ON DUAL AND THREE SPACE PROBLEMS FOR THE COMPACT APPROXIMATION PROPERTY

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We introduce the properties W*D and BW*D for the dual space of a Banach space. And then solve the dual problem for the compact approximation property(CAP): if X^* has the CAP and the W*D, then X has the CAP. Also, we solve the three space problem for the CAP: for example, if M is a closed subspace of a Banach space such that $M^\perp$ is complemented in X^* and X^* has the W*D, then X has the CAP whenever X/M has the CAP and M has the bounded CAP. Corresponding problems for the bounded compact approximation property are also addressed.

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1. Introduction

The approximation property(AP) was introduced at the early stage of the Banach space theory; it already appeared in Banach's book [B]. A systematic study of the AP was carried in his memoir by Grothendieck [G]. The AP, besides finding many uses in Banach spaces, plays a special role in the structure theory of Banach spaces. One important question about the AP is whether or not it passes to the dual space and subspaces; the question in the opposite direction is equally important.

Well known is that if the dual X^* has the AP, then so does X, in general, the converse does not hold. But, the corresponding dual problem for the CAP is open (See Casazza [C], Problem 8.5):

If X^* has the CAP, must X have the CAP ?

In general the converse is false. On the other hand, if M is a closed subspace of a Banach space X, then the pair (X, M) has the three space property for the AP whenever M is complemented in X. The three space problem for non-complemented subspaces is much harder. Godefroy and Saphar [GS] obtained significant results on the three space problem for the AP under the assumption that $M^\perp$ is complemented in X^*. Thus we are led to raise the following problem:

Does the pair (X, M) have the three space property for the CAP whenever $M^\perp$ is complemented in X^*?

In this paper we solve the above two problems under the extra assumption that X^* and M^* have certain density properties for the space of compact operators.

2. Preliminaries and the property W*D

In this section we first fix our notions and provide necessary definitions with comments. At the end of the section we study relationship between our property W*D and various concepts of approximation properties.

Notation 2.1. Let X be a Banach space and $\lambda > 0$. Throughout this paper, we use the following notations :

Id_X : The identity operator on X.
$\mathcal{B}(X)$: The collection of bounded linear operators on X.
$\mathcal{F}(X)$: The collection of bounded and finite rank linear operators on X.

$\mathcal{K}(X)$: The collection of compact operators on X.

$\mathcal{K}(X^*, w^*)$: The collection of compact and w^*-to-w^* continuous operators on X^*.

$\mathcal{K}(X, \lambda)$: The collection of compact operators T on X satisfying $\|T\| \leq \lambda$.

$\mathcal{K}(X^*, w^*, \lambda)$: The collection of compact and w^*-to-w^* continuous operators T on X^* satisfying $\|T\| \leq \lambda$.

Similarly we define $\mathcal{F}(X^*, w^*)$, $\mathcal{F}(X, \lambda)$ and $\mathcal{F}(X^*, w^*, \lambda)$.

Note that w^* means the weak* topology on X^*. And observe that

$$\mathcal{K}(X^*, w^*, \lambda) = \{T^* \in \mathcal{K}(X^*) \ : \ T \in \mathcal{K}(X, \lambda)\},$$

where T^* is the adjoint of T.

We introduce an topology on $\mathcal{B}(X)$, which is an important tool to study the approximation properties. For compact $K \subset X$, $\epsilon > 0$, and $T \in \mathcal{B}(X)$ we put

$$N(T, K, \epsilon) = \{R \in \mathcal{B}(X) : \sup_{x \in K} \|Rx - Tx\| < \epsilon\}.$$

Let $\mathcal{S}$ be the collection of all such $N(T, K, \epsilon)$'s. Now we denote by τ the topology on $\mathcal{B}(X)$ generated by $\mathcal{S}$. Observe that for T and a net (T_α) in $\mathcal{B}(X)$

$$T_\alpha \longrightarrow T \ \text{ in } \ (\mathcal{B}(X), \tau) \Longleftrightarrow \text{for each compact } K \subset X \ \sup_{x \in K} \|T_\alpha x - Tx\| \longrightarrow 0.$$

Grothendieck ([G]) showed the following lemma.

Lemma 2.2. *Let X be a Banach space. Then the topology τ on $\mathcal{B}(X)$ is a locally convex topology and $(\mathcal{B}(X), \tau)^*$ consists of all functionals f of the form $f(T) = \sum_n x_n^* T x_n$, where $(x_n) \subset X, (x_n^*) \subset X^*$ and $\sum_n \|x_n\| \|x_n^*\| < \infty$.*

Now we give definition of various kinds of approximation properties for Banach spaces. We say that X has the *approximation property* (in short, AP) if for every compact $K \subset X$ and $\epsilon > 0$ there is a $T \in \mathcal{F}(X)$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$. Also we say that X has the *λ-bounded approximation property* (in short, λ-BAP) if for every compact $K \subset X$ and $\epsilon > 0$ there is a $T \in \mathcal{F}(X, \lambda)$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$. If X has the λ-bounded approximation property for some $\lambda > 0$, then we say that X has the *bounded approximation property* (in short, BAP). We say that a Banach space X has the *compact approximation property* (in short, CAP) if for every compact $K \subset X$ and $\epsilon > 0$ there is a $T \in \mathcal{K}(X)$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$. Also we say that a Banach space X has the *λ-bounded compact approximation property* (in short, λ-BCAP) if for every compact $K \subset X$ and $\epsilon > 0$ there is a $T \in \mathcal{K}(X, \lambda)$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$. If X has the λ-bounded compact approximation property for some $\lambda > 0$, then we say that X has the *bounded compact approximation property* (in short, BCAP). Recently Choi and Kim [CK] introduced weak versions of the approximation property. We say that X has the *weak approximation property* (in short, WAP) if for every $T \in \mathcal{K}(X)$, compact $K \subset X$, and $\epsilon > 0$ there is a $T_0 \in \mathcal{F}(X)$ such that $\|T_0 x - Tx\| < \epsilon$ for all $x \in K$. Using the τ-topology we see the following :

X has the AP iff $Id_X \in \overline{\mathcal{F}(X)}^{\tau}$.
X has the λ-BAP iff $Id_X \in \overline{\mathcal{F}(X, \lambda)}^{\tau}$.
X has the CAP iff $Id_X \in \overline{\mathcal{K}(X)}^{\tau}$.
X has the λ-BCAP iff $Id_X \in \overline{\mathcal{K}(X, \lambda)}^{\tau}$.
X has the WAP iff $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}^{\tau}$.

Also we observe the following :

A Banach space has the AP iff it has both the CAP and the WAP.

To check the above statement we need that the AP implies the WAP. Indeed, if a Banach space X has the AP, then we can pick a net (T_α) in $\mathcal{F}(X)$ such that $T_\alpha \xrightarrow{\tau} Id_X$. Then for any $S \in \mathcal{K}(X)$, we have $T_\alpha S \xrightarrow{\tau} S$, hence $S \in \overline{\mathcal{F}(X)}^\tau$, which proves that X has the WAP.

We need two topologies on the space of operators. We define topologies by specifying convergent nets. Here X is a Banach space.

Definition 2.3. For T and a net (T_α) in $\mathcal{B}(X)$ we say that the net (T_α) converges to T in the ν-topology, or $T_\alpha \xrightarrow{\nu} T$ iff

$$\sum_n x_n^*(T_\alpha x_n) \longrightarrow \sum_n x_n^*(Tx_n)$$

for every $(x_n) \subset X$ and $(x_n^*) \subset X^*$ satisfying $\sum_n \|x_n\|\|x_n^*\| < \infty$.

Recall Lemma 2.2. Then on the space $\mathcal{B}(X)$ the τ-topology is stronger than the ν-topology. But by a convex combination argument we see the following :

X has the AP iff $Id_X \in \overline{\mathcal{F}(X)}^\nu$.
X has the λ-BAP iff $Id_X \in \overline{\mathcal{F}(X,\lambda)}^\nu$.
X has the CAP iff $Id_X \in \overline{\mathcal{K}(X)}^\nu$.
X has the λ-BCAP iff $Id_X \in \overline{\mathcal{K}(X,\lambda)}^\nu$.
X has the WAP iff $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}^\nu$.

Definition 2.4. For T and a net (T_α) in $\mathcal{B}(X^*)$ we say that the net (T_α) converges to T in the *weak**-topology, or $T_\alpha \xrightarrow{weak^*} T$ iff

$$\sum_n (T_\alpha x_n^*)x_n \longrightarrow \sum_n (Tx_n^*)x_n$$

for every $(x_n) \subset X$ and $(x_n^*) \subset X^*$ satisfying $\sum_n \|x_n\|\|x_n^*\| < \infty$.

The name, the *weak**-topology, comes from the fact that $\mathcal{B}(X^*)$ can be, in the canonical way, identified with $(X^*\hat{\otimes}_\pi X)^*$, the dual of the completed projective tensor product of X^* and X. On the space $\mathcal{B}(X^*)$ the ν-topology is stronger than the *weak**-topology. But they coincide when X is reflexive. Note that for T and a net (T_α) in $\mathcal{B}(X)$

$$T_\alpha \xrightarrow{\nu} T \quad \text{iff} \quad T_\alpha^* \xrightarrow{weak^*} T^*. \tag{2.1}$$

We finally define the properties which enable us to prove the dual and the three space problems for the CAP in our setting.

Definition 2.5. Let X be a Banach space.

(a) The dual space X^* is said to have the *weak** *density* for compact operators, in short, W*D if $\mathcal{K}(X^*) \subset \overline{\mathcal{K}(X^*,w^*)}^{weak^*}$.
(b) The dual space X^* is said to have the *bounded weak** *density* for compact operators, in short, BW*D if $\mathcal{K}(X^*,1) \subset \overline{\mathcal{K}(X^*,w^*,\lambda)}^{weak^*}$ for some $\lambda > 0$.

In Section 3 it is shown that not every Banach space has the BW*D. The question whether or not every Banach space has the W*D, which is not known, will be shown to be closely related to the dual problem for the CAP.

In this section we want to find, for the dual space X^*, the relations between W*D, or BW*D and the various kinds of approximation properties. For this we need a lemma which is originally due to Lindenstrauss and Tzafriri [LT] and Johnson ([J], Lemma 1). A proof of the following version is given in [CK].

Lemma 2.6. *Let X be a Banach space. Then we have the following.*

(a) $\mathcal{F}(X^*) \subset \overline{\mathcal{F}(X^*, w^*)}^{\tau} \subset \overline{\mathcal{F}(X^*, w^*)}^{weak^*}$.
(b) $\mathcal{F}(X^*, \lambda) \subset \overline{\mathcal{F}(X^*, w^*, \lambda)}^{\tau} \subset \overline{\mathcal{F}(X^*, w^*, \lambda)}^{weak^*}$ *for all* $\lambda > 0$.

Now we have a proposition about the properties W*D and BW*D.

Proposition 2.7. *Let X be a Banach space. Then the following statements hold.*

(a) *If X^* is reflexive, X^* has the W*D and the BW*D. But the converse is false in general.*
(b) *If X^* has the WAP, X^* has the W*D. But the converse is false in general.*
(c) *If X^* has the BAP, X^* has the BW*D. But the converse is false in general.*

Proof. (a). If X is reflexive, then every $T \in \mathcal{B}(X^*)$, being w-to-w continuous, is w^*-to-w^* continuous, hence $T \in \mathcal{B}(X^*, w^*)$ and we have $\mathcal{K}(X^*) = \mathcal{K}(X^*, w^*)$ and $\mathcal{K}(X^*, 1) = \mathcal{K}(X^*, w^*, 1)$, which implies that X^* has the W*D and the BW*D.

To show that the converse is false in general we consider $X = c_0$, a non-reflexive Banach space. Writing $X^* = l_1$, we claim that

$$\mathcal{K}(X^*, 1) \subset \overline{\mathcal{K}(X^*, w^*, 1)}^{\tau},$$

which obviously implies that X^* has the BW*D, hence the W*D as well. Indeed, if we let $T \in \mathcal{K}(X^*, 1)$, then for each $n \in \mathbf{N}$ the projection $P_n \in \mathcal{B}(l_1)$ given by

$$P_n((\alpha_i)) = (\alpha_1, ..., \alpha_n, 0, 0, ...)$$

is w^*-to-norm continuous because

$$\|P_n((\alpha_i))\| = |\alpha_1| + ... + |\alpha_n| = \sum_{j=1}^{n} |\langle e_j, (\alpha_i)\rangle|$$

where $e_j \in c_0$ is the jth standard basis vector. Obviously $\|P_n\| \le 1$ for each n. Put $T_n = TP_n$. Then each T_n is compact and w^*-to-norm continuous, hence it is w^*-to-w^* continuous. Having shown that each $T_n \in \mathcal{K}(X^*, w^*, 1)$, it remains to show that $T_n \xrightarrow{\tau} T$.

Let $K \subset X^*$ be compact and $\epsilon > 0$. There is a finite set $A \subset K$ such that for each $x^* \in K$ there is $y^* \in A$ with $\|x^* - y^*\| < \epsilon/3$. Since $T_n x^* \longrightarrow Tx^*$ for each $x^* \in X^* = l_1$, there is $N \in \mathbf{N}$ such that $n \ge N$ implies

$$\|T_n y^* - Ty^*\| < \frac{\epsilon}{3}$$

for all $y^* \in A$. One can check that $n \ge N$ implies $\|T_n x^* - Tx^*\| < \epsilon$ for all $x^* \in K$, which completes the proof.

(b). Assume that X^* has the WAP. Then we have

$$\mathcal{K}(X^*) \subset \overline{\mathcal{F}(X^*)}^{\tau} = \overline{\mathcal{F}(X^*, w^*)}^{\tau} \subset \overline{\mathcal{K}(X^*, w^*)}^{\tau} \subset \overline{\mathcal{K}(X^*, w^*)}^{weak^*},$$

where we used Lemma 2.6.(a) and the fact that the τ-topology is stronger than the *weak**-topology. Thus X^* has the W*D.

To prove that the converse is not true in general we consider the Willis space Z, which is a separable reflexive Banach space having the CAP, but not having the AP (See Willis [W]). Hence Z does not have the WAP. Since Z is reflexive, with $X = Z^*$, we have that X is reflexive, hence by (a), X^* has the W*D. But, X^*, being isometric to Z, fails to have the WAP.

(c). Assume that X^* has the BAP. Then there is $\lambda > 0$ and a net (T_α) in $\mathcal{F}(X^*, \lambda)$ such that $T_\alpha \xrightarrow{\tau} Id_{X^*}$. Now, if $S \in \mathcal{K}(X^*, 1)$, then (T_α) converges uniformly on the compact set $S(B_{X^*})$, the image of the unit ball of X^* under S, or $\|T_\alpha S - S\| \to 0$, hence $T_\alpha S \xrightarrow{\tau} S$, which implies that

$$S \in \overline{\mathcal{F}(X^*, \lambda)}^{\tau} \subset \overline{\mathcal{F}(X^*, w^*, \lambda)}^{weak^*}.$$

Here we used Lemma 2.6.(b). This proves that X^* has the BW*D. Of course, the counterexample in (b) serves as a counterexample for (c) also. □

The following diagram summarizes the relations we have found between the properties for the dual space X^*, including the W*D and the BW*D:

$$BAP \Longrightarrow AP \Longrightarrow WAP \underset{\not\Leftarrow}{\overset{\Longrightarrow}{}} W^*D$$

$$BAP \underset{\not\Leftarrow}{\overset{\Longrightarrow}{}} BW^*D \Longrightarrow W^*D$$

$$\text{Reflexivity} \underset{\not\Leftarrow}{\overset{\Longrightarrow}{}} BW^*D$$

3. The dual problem for the CAP

The following theorem shows that the properties W*D and BW*D are the right assumptions in solving the dual problem for the CAP.

Theorem 3.1. *Let X be a Banach space. Then we have the following.*
(a) *If X^* has the CAP and the W*D, then X has the CAP.*
(b) *If X^* has the BCAP and the BW*D, then X has the BCAP.*

Proof. (a). Assume that X^* has the CAP and the W*D. Then

$$Id_{X^*} \in \overline{\mathcal{K}(X^*)}^{\tau} \text{ and } \mathcal{K}(X^*) \subset \overline{\mathcal{K}(X^*, w^*)}^{weak^*}.$$

Since the τ-topology is stronger than the *weak**-topology, we have

$$Id_{X^*} \in \overline{\mathcal{K}(X^*)}^{weak^*} = \overline{\mathcal{K}(X^*, w^*)}^{weak^*}.$$

By (2.1) $Id_X \in \overline{\mathcal{K}(X)}^{\nu}$ which proves that X has the CAP.

(b). Assume that X^* has the BCAP and the BW*D. Then,

$$Id_{X^*} \in \overline{\mathcal{K}(X^*, \lambda)}^{\tau} \text{ and } \mathcal{K}(X^*, 1) \subset \overline{\mathcal{K}(X^*, w^*, \mu)}^{weak^*}$$

for some λ and $\mu > 0$. Since $\mathcal{K}(X^*, \lambda) \subset \overline{\mathcal{K}(X^*, w^*, \lambda\mu)}^{weak^*}$, as in (a), we have

$$Id_{X^*} \in \overline{\mathcal{K}(X^*, w^*, \lambda\mu)}^{weak^*}.$$

By (2.1) $Id_X \in \overline{\mathcal{K}(X, \lambda\mu)}^{\nu}$ which proves that X has the BCAP. □

It is known that there is a Banach space X such that X fails to have the BCAP but X^* has the BCAP. According to Theorem 3.1.(b) X^* cannot have the BW*D. It is not known whether all Banach spaces have the W*D. Thus we are led to the following:

Question. *Does the dual of every Banach space have the W*D ?*

If the above question has the affirmative answer, then the general dual problem for the CAP, in view of Theorem 3.1.(a), also has the affirmative answer.

4. The three space problem for the CAP

First we start with the following simple case when the subspace is complemented.

Proposition 4.1. *If M is a complemented subspace of a Banach space X, then the pair (X, M) have the three space property for the CAP and the BCAP.*

Proof. Assume that M is a complemented subspace of a Banach space X. Then there is a projection $P : X \longrightarrow M$ onto M. Let $\iota : M \longrightarrow X$ be the inclusion.

First assume that X has the CAP. We will show that both M and X/M have the CAP. By the assumption there is a net (T_α) be in $\mathcal{K}(X)$ such that $T_\alpha \xrightarrow{\tau} Id_X$, hence $T_\alpha \xrightarrow{\nu} Id_X$. Put $S_\alpha = PT_\alpha\iota$. Then (S_α) is a net in $\mathcal{K}(M)$ such that, whenever $(m_n) \subset M$ and $(m_n^*) \subset M^*$ with $\sum_n \|m_n\|\|m_n^*\| < \infty$, we have

$$\sum_n m_n^*(S_\alpha m_n) = \sum_n m_n^* P(T_\alpha \,\iota\, m_n) \longrightarrow \sum_n (m_n^* P)(\iota\, m_n) = \sum_n m_n^* m_n$$

because $\sum_n \|\iota\, m_n\|\|m_n^* P\| < \infty$. Hence $S_\alpha \xrightarrow{\nu} Id_M$ and M has the CAP.

For X/M observe that $Id_X - \iota\, P : X \longrightarrow X$ is a projection with kernel M. Thus if we put $N = (Id_X - \iota\, P)(X)$ and define $Q : X \longrightarrow N$ by $Q(x) = (Id_X - \iota\, P)x$, then Q is a projection onto N. Hence N is a complemented subspace of X. By the above argument N has the CAP. But X/M is isomorphic to N, hence it is easily checked that X/M also has the CAP.

The above argument also shows that if X has the BCAP, then so do M and X/M. Indeed, notice that, in the case that X has the BCAP, (T_α) can be chosen as a bounded net in $\mathcal{K}(X)$. Thus (S_α) becomes bounded too, which implies that M has the BCAP. For the same reason X/M has the BCAP.

Now assume that both M and X/M have the CAP. Let $j : N \longrightarrow X$ be the inclusion. Observe that $X = M \oplus N$, the sum of M and N. Since both M and N have the CAP, given a compact $K \subset X$ and $\epsilon > 0$ there are $S \in \mathcal{K}(M)$ and $R \in \mathcal{K}(N)$ such that

$$\|SPx - Px\| < \epsilon \quad \text{and} \quad \|RQx - Qx\| < \epsilon$$

for all $x \in K$. Put $Tx = \iota SPx + jRQx$ for $x \in X$. Then we observe that $T \in \mathcal{K}(X)$ and

$$\|Tx - x\| = \|\iota\,(SPx - Px) + j(RQx - Qx)\| < 2\epsilon$$

for all $x \in K$. Thus X has the CAP.

In the above, if M and N have the BCAP, then there are $\lambda, \mu > 0$ so that

$$Id_M \in \overline{\mathcal{K}(M, \lambda)}^{\tau} \quad \text{and} \quad Id_N \in \overline{\mathcal{K}(N, \mu)}^{\tau}.$$

Hence we could have chosen S and R in the above so that they also satisfy $S \in \mathcal{K}(M, \lambda)$ and $R \in \mathcal{K}(N, \mu)$. Then, since $\|T\| \leq \lambda\|P\| + \mu\|Q\|$, we have

$$Id_X \in \overline{\mathcal{K}(X, \lambda\|P\| + \mu\|Q\|)}^{\tau},$$

which proves that X has the BCAP. □

The following is a well-known fact (See Diestel [D, Exercises 1.6 and 2.6,(1)]).

Fact. Let (X_n) be a sequence of Banach spaces. If $1 \leq p < \infty$ and K is a relatively compact subset of $(\sum_n \bigoplus X_n)_{l_p}$, then for every $\epsilon > 0$ there is a positive integer N_ϵ such that

$$\sum_{n > N_\epsilon}^{\infty} \|k_n\|_{X_n}^p < \epsilon$$

for all $(k_n) \in K$. Also, if a subset K of $(\sum_n \bigoplus X_n)_{c_0}$ is relatively compact, then for every $\epsilon > 0$ there is a positive integer N_ϵ such that

$$\sup_{n > N_\epsilon} \|k_n\|_{X_n} < \epsilon$$

for all $(k_n) \in K$.

Now from the above fact and an argument of the proof (If M and N have the CAP (respectively BCAP), then $M \bigoplus N$ has the CAP (respectively BCAP)) of Proposition 4.1 we can easily check that the CAP and the BCAP pass through sums. More precisely, if (X_n) is a sequence of Banach spaces with the CAP, then the spaces $(\sum_n \bigoplus X_n)_{l_p}$ for every $1 \le p < \infty$ and $(\sum_n \bigoplus X_n)_{c_0}$ have the CAP. And, if $(X_k)_{k=1}^n$ is a finite sequence of Banach spaces with the BCAP, then the spaces $(\sum_{k=1}^n \bigoplus X_k)_{l_p}$ for every $1 \le p \le \infty$ have the BCAP.

Now we consider the general case when M is not necessarily complemented in X. Observe that if M is complemented in X, then $M^\perp$ is complemented in X^*. So it is reasonable for us to approach the three space problem with the weaker assumption that $M^\perp$ is complemented in X^*.

Theorem 4.2. *Let M be a closed subspace of a Banach space X such that $M^\perp$ is complemented in X^*.*

(a) *If X has the CAP and M^* has the W*D, then M has the CAP.*
(b) *If X has the BCAP and M^* has the BW*D, then M has the BCAP.*

Proof. Assume that $M^\perp$ is complemented in X^*. Then there is a projection $P : X^* \longrightarrow M^\perp$ onto $M^\perp$. Define a map $U : M^* \longrightarrow X^*$ by

$$Um^* = x^* - Px^*$$

where x^* is any linear functional in X^* with $x^* = m^*$ on M. Since P is a projection on $M^\perp$, one easily checks that U is well-defined and

$$(Um^*)m = m^*m$$

for all $m^* \in M^*$ and $m \in M$. Of course, U is a bounded operator.

Let $\iota : M \longrightarrow X$ be the inclusion.

(a). Assume that X has the CAP and M^* has the W*D. Since X has the CAP, there is a net (T_α) in $\mathcal{K}(X)$ such that $T_\alpha \xrightarrow{\tau} Id_X$, hence $T_\alpha \xrightarrow{\nu} Id_X$. By (2.1) $T_\alpha^* \xrightarrow{weak^*} Id_X^*$. Observe that $\iota^* T_\alpha^* U \in \mathcal{K}(M^*)$ and if $(m_n) \subset M$ and $(m_n^*) \subset M^*$ with $\sum_n \|m_n\| \|m_n^*\| < \infty$, then $\sum_n \|\iota\, m_n\| \|Um_n^*\| < \infty$, hence

$$\sum_n (\iota^* T_\alpha^* U m_n^*) m_n \longrightarrow \sum_n (Id_{X^*} U m_n^*) \iota\, m_n = \sum_n m_n^* m_n.$$

Thus $Id_{M^*} \in \overline{\mathcal{K}(M^*)}^{weak^*}$. Now because of the assumption that M^* has the W*D, we have $Id_{M^*} \in \overline{\mathcal{K}(M^*, w^*)}^{weak^*}$, hence $Id_M \in \overline{\mathcal{K}(M)}^{\nu}$, which proves that M has the CAP.

(b). Assume that X has the BCAP and M^* has the BW*D. Hence $Id_X \in \overline{\mathcal{K}(X, \lambda)}^{\tau}$ and $\mathcal{K}(M^*, 1) \subset \overline{\mathcal{K}(M^*, w^*, \mu)}^{weak^*}$ for some λ and $\mu > 0$. We proceed as in the proof of (a). This time we can arrange a net (T_α) in the above from $\mathcal{K}(X, \lambda)$ so that

$$\|\iota^* T_\alpha^* U\| \le \lambda \|U\|.$$

Thus we have $Id_{M^*} \in \overline{\mathcal{K}(M^*, w^*, \lambda\mu\|U\|)}^{weak^*}$, or $Id_M \in \overline{\mathcal{K}(M, \lambda\mu\|U\|)}^{\nu}$, which proves that M has the BCAP. ☐

Our last theorem is about the remaining part of the three space problem.

Theorem 4.3. *Let M be a closed subspace of a Banach space X such that $M^\perp$ is complemented in X^* and M has the BCAP.*

(a) *If X/M has the CAP and X^* has the W*D, then X has the CAP.*
(b) *If X/M has the BCAP and X^* has the BW*D, then X has the BCAP.*

Proof. Assume that $M^\perp$ is complemented in X^* and M has the BCAP. As in the proof of Theorem 4.2 we let $\iota : M \longrightarrow X$ be the inclusion, $P : X^* \longrightarrow M^\perp$ be the projection onto $M^\perp$ and $U : M^* \longrightarrow X^*$ be given by

$$Um^* = x^* - Px^*$$

where x^* is in X^* with $x^* = m^*$ on M. Recall that U is a well-defined bounded operator such that

$$(Um^*)m = m^*m$$

for all $m^* \in M^*$ and $m \in M$.

By the assumption that M has the BCAP, there is $\lambda > 0$ and a net (S_α) in $\mathcal{K}(M, \lambda)$ such that $S_\alpha \xrightarrow{\nu} Id_M$. Observe that $(US_\alpha^*\iota^*)$ is a bounded net in $\mathcal{K}(X^*)$, hence, in view of the Alaoglu theorem, it has a *weak**-cluster point W in $\mathcal{B}(X^*)$. If $x^* \in X^*$ and $m \in M$, then $(Wx^*)m$ is a cluster point of the net $((US_\alpha^*\iota^*x^*)m)$ and

$$(US_\alpha^*\iota^*x^*)m = (S_\alpha^*\iota^*x^*)m = \iota^*x^*(S_\alpha m) \longrightarrow (\iota^*x^*)m = x^*m,$$

thus we have

$$(Wx^*)m = x^*m.$$

Now define an opertor R on X^* by

$$Rx^* = Wx^* - x^*.$$

Then $R : X^* \longrightarrow M^\perp$ is a well-defined bounded operator.

For the proof of this theorem let $j : M^\perp \longrightarrow X^*$ be the inclusion map. We will identify $(X/M)^*$ with $M^\perp$ in the canonical way.

(a). Assume that X/M has the CAP and X^* has the W*D.

Observe that $Id_{X^*} = W - jR$ and notice that $W \in \overline{\mathcal{K}(X^*)}^{weak^*}$. Thus, if we show that $jR \in \overline{\mathcal{K}(X^*)}^{weak^*}$, then we have

$$Id_{X^*} \in \overline{\mathcal{K}(X^*)}^{weak^*} = \overline{\mathcal{K}(X^*, w^*)}^{weak^*}$$

because of the assumption that X^* has the W*D. This proves that X has the CAP.

It only remains to check that $jR \in \overline{\mathcal{K}(X^*)}^{weak^*}$. By the assumption that X/M has the CAP, there is a net (Q_β) in $\mathcal{K}(X/M)$ such that $Q_\beta \xrightarrow{\nu} Id_{X/M}$. Now consider the net (jQ_β^*R) in $\mathcal{K}(X^*)$. If $(x_n) \subset X$ and $(x_n^*) \subset X^*$ with $\sum_n \|x_n\|\|x_n^*\| < \infty$, then

$$\sum_n (jQ_\beta^*Rx_n^*)x_n = \sum_n (Q_\beta^*(Rx_n^*))(x_n + M) \longrightarrow \sum_n (Rx_n^*)(x_n + M) = \sum_n (jRx_n^*)x_n$$

because $Q_\beta^* \xrightarrow{weak^*} Id_{M^\perp}$ and $\sum_n \|x_n + M\|\|Rx_n^*\| < \infty$.

This proves $jQ_\beta^*R \xrightarrow{weak^*} jR$ and $jR \in \overline{\mathcal{K}(X^*)}^{weak^*}$.

(b). Assume that X/M has the BCAP and X^* has the BW*D. Thus there are $\mu, \eta > 0$ and a net (Q_β) in $\mathcal{K}(X/M, \eta)$ such that $Q_\beta \xrightarrow{\nu} Id_{X/M}$ and $\mathcal{K}(X^*, 1) \subset \overline{\mathcal{K}(X^*, w^*, \mu)}^{weak^*}$.

First, observe that $W \in \overline{\mathcal{K}(X^*, \lambda\|U\|)}^{weak^*}$. Also, if we proceed as in the proof of (a), we find that $jR \in \overline{\mathcal{K}(X^*, \eta\|R\|)}^{weak^*}$. Thus, we have

$$Id_{X^*} \in \overline{\mathcal{K}(X^*, w^*, \mu(\lambda\|U\| + \eta\|R\|))}^{weak^*},$$

which proves that X has the BCAP. □

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