WEAK AND QUASI APPROXIMATION PROPERTIES IN BANACH SPACES

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Abstract. In this paper, we introduce weak versions (the weak approximation property, the bounded weak approximation property, and the quasi approximation property) of the approximation property and derive various characterizations of these properties. And we show that if the dual of a Banach space X has the weak approximation property (respectively, the bounded weak approximation property), then X itself has the weak approximation property (respectively, the bounded weak approximation property). Also we observe that the bounded weak approximation property is closely related to the quasi approximation property.

1. Introduction and main results

We say that a Banach space X has the *approximation property* (in short, AP) if for every compact set $K \subset X$ and every $\epsilon > 0$, there is a finite rank operator $T \in \mathcal{F}(X)$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$. Also we say that a Banach space X has the λ-*bounded approximation property* (in short, λ-BAP) if for every compact set $K \subset X$ and every $\epsilon > 0$, there is a $T \in \mathcal{F}(X)$ such that $\|T\| \leq \lambda$ and $\|Tx - x\| < \epsilon$ for all $x \in K$. If X has the λ-bounded approximation property for some $\lambda > 0$, then X has the *bounded approximation property* (in short, BAP). Clearly, if a Banach space X has the BAP, then X has the AP. For the established notions of approximation properties we follow Casazza [1] and Lindenstrauss [5].

Throughout this paper, we will use the following notations :

$\mathcal{B}(X)$: The collection of bounded linear operators from X into X.
$\mathcal{F}(X)$: The collection of bounded and finite rank linear operators from X into X.
$\mathcal{K}(X)$: The collection of compact operators from X into X.

Grothendieck [2] showed the following 'beautiful' characterization of the AP :

X has the AP if and only if for every Banach space Y, every compact operator T from Y into X and every $\epsilon > 0$, there is a finite rank operator T_0 from Y into X such that $\|T_0 - T\| < \epsilon$.

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To be short, *X has the AP iff for every Banach space Y $\mathcal{K}(Y,X) \subset \overline{\mathcal{F}(Y,X)}$*, where the closure is in the operator norm topology. Here $\mathcal{K}(Y,X)$ is the collection of compact operators from Y into X. Similarly, $\mathcal{F}(Y,X)$ is defined.

Now one can consider a Banach space X satisfying the property

$$\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}. \tag{1.1}$$

Of course, Banach spaces having the AP satisfies the property (1.1). Also, in Section 3, we will introduce a topology τ on $\mathcal{B}(X)$ which is weaker than the operator norm topology; hence we are led to consider Banach space X satisfying the property

$$\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}^{\tau}. \tag{1.2}$$

Since τ is weaker than the operator norm topology, Banach spaces having the property (1.1) satisfy the property (1.2). The purpose of this paper is to research Banach spaces having the properties (1.1) and (1.2). Also we will consider a well-known open problem for the property (1.1).

Now let's formally introduce the weak versions of the approximation property.

Definition 1.1. Let X be a Banach space. We say that X has the *weak approximation property* (in short, WAP) if for every $T \in \mathcal{K}(X)$, every compact set $K \subset X$, and every $\epsilon > 0$, there is a $T_0 \in \mathcal{F}(X)$ such that $\|T_0x - Tx\| < \epsilon$ for all $x \in K$. Also we say that a Banach space X has the *bounded weak approximation property* (in short, BWAP) if for every $T \in \mathcal{K}(X)$, there exists a $\lambda_T > 0$ such that for every compact set $K \subset X$ and every $\epsilon > 0$, there is a $T_0 \in \mathcal{F}(X)$ such that $\|T_0\| \leq \lambda_T$ and $\|T_0x - Tx\| < \epsilon$ for all $x \in K$.

Clearly, if a Banach space X has the BWAP, then X has the WAP. In Section 3, we will employ a topology τ on $\mathcal{B}(X)$ which is weaker than the operator norm topology and we will see that *a Banach space X has the WAP iff $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}^{\tau}$*.

It is well-known that *if X^* has the AP (respectively, BAP), then X has the AP (respectively, BAP).* The converse is not true in general (See [1], p.275, proposition 1.4). In this paper, we obtain the same result for the WAP and BWAP by the following theorem.

Theorem 1.2. *Let X be a Banach space. If X^* has the WAP (respectively, BWAP), then X has the WAP (respectively, BWAP). Hence, if X is reflexive, then X has the WAP (respectively, BWAP) if and only if X^* has the WAP (respectively, BWAP).*

The proof of Theorem 1.2 will be given in Section 4 after we characterize Banach spaces having the WAP and BWAP in Section 3.

Definition 1.3. Let X be a Banach space. We say that X has the *quasi approximation property* (in short, QAP) if for every $T \in \mathcal{K}(X)$ and every $\epsilon > 0$, there is a $T_0 \in \mathcal{F}(X)$ such that $\|T_0 - T\| < \epsilon$.

So *a Banach space X has the QAP iff $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}$*, that is, *$X$ has the QAP iff for every $T \in \mathcal{K}(X)$, there is a sequence $(T_n) \in \mathcal{F}(X)$ such that $\|T_n - T\| \longrightarrow 0$.* Note that $\{T_n\}$ is bounded. In Section 3 (Theorem 3.9), we show that *a Banach space X has the BWAP iff for every $T \in \mathcal{K}(X)$, there exists a bounded net (T_α) in $\mathcal{F}(X)$ such that $T_\alpha x \longrightarrow Tx$ for all $x \in X$.* Thus the QAP implies the BAP. Since the AP implies the QAP, we get the following diagram :

$$BAP \Longrightarrow AP \Longrightarrow QAP \Longrightarrow BWAP \Longrightarrow WAP. \quad (1.3)$$

We will observe that not all Banach spaces (even though separable and reflexive) have the WAP in Section 2.

Long ago J. Lindenstrauss had the following question ([5], p.37, Problem 1.e.9) :

If a Banach space X has the QAP, then does X have the AP ? (1.4)

This question has not been solved yet. Trying to answer this question in Section 3, we obtain a characterization of reflexive Banach spaces having the QAP, which will enable us to prove the following results.

Theorem 1.4. *Let X be a reflexive Banach space. Then X has the QAP if and only if X^* has the QAP.*

Theorem 1.5. *Let X be a separable reflexive Banach space. Then X has the QAP if and only if X has the BWAP.*

2. Relations between approximation properties

In this section, we observe relations between various approximation properties and introduce some examples.

Example 2.1. (Banach spaces having the WAP, BWAP, QAP, and AP) Note that every Banach space X having a Schauder basis has the BAP (See Megginson [6], Theorem 4.1.33). It is well-known that the classical Banach spaces $l_p(1 \le p < \infty)$ and c_0 have the Schauder basis. Hence the diagram (1.3) says that $l_p(1 \le p < \infty)$ and c_0 have the WAP, BWAP, QAP, and AP.

We say that a Banach space X has the *compact approximation property* (in short, CAP) if for every compact set $K \subset X$ and every $\epsilon > 0$, there is a compact operator $T \in \mathcal{K}(X)$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$. Also we say that a Banach space X has the λ-*bounded compact approximation property* (in short, λ-BCAP) if for every compact set $K \subset X$ and every $\epsilon > 0$, there is a $T \in \mathcal{K}(X)$ such that $\|T\| \le \lambda$ and $\|Tx - x\| < \epsilon$ for all $x \in K$. If X has the λ-bounded compact approximation property for some $\lambda > 0$, then X has the *bounded compact approximation property* (in short, BCAP). Clearly, if a Banach space X has the BCAP, then X has the CAP. Observe that if a Banach space X has the AP (respectively, λ-BAP), then X has the CAP (respectively, λ-BCAP). Thus, using the definitions and the diagram (1.3), one can easily get the following proposition.

Proposition 2.2. Let X be a Banach space. Then the following are equivalent.

(1) X has the AP.
(2) X has both the CAP and the WAP.
(3) X has both the CAP and the BWAP.
(4) X has both the CAP and the QAP.

We need note (2)$\Longleftrightarrow$(3)$\Longleftrightarrow$(4) in Proposition 2.2. 'hitting' the CAP, then the WAP, the BWAP, and the QAP are equivalent.

Example 2.3. (A Banach space failing to have the WAP, BWAP, QAP, and AP)

There is a separable reflexive Banach space Z having the CAP but failing to have the AP (See Willis [7]). Proposition 2.3 implies that the Banach space Z does not have the WAP; hence, by the diagram (1.3) Z does not have BWAP and QAP either.

The following proposition is the bounded version of Proposition 2.2.

Proposition 2.4. Let X be a Banach space. Then the following are equivalent.

(1) X has the λ-BAP.
(2) X has both the QAP and the λ-BCAP.
(3) X has both the AP and the λ-BCAP.

Recall the question (1.4). The equivalence (2)$\Longleftrightarrow$(3) in Proposition 2.4 suggests that the AP is closely related to the QAP. But we don't know whether or not the QAP implies the AP. But with the extra assumption of the λ-BCAP the AP and the QAP are equivalent. Roughly speaking, the AP and QAP are 'weakly' equivalent.

To prove Proposition 2.4 we need the following lemma.

Lemma 2.5. *Let X be a Banach space. If X has both the QAP and λ-BCAP, then for every compact set $K \subset X$, every $\delta > 0$, and $\epsilon > 0$, there is a $T \in \mathcal{F}(X)$ such that $\|T\| \leq \lambda + \delta$ and $\|Tx - x\| < \epsilon$ for all $x \in K$.*

Proof. Let K be compact in X, $\delta > 0$, and $\epsilon > 0$. Then there is a $M > 0$ such that $K \subset MB_X$, where B_X is the unit ball in X. Since X has the λ-BCAP, there is a $S \in \mathcal{K}(X)$ such that

$$\|S\| \leq \lambda \quad \text{and} \quad \sup_{x\in K} \|Sx - x\| < \frac{\epsilon}{2}.$$

Since X has the QAP, there is a $T \in \mathcal{F}(X)$ such that

$$\|T - S\| < \min(\frac{\epsilon}{2M}, \delta).$$

Now we have

$$\|T\| \leq \lambda + \delta$$

and

$$\sup_{x\in K} \|Tx - x\| < M\|T - S\| + \frac{\epsilon}{2} < \epsilon.$$

This completes the proof. □

Proof of Proposition 2.4. Observing the diagram (1.3) and definitions, we see that it is enough to show (2)$\Longrightarrow$(1).

Now suppose (2) and let K be compact in X, $\epsilon > 0$. Let $M = \sup_{x\in K} \|x\|$ and choose $\delta > 0$ so that

$$\frac{\delta M}{\lambda + \delta} < \frac{\epsilon}{2}.$$

By Lemma 2.6 there is a $S \in \mathcal{F}(X)$ such that

$$\|S\| \leq \lambda + \delta \quad \text{and} \quad \sup_{x\in K} \|Sx - x\| < \frac{\epsilon}{2}.$$

Put $T = \frac{\lambda}{\lambda+\delta} S$. Then $\|T\| \leq \lambda$ and we have

$$\sup_{x\in K} \|Tx - x\| \leq \frac{\lambda}{\lambda + \delta} \sup_{x\in K} \|Sx - x\| + \frac{\delta}{\lambda + \delta} \sup_{x\in K} \|x\| < \epsilon.$$

Thus T is a desired finite rank operator. Hence X has the λ-BAP.

□

3. Characterizations of wap, bwap, and qap

For a Banach space X, we will employ an important topology on $\mathcal{B}(X)$. For compact $K \subset X$, $\epsilon > 0$, and $T \in \mathcal{B}(X)$ we put

$$B(T, K, \epsilon) = \{R \in \mathcal{B}(X) : \sup_{x \in K} \|Rx - Tx\| < \epsilon\}. \tag{3.1}$$

Let $\mathcal{S}$ be the collection of all such $B(T, K, \epsilon)$'s. Now we denote by τ the topology on $\mathcal{B}(X)$ generated by $\mathcal{S}$. The following lemma 3.1 comes from Grothendieck [2]. Since Lemma 3.1 is important in our characterizations, we provide our own proof.

Lemma 3.1. *Let X be a Banach space. Then the topology τ on $\mathcal{B}(X)$ is a locally convex topology and $(\mathcal{B}(X), \tau)^*$ consists of all functionals f of the form $f(T) = \sum_n x_n^*(Tx_n)$, where $(x_n) \subset X, (x_n^*) \subset X^*$ are sequences with $\sum_n \|x_n\|\|x_n^*\| < \infty$.*

Proof. For each compact $K \subset X$ consider the seminorm p_K on $\mathcal{B}(X)$ given by

$$p_K(T) = \sup\{\|Tx\| : x \in K\}.$$

The τ is also generated by the family $\{p_K : K \subset X \text{ compact}\}$ of seminorms. Hence $(\mathcal{B}(X), \tau)$ is a locally convex space. If a linear functional f on $\mathcal{B}(X)$ is given by

$$f(T) = \sum_{n=1}^{\infty} x_n^*(Tx_n)$$

where (x_n) and $(x_n^*) \subset X^*$ are sequences with $\sum_n \|x_n\|\|x_n^*\| < \infty$, then we will show that there is a compact $K \subset X$ and $B > 0$ such that

$$|f(T)| \leq Bp_K(T)$$

for all $T \in \mathcal{B}(X)$, which proves that f is τ-continuous. Indeed, we may find positive λ_n's such that

$$\lambda_n \uparrow \infty \quad \text{and} \quad \sum_n \lambda_n \|x_n\|\|x_n^*\| < \infty.$$

Of course, we may assume $x_n \neq 0$ for all n. Thus considering sequences

$$\left(\frac{x_n}{\lambda_n \|x_n\|}\right) \quad \text{and} \quad (\lambda_n \|x_n\| x_n^*)$$

instead of (x_n) and (x_n^*) we may assume that $\|x_n\| \longrightarrow 0$ and $B = \sum_n \|x_n^*\| < \infty$. Put $K = \{0\} \bigcup \{x_n : n \in \mathbf{N}\}$. Observe that K is a compact subset of X and $|f(T)| \leq Bp_K(T)$ holds for all $T \in \mathcal{B}(X)$.

Now for the converse assume that $f \in (\mathcal{B}(X), \tau)^*$. Then there are compact sets $K_1, ..., K_m$ and $\epsilon_1, ..., \epsilon_m > 0$ such that $T \in \bigcap_{i=1}^m B(0, K_i, \epsilon_i)$ implies $|f(T)| \leq 1$. Put $K = \bigcup_{i=1}^m K_i$ and $B = \max\{1/\epsilon_i : 1 \leq i \leq m\}$. Then

$$|f(T)| \leq Bp_K(T)$$

for all $T \in \mathcal{B}(X)$. Here the compact set K is contained in the closed convex hull $\overline{\text{co}}(\{x_n\})$ of a sequence (x_n) such that $\|x_n\| \longrightarrow 0$ (See [6], Lemma 3.4.30). Of course, $\overline{\text{co}}(\{x_n\})$ itself is compact by Mazur's theorem. Hence in the above inequality we may assume that $K = \overline{\text{co}}(\{x_n\})$ and that $\|x_n\| \longrightarrow 0$. Observe that $p_K(T) = \sup_n \|Tx_n\|$.

Let's consider the Banach space $c_0(X)$ of all sequences (x_n) in X which converges to 0. Here $\|(x_n)\| = \sup_n \|x_n\|$. Also consider the linear map $\varphi : \mathcal{B}(X) \longrightarrow c_0(X)$ given by $\varphi(T) = (Tx_n)$. Observe that $\varphi(T) = 0$ implies $p_K(T) = \sup_n \|Tx_n\| = 0$, hence $f(T) = 0$. Thus the linear functional $(Tx_n) \longrightarrow f(T)$ is well-defined. The above linear functional, being bounded on the subspace $\varphi(\mathcal{B}(X))$ of $c_0(X)$, can be extended to a $\psi \in c_0(X)^*$ by the Hahn-Banach theorem. So, for the scalar sequences, ψ can be represented by $(x_n^*) \in l_1(X^*)$. Thus, $\sum_n \|x_n^*\| < \infty$ and for each $T \in \mathcal{B}(X)$ we have

$$f(T) = \psi(Tx_n) = \sum_{n=1}^{\infty} x_n^*(Tx_n).$$

Since $\sum_n \|x_n\|\|x_n^*\| < \infty$, this proves our lemma.

□

We also need the following lemma in a characterization of WAP.

Lemma 3.2. *Let X be a Banach space. For every $(x_n) \subset X$ and $(x_n^*) \subset X^*$ with $\sum_n \|x_n\|\|x_n^*\| < \infty$, the following are equivalent.*

(1) *$\sum_n (x_n^* x)x_n = 0$ for all $x \in X$.*
(2) *$\sum_n x_n^*(Tx_n) = 0$ for all $T \in \mathcal{F}(X)$.*

Proof. (1)$\Longrightarrow$(2). Let $T \in \mathcal{F}(X)$. If $z_1, ..., z_m$ is a basis for $T(X)$=Z, then it has coordinate functionals $z_1^*, ..., z_m^*$ in Z^*. Then for each $x \in X$, Tx=$\sum_{k=1}^m z_k^*(Tx)z_k$. Let y_k^*=$z_k^*T \in X^*$. Then for each $x \in X$, Tx=$\sum_{k=1}^m (y_k^* x)z_k$. Hence we have

$$\sum_{n=1}^{\infty} x_n^*(Tx_n) = \sum_{n=1}^{\infty} x_n^*\Big(\sum_{k=1}^{m}(y_k^* x_n)z_k\Big) = \sum_{k=1}^{m} y_k^*\Big(\sum_{n=1}^{\infty}(x_n^* z_k)x_n\Big) = 0.$$

(2)$\Longrightarrow$(1). Let $x \in X$. For each $x^* \in X^*$, consider $T_{x^*} \in \mathcal{F}(X)$ given by

$$T_{x^*}y = (x^*y)x$$

for $y \in X$. Now for each $x^* \in X^*$ we have

$$x^*\Big(\sum_{n=1}^{\infty}(x_n^* x)x_n\Big) = \sum_{n=1}^{\infty} x_n^*(T_{x^*}x_n) = 0.$$

Hence $\sum_{n=1}^{\infty}(x_n^* x)x_n = 0$. □

By the locally convex space version of the Hahn-Banach theorem, we can get the following lemma (See [6], Corollary 2.2.20), which gives a characterization of WAP in Theorem 3.4.

Lemma 3.3. *Let X be a Banach space. Suppose that $\mathcal{Z}$ is a subspace of $\mathcal{K}(X)$. Then the following are equivalent.*

(1) *$\mathcal{Z}$ is τ-dense in $\mathcal{K}(X)$.*
(2) *For every $f \in (\mathcal{B}(X), \tau)^*$ such that $f(T) = 0$ for all $T \in \mathcal{Z}$, we have $f(T) = 0$ for all $T \in \mathcal{K}(X)$.*

Now we are ready to characterize Banach spaces having the WAP by the following theorem.

Theorem 3.4. *Let X be a Banach space. Then the following are equivalent.*

(1) *X has the WAP.*
(2) *For every $T \in \mathcal{K}(X)$, there is a net (T_α) in $\mathcal{F}(X)$ such that $T_\alpha \longrightarrow T$ in $(\mathcal{B}(X), \tau)$.*
(3) *For every $(x_n) \subset X, (x_n^*) \subset X^*$ such that $\sum_n \|x_n\|\|x_n^*\| < \infty$ and $\sum_n (x_n^* x)x_n = 0$ for all $x \in X$, we have $\sum_n x_n^*(Tx_n) = 0$ for all $T \in \mathcal{K}(X)$.*

(4) *For every* $(x_n) \subset X, (x_n^*) \subset X^*$ *such that* $\sum_n \|x_n\|\|x_n^*\| < \infty$ *and* $\sum_n x_n^*(Tx_n) = 0$ *for all* $T \in \mathcal{F}(X)$*, we have* $\sum_n x_n^*(Tx_n) = 0$ *for all* $T \in \mathcal{K}(X)$.

Proof. Note that by the definition of WAP and τ-topology, X has the WAP if and only if $\mathcal{F}(X)$ is τ-dense in $\mathcal{K}(X)$. Hence the equivalence of (1) and (2) is immediate.

(1)$\Longleftrightarrow$(4). By Lemma 3.1. and Lemma 3.3.

(3)$\Longleftrightarrow$(4). By Lemma 3.2.

□

Let X be a Banach space. Now we want to characterize when the dual X^* has the WAP. For this we consider the subspace $\mathcal{Y}_X$ of $\mathcal{B}(X^*)$ given by

$$\begin{aligned}\mathcal{Y}_X = \{T \in \mathcal{B}(X^*) : \text{ there exist } (x_k)_{k=1}^m \subset X \text{ and } (x_k^*)_{k=1}^m \subset X^* \\ \text{such that } Tx^* = \textstyle\sum_{k=1}^m x^*(x_k)x_k^* \text{ for } x^* \in X^*\}.\end{aligned} \tag{3.2}$$

The following lemma is found in Lindenstrauss [5]. For the completeness of the presentation we include our proof.

Lemma 3.5. *Let* X *be a Banach space and* $\mathcal{Y} = \mathcal{Y}_X$ *be as in (3.2). Then* $\mathcal{Y}$ *is* τ*-dense in* $\mathcal{F}(X^*)$.

Proof. It suffices to show that every $T_0 \in \mathcal{F}(X^*)$ of rank one belongs to the τ-closure of $\mathcal{Y}$. So let $x_0^* \in X^*, x_0^{**} \in X^{**}$ and write $T_0x^* = x_0^{**}(x^*)x_0^*$ for $x^* \in X^*$.

Let $K \subset X^*$ be compact and $\epsilon > 0$. First choose a small $\delta > 0$ so that $\delta\|x_0^*\|(1+2\|x_0^{**}\|) < \epsilon$. Since K is compact, there are $x_1^*, ..., x_m^*$ in K such that for each $x^* \in K$, $\|x^* - x_i^*\| < \delta$ for some $1 \le i \le m$. By Goldstine's theorem (See [6], Corollary 2.6.28) there is $x_0 \in X$ such that $\|x_0\| \le \|x_0^{**}\|$ and $\|x_i^*x_0 - x_0^{**}x_i^*\| < \delta$ for all $1 \le i \le m$. Now consider $T \in \mathcal{Y}$ given by $Tx^* = (x^*x_0)x_0^*$ for $x^* \in X^*$. Using the triangle inequality one checks that $T \in B(T_0, K, \epsilon)$, which proves the lemma.

□

Now for X^*, we can get a sharper characterization of WAP than in Theorem 3.4.

Theorem 3.6. *Let* X *be a Banach space and* $\mathcal{Y} = \mathcal{Y}_X$ *be as in (3.2). Then the following are equivalent.*

(1) X^* *has the WAP.*
(2) *For every* $T \in \mathcal{K}(X^*)$*, there is a net* $(T_\alpha) \subset \mathcal{Y}$ *such that* $T_\alpha \longrightarrow T$ *in* $(\mathcal{B}(X^*), \tau)$.

Proof. Note that by the definition of WAP, τ-topology and Lemma 3.5 X^* has the WAP if and only if $\mathcal{Y}$ is τ-dense in $\mathcal{K}(X^*)$. Hence the theorem follows. □

Now we characterize Banach spaces having the BWAP. For this we need the following two lemmas.

The following lemma 3.7 shows that the compact sets in the definition of BWAP can be replaced by finite sets.

Lemma 3.7. *Let* X *be a Banach space. Then the following are equivalent.*

(1) X *has the BWAP.*
(2) *For every* $T \in \mathcal{K}(X)$*, there is a* $\lambda_T > 0$ *such that for every finite set* $A \subset X$ *and every* $\epsilon > 0$*, there is a* $T_0 \in \mathcal{F}(X)$ *such that* $\|T_0\| \le \lambda_T$ *and* $\|T_0x - Tx\| < \epsilon$ *for all* $x \in A$.

Proof. It only requires to prove that (2)$\Longrightarrow$(1). Let $T \in \mathcal{K}(X)$ and $\lambda_T > 0$ be as in (2). Let $K \subset X$ be compact and $\epsilon > 0$. Choose $\delta > 0$ so that $(\lambda_T + \|T\|)\delta < \epsilon/2$. Since K is compact, there is a finite $A \subset K$ such that whenever $x \in K$ we have $\|x - y\| < \delta$ for some $y \in A$. Now (2) gives $T_0 \in \mathcal{F}(X)$ such that $\|T_0\| \leq \lambda_T$ and $\|T_0 x - Tx\| < \epsilon/2$ for all $x \in A$. Now an easy application of the triangle inequality gives $T_0 \in B(T, K, \epsilon)$, which completes the proof. □

Lemma 3.8. *Let X be a Banach space. Suppose that $\mathcal{C}$ is a balanced convex subset of $\mathcal{K}(X)$. Let $T \in \mathcal{K}(X)$. Then the following are equivalent.*

(1) *T belongs to the τ-closure of $\mathcal{C}$.*
(2) *For every $f \in (\mathcal{B}(X), \tau)^*$ such that $|f(S)| \leq 1$ for all $S \in \mathcal{C}$, we have $|f(T)| \leq 1$.*

Proof. (1)$\Longrightarrow$(2). By continuity.

(2)$\Longrightarrow$(1). Suppose that T dose not belongs to the τ-closure of $\mathcal{C}$.

By an application of the separation theorem (See [6], Theorem 2.2.28), there is a $f \in (\mathcal{B}(X), \tau)^*$ such that for all S in the τ-closure of $\mathcal{C}$ we have

$$\mathrm{Re} f(S) \leq 1 < \mathrm{Re} f(T).$$

Observe that $|f(S)| \leq 1$ for all S in $\mathcal{C}$ because $\mathcal{C}$ is balanced. This contradicts (2). □

Now we are ready to characterize Banach spaces having the BWAP by the following theorem.

Theorem 3.9. *Let X be a Banach space. Then the following are equivalent.*

(1) *X has the BWAP.*
(2) *For every $T \in \mathcal{K}(X)$, there exists a bounded net (T_α) in $\mathcal{F}(X)$ such that $T_\alpha x \longrightarrow Tx$ for all $x \in X$.*
(3) *For every $T \in \mathcal{K}(X)$, there is a $\lambda_T > 0$ such that for every $(x_n) \subset X, (x_n^*) \subset X^*$ such that $\sum_n \|x_n\|\|x_n^*\| < \infty$, if $|\sum_n x_n^*(Sx_n)| \leq 1$ for all S in $\{R \in \mathcal{F}(X) : \|R\| \leq \lambda_T\}$, then $|\sum_n x_n^*(Tx_n)| \leq 1$.*

Proof. Note that by the definition of BWAP and τ-topology, X has the BWAP if and only if for every $T \in \mathcal{K}(X)$, there is a $\lambda_T > 0$ such that T belongs to the τ-closure of $\{R \in \mathcal{F}(X) : \|R\| \leq \lambda_T\}$.

(1)$\Longrightarrow$(2). Clear.

(2)$\Longrightarrow$(1). Let $T \in \mathcal{K}(X)$, A be a finite set in X and $\epsilon > 0$. Then by (2) there is $\lambda_T > 0$ and a net (T_α) in $\mathcal{F}(X)$ such that $\|T_\alpha\| \leq \lambda_T$ for all α and $T_\alpha x \longrightarrow Tx$ for all $x \in X$. Hence there is a β such that $\|T_\beta x - Tx\| < \epsilon$ for all $x \in A$. Hence by Lemma 3.7, X has the BWAP.

(1)$\Longleftrightarrow$(3). Since, for $\lambda > 0$, $\{R \in \mathcal{F}(X) : \|R\| \leq \lambda\}$ is balanced and convex, Lemma 3.1 and Lemma 3.8 prove the equivalence. □

In [3], Johnson showed the following lemma. Note that w^* means w^*-topology.

Lemma 3.10. *Let X be a Banach space. Let F be a finite-dimensional Banach space, $A \subset X^*$ a finite set, $S : X^* \longrightarrow F$ a bounded linear operator, and $\epsilon > 0$. Then there is a w^*-continuous linear operator $T : X^* \longrightarrow F$ such that $Tx^* = Sx^*$ for all $x^* \in A$ and $\|T\| \leq \|S\| + \epsilon$.*

From Lemma 3.10 we obtain the following lemma which will give another characterization of BWAP for dual spaces in Theorem 3.12.

Lemma 3.11. *Let X be a Banach space and $\lambda > 0$. Then $\{S \in \mathcal{F}(X^*) : \|S\| \leq \lambda$ and S is w^*-to-w^* continuous $\}$ is τ-dense in $\{S \in \mathcal{F}(X^*) : \|S\| \leq \lambda\}$.*

Proof. Let $S \in \mathcal{F}(X^*)$ with $\|S\| \leq \lambda$, $K \subset X^*$ a compact set, and $\epsilon > 0$. Put $M = \sup_{x^* \in K} \|Sx^*\|$ and choose a $\delta > 0$ so that

$$(2\lambda + \delta)\delta < \frac{\epsilon}{2} \quad \text{and} \quad \frac{\delta M}{\lambda + \delta} < \frac{\epsilon}{2}.$$

Since K is compact, we have a finite set $A \subset K$ such that for each $x^* \in K$ there is $y^* \in A$ satisfying $\|x^* - y^*\| < \delta$.

Now by Lemma 3.10 we have a w^*-to-w^* continuous $T_1 \in \mathcal{F}(X^*)$ such that $T_1 y^* = S y^*$ for all $y^* \in A$ and $\|T_1\| \leq \|S\| + \delta$. Then one can checks that

$$\sup_{x^* \in K} \|T_1 x^* - Sx^*\| < \frac{\epsilon}{2} \quad \text{and} \quad \|T_1\| \leq \lambda + \delta.$$

Put $T = \frac{\lambda}{\lambda + \delta} T_1$. Then we have

$$\sup_{x^* \in K} \|Tx^* - Sx^*\| \leq \frac{\lambda}{\lambda + \delta} \sup_{x^* \in K} \|T_1 x^* - Sx^*\| + \frac{\delta}{\lambda + \delta} \sup_{x^* \in K} \|Sx^*\| < \epsilon.$$

Since $T \in \{S \in \mathcal{F}(X^*) : \|S\| \leq \lambda$ and S is w^*-to-w^* continuous $\}$, we get a proof of the lemma. □

Now for X^*, we get a sharper characterization of BWAP.

Theorem 3.12. *Let X be a Banach space. Then the following are equivalent.*

1. *X^* has the BWAP.*
2. *For every $T \in \mathcal{K}(X^*)$, there exists a bounded net (T_α) of w^*-to-w^* continuous finite rank operators on X^* such that $T_\alpha x^* \longrightarrow Tx^*$ for all $x^* \in X^*$.*
3. *For every $T \in \mathcal{K}(X^*)$, there is a $\lambda_T > 0$ such that for every $(x_n^*) \subset X^*, (x_n^{**}) \subset X^{**}$ such that $\sum_n \|x_n^*\| \|x_n^{**}\| < \infty$, if $|\sum_n x_n^{**}(Sx_n^*)| \leq 1$ for all S in $\{R \in \mathcal{F}(X^*) : \|R\| \leq \lambda_T$ and R is w^*-to-w^* continuous $\}$, then $|\sum_n x_n^{**}(Tx_n^*)| \leq 1$.*

Proof. Note that by the definition of BWAP and τ-topology, and Lemma 3.11, X^* has the BWAP if and only if for every $T \in \mathcal{K}(X^*)$, there is a $\lambda_T > 0$ such that T belongs to the τ-closure of $\{R \in \mathcal{F}(X^*) : \|R\| \leq \lambda_T$ and R is w^*-to-w^*continuous$\}$. This and Theorem 3.9 prove (1)$\Longleftrightarrow$(2).

(1)$\Longleftrightarrow$(3). Since, for $\lambda > 0$, $\{R \in \mathcal{F}(X^*) : \|R\| \leq \lambda$ and R is w^*-to-w^*continuous$\}$ is balanced and convex, Lemma 3.1 and Lemma 3.8 give the equivalence.

□

Now we have a characterization of reflexive Banach spaces having the QAP due to the following lemma (Kalton [4]). Recall that for a sequence $(T_n) \subset \mathcal{B}(X)$ and $T \in \mathcal{B}(X)$ $T_n \longrightarrow T$ *in the weak operator topology iff for each $x \in X$ and $x^* \in X^*$, $x^* T_n x \longrightarrow x^* T x$.*

Lemma 3.13. *Let X be a reflexive Banach space. If $T_n \longrightarrow T$ in the weak operator topology, where (T_n) is a sequence in $\mathcal{K}(X)$ and $T \in \mathcal{K}(X)$, then there is a sequence (S_n) of convex combinations of $\{T_n\}$ such that $\|S_n - T\| \longrightarrow 0$.*

Theorem 3.14. *Let X be a reflexive Banach space. Then X has the QAP if and only if for every $T \in \mathcal{K}(X)$ there is a sequence (T_n) in $\mathcal{F}(X)$ such that $x^* T_n x \longrightarrow x^* T x$ for each $x \in X$ and $x^* \in X^*$.*

Proof. Note that X has the QAP if and only if for every $T \in \mathcal{K}(X)$ there is a $(S_n) \subset \mathcal{F}(X)$ such that $\|S_n - T\| \longrightarrow 0$. Thus "only if" part is clear. Since $\mathcal{F}(X) \subset \mathcal{K}(X)$, by Lemma 3.13 we get "if" part also. □

4. Proof of main results

We start with showing that the WAP, BWAP, and QAP are inherited to the complemented subspaces.

Theorem 4.1. *Let X be a Banach space and Y a complemented subspace of X. If X has the WAP (respectively, BWAP), then Y has the WAP (respectively, BWAP).*

Proof. Suppose that X has the BWAP. Let P be a projection from X onto Y. Let $T \in \mathcal{K}(Y)$, K be a compact set in Y and $\epsilon > 0$. Since $TP \in \mathcal{K}(X)$, there is a $\lambda_{TP} > 0$ and $T_0 \in \mathcal{F}(X)$ such that $\|T_0\| \leq \lambda_{TP}$ and $\|P\| \sup_{x \in K} \|TPx - T_0 x\| < \epsilon$. Now consider $PT_0 I_Y \in \mathcal{F}(Y)$, where I_Y is the inclusion of Y into X. Put $\lambda_T = \|P\|\lambda_{TP}$. Then $\|PT_0 I_Y\| \leq \lambda_T$ and

$$\sup_{x \in K} \|Tx - PT_0 I_Y x\| = \sup_{x \in K} \|PTPx - PT_0 x\| \leq \|P\| \sup_{x \in K} \|TPx - T_0 x\| < \epsilon.$$

Hence $PT_0 I_Y$ is the desired finite-rank operator and so Y has the BWAP.

Similarly, we can argue for the WAP. □

Similarly, we can also prove the following theorem.

Theorem 4.2. *Let X be a Banach space and Y a complemented subspace of X. If X has the QAP, then Y has the QAP.*

Now we return to the proof of main results.

Proof of Theorem 1.2. Suppose that X^* has the WAP and let $(x_n) \subset X$ and $(x_n^*) \subset X^*$ satisfy that $\sum_n \|x_n\|\|x_n^*\| < \infty$ and $\sum_n (x_n^* x) x_n = 0$ for all $x \in X$. Observe $\sum_n \|x_n^*\|\|Q_X(x_n)\| < \infty$, where Q_X is the natural map from X into X^{**}. Since $\sum_n (x_n^* x) x_n = 0$ for all $x \in X$, we have

$$\Big(\sum_n Q_X(x_n) x^* \cdot x_n^*\Big) x = x^* \Big(\sum_n (x_n^* x) x_n\Big) = 0$$

for all $x \in X$ and $x^* \in X^*$; hence

$$\sum_n Q_X(x_n) x^* \cdot x_n^* = 0$$

for each $x^* \in X^*$. Since X^* has the WAP, Theorem 3.4 implies that

$$\sum_n Q_X(x_n)(Tx^*) = 0$$

for all $T \in \mathcal{K}(X^*)$. Now let $S \in \mathcal{K}(X)$. Then $S^* \in \mathcal{K}(X^*)$ (See [6], Theorem 3.4.15), where S^* is the adjoint of S; hence we have

$$\sum_n x_n^*(Sx_n) = \sum_n Q_X(x_n)(S^* x_n^*) = 0.$$

Thus, in virtue of Theorem 3.4, we conclude that X has the WAP.

Now suppose that X^* has the BWAP and let $T \in \mathcal{K}(X)$. Then $T^* \in \mathcal{K}(X^*)$. Since X^* has the BWAP, Theorem 3.12 gives a $\lambda = \lambda_{T^*} > 0$ such that for every

$(x_n^*) \subset X^*, (x_n^{**}) \subset X^{**}$ with $\sum_n \|x_n^*\|\|x_n^{**}\| < \infty$, if $|\sum_n x_n^{**}(Ux_n^*)| \leq 1$ for all U in $\{R \in \mathcal{F}(X^*) : \|R\| \leq \lambda$ and R is w^*-to-w^* continuous $\}$, then $|\sum_n x_n^{**}(T^*x_n^*)| \leq 1$.

Now suppose that $(x_n) \subset X$ and $(x_n^*) \subset X^*$ satisfy $\sum_n \|x_n\|\|x_n^*\| < \infty$. Assume $|\sum_n x_n^*(Sx_n)| \leq 1$ for all S in $\{R \in \mathcal{F}(X) : \|R\| \leq \lambda\}$. Let $U \in \mathcal{F}(X^*)$ be w^*-to-w^* continuous and $\|U\| \leq \lambda$. Then there is $S \in \mathcal{B}(X)$ such that $S^* = U$ (See [6], Theorem 3.1.11). Observe that $S \in \mathcal{F}(X)$ and $\|S\| \leq \lambda$. Thus

$$\Big|\sum_n Q_X(x_n)(Ux_n^*)\Big| = \Big|\sum_n x_n^*(Sx_n)\Big| \leq 1.$$

Then, by the condition on λ, we have

$$\Big|\sum_n x_n^*(Tx_n)\Big| = \Big|\sum_n Q_X(x_n)(T^*x_n^*)\Big| \leq 1.$$

Hence X has the BWAP by Theorem 3.9. □

Proof of Theorem 1.4. It is enough to show that if X has the QAP, then X^* has the QAP. Suppose that X has the QAP. To show that X^* has the QAP we will use Theorem 3.14. Let $T \in \mathcal{K}(X^*)$. Since X is reflexive, T is w^*-to-w^* continuous, hence $T = S^*$ for some $S \in \mathcal{K}(X)$. Since X has the QAP, Theorem 3.14 gives (S_n) in $\mathcal{F}(X)$ such that $x^*S_nx \longrightarrow x^*Sx$ for all $x \in X$ and $x^* \in X^*$. Observe that $(S_n^*) \subset \mathcal{F}(X^*)$. Now for each $x^* \in X^*$ and $x^{**} \in X^{**}$ we show that $x^{**}S_n^*x^* \longrightarrow x^{**}Tx^*$, which, along with Theorem 3.14, implies that X^* has the QAP. Indeed, since X is reflexive, for each $x^{**} \in X^{**}$ there is a $x \in X$ such that $Q_X(x) = x^{**}$, hence for each $x^* \in X^*$,

$$x^{**}Tx^* = x^{**}S^*x^* = x^*Sx$$

and

$$x^{**}S_n^*x^* = x^*S_nx,$$

from which we see that $x^{**}S_n^*x^* \longrightarrow x^{**}Tx^*$. □

In order to prove Theorem 1.5 we need the following characterization of separable Banach spaces having the BWAP.

Theorem 4.3. *Let X be a separable Banach space. Then X has the BWAP if and only if for every $T \in \mathcal{K}(X)$ there is a $(T_n) \subset \mathcal{F}(X)$ such that $\|T_nx - Tx\| \longrightarrow 0$ for each $x \in X$.*

Proof. Since X is separable, there is a compact $K \subset X$ such that $X = [K]$. Suppose that X has the BWAP and let $T \in \mathcal{K}(X)$. Then for each $n \in \mathbf{N}$ there is a $T_n \in \mathcal{F}(X)$ such that

$$\sup_{x\in K} \|T_nx - Tx\| < \frac{1}{n} \quad \text{and} \quad \sup_n \|T_n\| \leq \lambda$$

for some $\lambda > 0$. This shows $\sup_{x\in K} \|T_nx - Tx\| \longrightarrow 0$ and so $\|T_nx - Tx\| \longrightarrow 0$ for each $x \in \langle K\rangle$. Now let $x \in X$ and $\epsilon > 0$. Then there is a $x_0 \in \langle K\rangle$ such that

$$\|x - x_0\| < \min(\frac{\epsilon}{3\lambda}, \frac{\epsilon}{3\|T\|}).$$

Since $\|T_nx_0 - Tx_0\| \longrightarrow 0$, there is a $N \in \mathbf{N}$ such that $n \geq N$ implies $\|T_nx_0 - Tx_0\| < \epsilon/3$. Using the triangle inequality one can checks that $n \geq N$ implies $\|T_nx - Tx\| < \epsilon$. This proves "only if" part of the theorem. "if" part of the theorem is proved by the uniform boundedness principle and Theorem 3.9. □

Now we can prove Theorem 1.5.

Proof of Theorem 1.5. By the diagram (1.3) it is enough to show that the BWAP implies the QAP.

Now suppose that X has the BWAP and $T \in \mathcal{K}(X)$. Then by Theorem 4.3 there is a $(T_n) \subset \mathcal{F}(X)$ such that $\|T_n x - Tx\| \longrightarrow 0$ for each $x \in X$. So $x^*T_n x \longrightarrow x^*Tx$ for each $x \in X$ and x^*. Hence Theorem 3.14 says that X has the QAP. □

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