac{(j-1)(k-1)2\pi}{m_i}}.$$

If we let $T(\xi) = S\tilde{A}(\xi)$, then the (j,k) entry $T_{j,k}(\xi)$ of $T(\xi)$ is obtained as follows.
For $1\le j\le m_1$,

$$\begin{aligned} T_{jk}(\xi) &= \sum_{l=1}^{m_1}\frac{1}{m_1}e^{-i\frac{(j-1)(l-1)2\pi}{m_1}}e^{i\frac{(l-1)(k-1)2\pi}{m_1}}A_1(\xi+(k-1)d) \\ &= \sum_{l=1}^{m_1}\frac{1}{m_1}\Big(e^{i\frac{(k-j)2\pi}{m_1}}\Big)^{l-1}A_1(\xi+(k-1)d). \end{aligned}$$

Since $e^{i\frac{(k-j)2\pi}{m_1}} = 1$ if and only if $j = k \pmod{m_1}$,

$$T_{jk}(\xi) = A_1(\xi+(k-1)d)\delta_{0,(k-j)\ mod\ m_1} \text{ for } 1\le j\le m_1.$$

Similarly, we have

$$T_{jk}(\xi) = A_n(\xi+(k-1)d)\delta_{0,(k-j+\sum_{i=1}^{n-1}m_i)\ mod\ m_n} \quad \text{for } \sum_{i=1}^{n-1}m_i+1\le j\le\sum_{i=1}^{n}m_i.$$

where $1\le n\le N$. Since S is a non-singular constant matrix and $|\det A(\xi)| = |\det\tilde{A}(\xi)|$, the condition (2.4) is equivalent to the condition

$$|\det T(\xi)| \ge \tilde{\alpha} := \alpha|\det S|,\ \xi\in I. \tag{2.10}$$

If all channels $A_j(\xi)$ for $1\le j\le N$ are continuous on $[-\pi\omega,\pi\omega]$. then the condition (2.10) is also equivalent to $\det T(\xi)\ne 0$ on I since in this case $\det T(\xi)$ is also continuous on I. Furthermore, in case of 2 channels, the condition (2.10) can be simplified as:

Proposition 2.3. *Assume $N = 2$ and $m_1\ge m_2$. Let*

$$h(\xi) = \Big[\prod_{j=m_2}^{m_1-1}A_1(\xi+jd)\Big]\times\begin{vmatrix}\prod_{j=0}^{m_2-1}A_1(\xi+jd) & \prod_{j=m_1}^{M-1}A_1(\xi+jd)\\ \prod_{j=0}^{m_2-1}A_2(\xi+jd) & \prod_{j=m_1}^{M-1}A_2(\xi+jd)\end{vmatrix}$$

($\prod_1^0 := 1$ *when* $m_1 = m_2 = 1$). *Then the condition (2.10) is equivalent to the condition that there is a constant* $\tilde{\alpha} > 0$ *such that* $|h(\xi)| \geq \tilde{\alpha}$, $\xi \in I$.

Proof. If $m_1 = m_2 = 1$, then it is trivial. Hence, assume that $m_1 > m_2$. It is enough to show that $|h(\xi)| = |\det T(\xi)|$. Since $T_{jk}(\xi) = A_1(\xi + (j-1)d)\delta_{jk}$ for $m_2 + 1 \leq j \leq m_1$, we have by taking Laplace expansions

$$\det T(\xi) = \prod_{j=m_2}^{m_1-1} A_1(\xi + jd) \det \tilde{T}(\xi)$$

where $\tilde{T}(\xi) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a block matrix with

$$\begin{aligned} A &= diag[A_1(\xi), A_1(\xi + d), \cdots, A_1(\xi + (m_2 - 1)d)] \\ B &= diag[A_1(\xi + m_1 d), A_1(\xi + (m_1 + 1)d), \cdots, A_1(\xi + (M - 1)d)] \\ C &= diag[A_2(\xi), A_2(\xi + d), \cdots, A_2(\xi + (m_2 - 1)d)] \\ D &= [d_{jk}]_{j,k=1}^{m_2} \end{aligned}$$

and

$$d_{j,k} = A_2(\xi + (k + m_1 - 1)d)\delta_{0,(k-j+m_1)\ mod\ m_2}.$$

If $m_2 = 1$, $\tilde{T}(\xi)$ is a 2×2 matrix and

$$\det T(\xi) = \Big[\prod_{j=2}^{m_1} A_1(\xi + (j-1)d)\Big] \times \Big[A_1(\xi)A_2(\xi + (M-1)d) - A_2(\xi)A_1(\xi + (M-1)d)\Big]$$

so that $|h(\xi)| = |\det T(\xi)|$.
If $m_2 \geq 2$, we let $r := m_1\ mod\ m_2$, then $A_2(\xi + m_1 d)$ occurs in $(r+1)$-th row and $gcd(m_2, r) = 1$. Hence $|\det \tilde{T}(\xi)| = |h(\xi)|$ follows from the following lemma. ∎

Lemma 2.4. *Let* $N > r \geq 1$ *be integers with* $gcd(N, r) = 1$. *Let* $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be a block matrix where*

$$A = diag[a_1, a_2, \cdots, a_N],\ B = diag[b_1, b_2, \cdots, b_N],\ C = diag[c_1, c_2, \cdots, c_N]$$

and $D = \begin{bmatrix} 0 & D_1 \\ D_2 & 0 \end{bmatrix}$ *is a block matrix where*

$$D_1 = diag[d_{N-r+1}, d_{N-r+2}, \cdots, d_N] \text{ and } D_2 = diag[d_1, d_2, \cdots, d_{N-r}].$$

Then

$$\det X = (-1)^{N-1}\Big[\prod_{k=1}^{N} a_k d_k - \prod_{k=1}^{N} b_k c_k\Big].$$

Proof. Let $x_{i,j}$ be the (i,j) entry of the matrix X. By the definition of determinant,

$$\det X = \sum_{\sigma} \epsilon(\sigma) x_{1,\sigma(1)} x_{2,\sigma(2)} \cdots x_{2N,\sigma(2N)}$$

where the sum is taken over all permutations σ of $\{1, 2, \cdots, 2N\}$ and $\epsilon(\sigma)$ is the sign of the permutation σ. Note that $x_{1,\sigma(1)}x_{2,\sigma(2)}\cdots x_{2N,\sigma(2N)}$ is trivial unless

$$\begin{aligned}\sigma(k) &= \begin{cases} k & \text{or } k+N \quad \text{for } 1 \le k \le N \\ k-N & \text{or } k+N-r \quad \text{for } N+1 \le k \le N+r \\ k-N & \text{or } k-r \quad \text{for } N+r+1 \le k \le 2N \end{cases} \\ &= \begin{cases} k & \text{or } k+N \quad \text{for } 1 \le k \le N \\ k-N & \text{or } N+\tau(k-r) \quad \text{for } N+1 \le k \le 2N, \end{cases} \end{aligned} \tag{2.11}$$

where $\tau(n)$ is an integer such that $1 \le \tau(n) \le N$ and $\tau(n) = n \ mod \ N$.
We claim that if $\sigma(1) = 1$ and σ satisfies (2.11), then

$$\sigma(k) = \begin{cases} k & \text{for } \ 1 \le k \le N, \\ N+\tau(k-r) & \text{for } \ N+1 \le k \le 2N. \end{cases} \tag{2.12}$$

Now, (2.12) is equivalent to

$$\sigma(\tau(1-kr)) = \tau(1-kr) \text{ and } \sigma(N+\tau(1-kr)) = N+\tau(1-(k+1)r) \tag{2.13}$$

for $0 \le k \le N-1$ since $\{\tau(1-kr) | 0 \le k \le N-1\} = \{1, 2, \cdots, N\}$ and $\tau(N+\tau(1-kr)-r) = \tau(1-(k+1)r)$. We prove (2.13) by induction on $k = 0, 1, \cdots, N-1$.
When $\sigma(1) = 1$, $\sigma(N+1) = N+\tau(1-r)$ since $\sigma(N+1) \ne \sigma(1) = 1$. Hence, (2.13) holds for $k = 0$. Assume that (2.13) holds for $0 \le k \le n < N-1$. By (2.11),

$$\sigma(\tau(1-(n+1)r)) = \tau(1-(n+1)r) \text{ or } N+\tau(1-(n+1)r).$$

Since $N+\tau(1-nr) \ne \tau(1-(n+1)r)$ and $\sigma(N+\tau(1-nr)) = N+\tau(1-(n+1)r)$, $\sigma(\tau(1-(n+1)r)) = \tau(1-(n+1)r)$. Again, by (2.11)

$$\sigma(N+\tau(1-(n+1)r)) = \tau(1-(n+1)r) \text{ or } N+\tau(1-(n+2)r).$$

Since $\tau(1-(n+1)r) \ne N+\tau(1-(n+1)r)$ and $\sigma(\tau(1-(n+1)r)) = 1-(n+1)r$, $\sigma(N+\tau(1-(n+1)r)) = N+\tau(1-(n+2)r)$, that is, (2.13) is true also for $n+1$. Hence the claim is proved. Similarly, if $\sigma(1) = N+1$ and σ satisfies (2.11),

$$\sigma(k) = \begin{cases} k+N & \text{for } \ 1 \le k \le N \\ k-N & \text{for } \ N+1 \le k \le 2N. \end{cases}$$

Let

$$\sigma_1 = (N+1, N+\tau(1-r), \cdots, N+\tau(1-(N-1)r))$$

and

$$\sigma_2 = (1, N+1)(2, N+2)\cdots(N, 2N)$$

be two permutations of $\{1, 2, \cdots, 2N\}$. Then $\epsilon(\sigma_1) = (-1)^{N-1}$ and $\epsilon(\sigma_2) = (-1)^N$ so that

$$\begin{aligned}\det X &= \epsilon(\sigma_1)x_{1,1}x_{2,2}\cdots x_{N,N}x_{N+1,2N-r+1}\cdots x_{N+\alpha,2N}x_{N+r+1,N+1}\cdots x_{2N,2N-r} \\ &\quad + \epsilon(\sigma_2)x_{1,N+1}x_{2,N+2}\cdots x_{N,2N}x_{N+1,1}\cdots x_{2N,N} \\ &= (-1)^{N-1}a_1\cdots a_N d_1\cdots d_N + (-1)^N b_1\cdots b_N c_1\cdots c_N.\end{aligned}$$

Hence the lemma is proved. □

Example 2.2. (Derivative sampling)
Take $A_1(\xi) = 1$ and $A_2(\xi) = i\xi$ so that $c_1(f)(t) = f(t)$ and $c_2(f)(t) = f'(t)$. We then claim that the determinant condition (2.4) holds if and only if $m_1 \ge m_2$ and m_2 is a positive odd integer. In fact, by Proposition 2.3 and the continuity of $h(\xi)$ on I, we only need to show

that $h(\xi) \neq 0$, $\xi \in I$ if and only if $m_1 \geq m_2$ and m_2 is a positive integer. First assume that $m_1 < m_2$. Then

$$\begin{aligned} h(\xi) &= \prod_{j=m_1}^{m_2-1} A_2(\xi + jd)\Big(\prod_{j=0}^{m_2-1} A_2(\xi + jd) - \prod_{j=m_1}^{M-1} A_2(\xi + jd)\Big) \\ &= i^{m_2} \prod_{m_1}^{m_2-1} (\xi + jd)\Big(\prod_{j=0}^{m_2-1} (\xi + jd) - \prod_{j=m_1}^{M-1} (\xi + jd)\Big). \end{aligned}$$

Since $-\pi\omega + m_1 d = \frac{-(M-2m_1)\pi\omega}{M} < 0$ and $-\pi\omega + m_2 d = \frac{(2m_2-M)\pi\omega}{M} > 0$, $0 \in [-\pi\omega + jd, -\pi\omega + (j+1)d]$ for some $j = m_1, m_1+1, \cdots, m_2-1$ so that $\prod_{j=m_1}^{m_2-1} (\xi + jd) = 0$ for some ξ in I. Hence $h(\xi) = 0$ for some ξ in I.

When $m_1 = m_2 = 1$, $h(\xi) = i\pi\omega \neq 0$.

Finally assume $m_1 > m_2$. Then

$$h(\xi) = i^{m_2}\Big(\prod_{j=m_1}^{M-1} (\xi + jd) - \prod_{j=0}^{m_2-1} (\xi + jd)\Big).$$

Since $-\pi\omega + m_1 d = \frac{(2m_1-M)\pi\omega}{M} > 0$ and $-\pi\omega + m_2 d = \frac{(2m_2-M)\pi\omega}{M} < 0$, $\prod_{j=m_1}^{M-1} (\xi + jd) > 0$ on I and $sgn \prod_{j=0}^{m_2-1} (\xi + jd) = (-1)^{m_2}$ on I. Hence if m_2 is odd, then $|h(\xi)| > 0$ on I.

For m_2 even,

$$\begin{aligned} h\Big(-\pi\omega + \frac{d}{2}\Big) &= i^{m_2}\Big(\prod_{j=m_1}^{M-1} (-\pi\omega + \frac{d}{2} + jd) - \prod_{j=0}^{m_2-1} (-\pi\omega + \frac{d}{2} + jd)\Big) \\ &= i^{m_2}\Big(\prod_{j=0}^{m_2-1} (-\pi\omega + \frac{d}{2} + (M-1-j)d) - \prod_{j=0}^{m_2-1} (-\pi\omega + \frac{d}{2} + jd)\Big) \\ &= i^{m_2}\Big(\prod_{j=0}^{m_2-1} (\pi\omega - \frac{d}{2} - jd) - \prod_{j=0}^{m_2-1} (-\pi\omega + \frac{d}{2} + jd)\Big) = 0. \end{aligned}$$

3. Aliasing error of asymmetric multi-channel sampling

In this section, we give an upper bound for the aliasing error, which occurs when we apply the asymmetric sampling formula (3.4) to a non-bandlimited signal. Let $F := \{f(t) \in L^2(\mathbb{R}) : \hat{f}(\xi) \in L^1(\mathbb{R})\}$. Then any signal $f(t)$ in F is, in fact, continuous on $\mathbb{R}$. Let $A_j(\xi)$, $1 \leq j \leq N$, be bounded measurable functions on $\mathbb{R}$ and let

$$c_j(f)(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) A_j(\xi) e^{it\xi} d\xi,\ f \in F.$$

Then $c_j(f)(t)$'s are continuous functions on $\mathbb{R}$ since $A_j(\xi)\hat{f}(\xi) \in L^1(\mathbb{R})$.
For any $\omega > 0$, take $[-\pi\omega, \pi\omega]$ to be an assumed band region for $f(t)$ in F and let

$$\begin{aligned} I(\omega) &:= [-\pi\omega, -\pi\omega + d(\omega)] \ (d(\omega)) = \frac{2\pi\omega}{M}, \ M \text{ as in section 2}), \\ E_\omega(t) &:= (1, e^{id(\omega)t}, \cdots, e^{i(M-1)d(\omega)t})^T, \\ A_{\omega,j}(\xi) &:= A_j(\xi)\chi_{[-\pi\omega,\pi\omega]}(\xi) \ (1 \le j \le N), \end{aligned}$$

where $\chi_{[-\pi\omega,\pi\omega]}(\xi)$ is the characteristic function of $[-\pi\omega, \pi\omega]$. Let $A_\omega(\xi)$ be the $M \times M$ transfer matrix on $I(\omega)$ as in Section 2, where $A_j(\xi)$'s are replaced by $A_{\omega,j}(\xi)$'s. Assuming that $A_\omega(\xi)$ is non-singular in $I(\omega)$, let

$$\begin{aligned} Y_\omega(\xi, t) &:= [A_\omega(\xi)^{-1}]^T E_\omega(t) \\ &= [Y_{\omega,j,k}(\xi, t)]_{j=1,k=0}^{N\ m_j-1} \end{aligned}$$

is the $M \times 1$ column vector as $Y(\xi, t)$ in section 2.

Lemma 3.1. *Let $g(\xi) \in L^1[-\pi, \pi]$ be periodic of period 2π and $g(\xi) \sim \sum_n \alpha_n e^{in\xi}$ be the Fourier series of $g(\xi)$. Then for any function $\phi(\xi)$ of bounded variation on $\mathbb{R}$,*

$$\int_a^b \phi(\xi)g(\xi)d\xi = \sum_n \alpha_n \int_a^b e^{in\xi}\phi(\xi)d\xi \ (-\infty < a < b < \infty).$$

Proof. See [4, p. 16]. □

Theorem 3.2. *Assume further that $A_j(\xi)$'s are of bounded variations on $\mathbb{R}$ and there are positive constants β_j for $1 \le j \le N$ such that*

$$|A_j(\xi)| \le \beta_j < \infty \ \textit{for}\ \xi \in \mathbb{R}$$

and for any $\omega > 0$, there is a positive constant $\alpha(\omega)$ such that

$$\alpha(\omega) \le |\det A_\omega(\xi)| \ \textit{for}\ \xi \in I(\omega).$$

Let

$$f_\omega(t) := \frac{M}{\sqrt{2\pi}\omega} \sum_{j=1}^{N} \sum_{k=1}^{m_j-1} \sum_{n=-\infty}^{\infty} c_j(f)\Big(\frac{M}{m_j\omega}(m_j n + k)\Big) y_{\omega,j,k}\Big(t - \frac{nM}{\omega}\Big) \tag{3.1}$$

be the multi-channel alias of $f(t)$ over the assumed band region $[-\pi\omega, \pi\omega]$, where

$$y_{\omega,j,k}(t) := \frac{1}{\sqrt{2\pi}} \int_{I(\omega)} Y_{\omega,j,k}(\xi, t)e^{it\xi}d\xi.$$

Then the series in (3.1) converges to a continuous function on $\mathbb{R}$:

$$f_\omega(t) = \sum_{j=1}^{N} \sum_{k=0}^{m_j-1} \int_{I(\omega)} \psi_{j,k}(\xi)Y_{\omega,j,k}(\xi, t)e^{it\xi}d\xi \tag{3.2}$$

where

$$\psi_{j,k}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} A_j(\xi + nd(\omega))\hat{f}(\xi + nd(\omega))e^{i\frac{Mk}{m_j\omega}(\xi + nd(\omega))} \tag{3.3}$$

converges in $L^1(I(\omega))$*. Moreover, we have an aliasing error bound:*

$$\sqrt{2\pi}|f(t)-f_\omega(t)| \le (1+\sum_{j=1}^{N}\sum_{k=1}^{m_j}\beta_j\gamma_{\omega,j,k})\int_{|\xi|\ge\pi\omega}|\hat f(\xi)|d\xi. \tag{3.4}$$

where $\gamma_{\omega,j,k} = \sup_{I(\omega)\times\mathbb{R}} |Y_{\omega,j,k}(\xi,t)|$.

If furthermore,

$$\liminf_{\omega>0} \alpha(\omega) > 0 \tag{3.5}$$

then $f_\omega(t)$ *converges to* $f(t)$ *uniformly on* $\mathbb{R}$ *as* $\omega \longrightarrow \infty$.

Proof. Since each $A_j(\xi)$ is bounded on $\mathbb{R}$ and $\hat f(\xi)\in L^1(\mathbb{R})$,

$$\begin{aligned}\int_{I(\omega)}|\psi_{j,k}(\xi)|d\xi &\le \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\int_{I(\omega)}|A_j(\xi+nd(\omega))\hat f(\xi+nd(\omega))|d\xi\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|A_j(\xi)\hat f(\xi)|d\xi<\infty.\end{aligned}$$

Hence, each $\psi_{j,k}(\xi)\in L^1(I(\omega))$ of which the Fourier series is

$$\begin{aligned}\psi_{j,k}(\xi) &\sim \frac{M}{2\pi\omega}\sum_{n=-\infty}^{\infty}\Big(\int_{I(\omega)}\psi_{j,k}(\xi)e^{i\frac{nM}{\omega}\xi}d\xi\Big)e^{-i\frac{nM}{\omega}\xi}\\ &= \frac{M}{2\pi\omega}\sum_{n=-\infty}^{\infty}c_j(f)\Big(\frac{M}{m_j\omega}(m_jn+k)\Big)e^{-i\frac{nM}{\omega}\xi}\end{aligned}$$

since

$$\begin{aligned}&\int_{I(\omega)}\psi_{j,k}(\xi)e^{i\frac{nM}{\omega}\xi}d\xi\\ &= \frac{1}{\sqrt{2\pi}}\sum_{l=-\infty}^{\infty}\int_{I(\omega)}A_j(\xi+ld(\omega))\hat f(\xi+ld(\omega))e^{\frac{Mk}{m_j\omega}(\xi+ld(\omega))}e^{i\frac{nM}{\omega}\xi}d\xi\\ &= \frac{1}{\sqrt{2\pi}}\sum_{l=-\infty}^{\infty}\int_{I(\omega)+ld(\omega)}A_j(\xi)\hat f(\xi)e^{i\frac{M}{m_j\omega}(m_jn+k)\xi}d\xi\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}A_j(\xi)\hat f(\xi)e^{i\frac{M}{m_j\omega}(m_jn+k)\xi}d\xi\\ &= c_j(f)(\frac{M}{m_j\omega}(m_jn+k)).\end{aligned}$$

Hence, by Lemma 3.1 and the fact that $Y_{\omega,j,k}(\xi,t)e^{it\xi}$ is of bounded variation on $I(\omega)$ as a function of ξ, we have

$$\begin{aligned}&\int_{I(\omega)}\psi_{j,k}(\xi)Y_{\omega,j,k}(\xi,t)e^{it\xi}d\xi\\ &= \frac{M}{2\pi\omega}\sum_{n=-\infty}^{\infty}c_j(f)\Big(\frac{M}{m_j\omega}(m_jn+k)\Big)\int_{I(\omega)}Y_{\omega,j,k}(\xi,t)e^{i(t-\frac{nM}{\omega})\xi}d\xi\\ &= \frac{M}{\sqrt{2\pi}\omega}\sum_{n=-\infty}^{\infty}c_j(f)\Big(\frac{M}{m_j\omega}(m_jn+k)\Big)y_{\omega,j,k}\Big(t-\frac{nM}{\omega}\Big)\end{aligned}$$

from which (3.2) follows.

Now since the series (3.3) converges in $L^1(I(\omega))$, we have

$$\begin{aligned}&\int_{I(\omega)} \psi_{j,k}(\xi) Y_{\omega,j,k}(\xi,t) e^{it\xi} d\xi \\ = \;& \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \int_{I(\omega)+nd(\omega)} A_j(\xi)\hat{f}(\xi) Y_{\omega,j,k}(\xi - nd(\omega), t) e^{i\frac{Mk}{m_j\omega}\xi} e^{it(\xi - nd(\omega))} d\xi,\end{aligned}$$

which converges absolutely and uniformly to a continuous function on $\mathbb{R}$ since $Y_{\omega,j,k}(\xi,t)$ is bounded on $I(\omega)\times\mathbb{R}$ and is continuous in t. Hence

$$f_\omega(t) = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{N} \sum_{k=0}^{m_j-1} \int_{I(\omega)+nd(\omega)} A_j(\xi)\hat{f}(\xi) Y_{\omega,j,k}(\xi - nd(\omega), t) e^{i\frac{Mk}{m_j\omega}\xi} e^{it(\xi - nd(\omega))} d\xi \tag{3.6}$$

is also continuous on $\mathbb{R}$.

On the other hand,

$$\begin{aligned} f(t) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{it\xi} d\xi = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \int_{I(\omega)+nd(\omega)} \hat{f}(\xi) e^{it\xi} d\xi \\ =& \frac{1}{\sqrt{2\pi}} (\sum_{n\in Q} + \sum_{n\notin Q}) \int_{I(\omega)+nd(\omega)} \hat{f}(\xi) e^{it\xi} d\xi\end{aligned}$$

where $Q := \{0,1,2,\cdots,M-1\}$. Then

$$\begin{aligned}&\sum_{n\in Q} \int_{I(\omega)+nd(\omega)} \hat{f}(\xi) e^{it\xi} d\xi \\ =\;& \sum_{n\in Q} \int_{I_\omega} \hat{f}(\xi + nd(\omega)) e^{it(\xi+nd(\omega))} d\xi \\ =\;& \int_{I(\omega)} E_\omega(t)^T D(\hat{f}(\xi)\chi_{[-\pi\omega,\pi\omega](\xi)}) e^{it\xi} d\xi \\ =\;& \int_{I(\omega)} E_\omega(t)^T A_\omega(\xi)^{-1} G_\omega(\xi) e^{it\xi} d\xi \\ =\;& \int_{I(\omega)} Y_\omega(\xi,t)^T G_\omega(\xi) e^{it\xi} d\xi \\ =\;& \sum_{j=1}^{N} \sum_{k=0}^{m_j-1} \sum_{l=1}^{M} \int_{I(\omega)} \Big[A_{\omega,j}(\xi + (l-1)d(\omega)) \hat{f}(\xi + (l-1)d(\omega)) \\ & \qquad \times Y_{\omega,j,k}(\xi,t) e^{i\frac{Mk}{m_j\omega}(\xi+(l-1)d(\omega))} e^{it\xi} \Big] d\xi\end{aligned}$$

where

$$G_\omega(\xi) := A_\omega(\xi) D(\hat{f}(\xi)\chi_{[-\pi\omega,\pi\omega]}(\xi)) = (G_{\omega,j,k}(\xi))_{j=1,k=0}^{N\;\; m_j-1}$$

is the $N\times 1$ column vector as in section 2 and

$$G_{\omega,j,k}(\xi) := \sum_{l=1}^{M} A_{\omega,j}(\xi + (l-1)d(\omega)) \hat{f}(\xi + (l-1)d(\omega)) e^{i\frac{Mk}{m_j\omega}(\xi+(l-1)d(\omega))}.$$

Hence

$$\sum_{n\in Q}\int_{I(\omega)+nd(\omega)}\hat{f}(\xi)e^{it\xi}d\xi \tag{3.7}$$
$$= \sum_{j=1}^{N}\sum_{k=0}^{m_j-1}\sum_{l=1}^{M}\int_{I(\omega)+(l-1)d(\omega)} A_j(\xi)\hat{f}(\xi)Y_{\omega,j,k}(\xi-(l-1)d(\omega),t)e^{i\frac{Mk}{m_j\omega}\xi}e^{it(\xi-(l-1)d(\omega))}d\xi$$

since $A_{\omega,j}(\xi)=A_j(\xi)$ on $[-\pi\omega,\pi\omega]$. We then have from (3.6) and (3.7)

$$\begin{aligned}
&\sqrt{2\pi}[f(t)-f_\omega(t)]\\
=\;&\Big(\sum_{n\in Q}+\sum_{n\notin Q}\Big)\int_{I(\omega)+nd(\omega)}\hat{f}(\xi)e^{it\xi}d\xi\\
&-\sum_{j=1}^{N}\sum_{k=0}^{m_j-1}\sum_{n=-\infty}^{\infty}\int_{I(\omega)+nd(\omega)}A_j(\xi)\hat{f}(\xi)Y_{\omega,j,k}(\xi-nd(\omega),t)e^{i\frac{Mk}{m_j\omega}\xi}e^{it(\xi-nd(\omega))}d\xi\\
=\;&\sum_{n\notin Q}\int_{I(\omega)+nd(\omega)}\hat{f}(\xi)e^{it\xi}\Big[1-\sum_{j=1}^{N}\sum_{k=0}^{m_j-1}A_j(\xi)Y_{\omega,j,k}(\xi-nd(\omega),t)e^{i\frac{Mk}{m_j\omega}\xi}e^{-ind(\omega)t}\Big]d\xi.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sqrt{2\pi}|f(t)-f_\omega(t)|\\
\le&\sum_{n\notin Q}\int_{I(\omega)+nd(\omega)}|\hat{f}(\xi)|\Big[1+\sum_{j=1}^{N}\sum_{k=0}^{m_j-1}|A_j(\xi)||Y_{\omega,j,k}(\xi-md,t)|\Big]d\xi\\
=&(1+\sum_{1}^{N}\sum_{k=0}^{m_j-1}\beta_j\gamma_{\omega,j,k})\int_{|\xi|\ge\pi\omega}|\hat{f}(\xi)|d\xi
\end{aligned}$$

which proves (3.4). Finally assume that the condition (3.5) holds. Choose a positive constant α with $0<\alpha<\liminf_{\omega>0}\alpha(\omega)$. Then $\alpha<\alpha(\omega)\le|\det A_\omega(\xi)|$ for ω large enough. Let $A_\omega(\xi)^{-1}=\frac{1}{\det A_\omega(\xi)}[c_{\omega,j,k}(\xi)]$, where $c_{\omega,j,k}(\xi)$ is the (j,k) cofactor of $A_\omega(\xi)$. Then $c_{\omega,j,k}(\xi)$'s are uniformly bounded in ξ and ω, that is, there is a positive constant K, independent of ξ and $\omega>0$, such that

$$|c_{\omega,j,k}(\xi)|\le K$$

for any j,k and $\omega>0$, $\xi\in I(\omega)$. Then we have from $Y_\omega(\xi,t)=[A_\omega(\xi)^{-1}]^T E_\omega(t)$

$$|Y_{\omega,j,k}(\xi,t)|\le\frac{MK}{\alpha}$$

for any j,k, any t in $\mathbb{R}$, and ω large enough. Hence

$$\gamma_{\omega,j,k}\le\frac{MK}{\alpha}\text{ for any } j,k \text{ and } \omega \text{ large enough}$$

so that

$$\sqrt{2\pi}|f(t)-f_\omega(t)|\le\Big(1+\frac{MK}{\alpha}\sum_{j=1}^{N}m_j\beta_j\Big)\int_{|\xi|\ge\pi\omega}|\hat{f}(\xi)|d\xi\longrightarrow 0$$

as $\omega\longrightarrow\infty$. Hence, $f_\omega(t)\longrightarrow f(t)$ uniformly on $\mathbb{R}$ as $\omega\longrightarrow\infty$. □

Example 3.1. In case of Example 2.1, $|A_1(\xi)| = 1$, $|A_2(\xi)| = 1$ and $\det A_\omega(\xi) = 4i$. We also have $|Y_{\omega,1,0}(\xi,t)| \le \frac{3}{2}$, $|Y_{\omega,1,1}(\xi,t)| \le \frac{3}{2}$ and $|Y_{\omega,2,0}(\xi,t)| \le \frac{1}{2}$ on $I(\omega) \times \mathbb{R}$. Hence, we obtain an upper bound for aliasing error as

$$|f(t) - f_\omega(t)| \le \frac{9}{2\sqrt{2\pi}} \int_{|\xi| \ge \pi\omega} |\hat{f}(\xi)| d\xi.$$

Acknowledgement This work is partially supported by BK-21 project and KOSEF(99–2–101–001–5).

References

[1] J. L. Brown, *On the error in reconstructing a nonband-limited function by means of the band pass sampling theorem* J. Math. Anal. Appl., 18, (1967), 75-84,

[2] P. L. Butzer, W. Splettstösser and R. L. Stens, *The sampling theorem and linear prediction in signal analysis* Jber. d. Dt. Math. Verein., 90, (1988), 1-70

[3] J. R. Higgins, *Sampling theory for Paley-Wiener spaces in the Riesz basis setting*, Proc. R. Irish. Acad., 94A, (1994), 219-236

[4] J. R. Higgins, *Sampling theory in Fourier and signal analysis*, Oxford University Press, (1996)

[5] J. R. Higgins and R. L. Stens *Sampling theory in Fourier and signal analysis: Advanced topics*, Oxford University Press (2000)

[6] A. Papoulis, *Signal Analysis*, McGraw-Hill, (1977)

[7] A. Papoulis *Generalized sampling expansion*, IEEE Trans. on circuits and systems, (1977), 652-654

[8] C. E. Shannon *Communication in the presence of noise*, Proc.IRE, 37, (1049) 10-21

[9] P. Weiss *An estimation of the error arising from misapplication of the sampling theorem*, Amer. Math. Soc. Notices, 10, (1963), 351

[10] A. I. Zayed, *Advances in Shannon's sampling theory*, CRC Press, (1993)

Division of Applied Mathematics, KAIST
Daejeon 305-701, Korea
e-mail: khkwon@amath.kaist.ac.kr