ABSOLUTE CONTINUITY OF VECTOR MEASURES AND OPERATORS

CHANGSUN CHOI AND M. DE KOCK

Abstract. In this paper we show that, for two regular Borel probabilities μ and ν on a compact Hausdorff space Ω, absolute continuity of measures $\mu \ll \nu$ is equivalent to absolute continuity of inclusion operators $[C(\Omega) \hookrightarrow L^1(\mu)] \ll [C(\Omega) \hookrightarrow L^1(\nu)]$. We then generalize this to vector valued cases.

1. Introduction

The notion of (absolute) continuity of one measure with respect to another has its roots in classical real variables, of course. The absolute continuity of point functions defined on intervals goes back at least as far as Harnack in his study of integration. However, it was G. Vitali who truly established absolutely continuous functions as objects of great importance in a sequence of fundamental properties in the first decade of the twentieth century. A decisive step in the evolution of absolute continuity was taken with the famous theorem of S. Banach and M. Zarecki to the effect that a continuous function $f : [0, 1] \to \mathbb{R}$ of bounded variation is absolutely continuous if and only if $f(E)$ has Lebesgue measure zero whenever E does. The emergence of general measure theory was accompanied by the important role to be played in that theory by absolute continuity. Already by 1930, Nikodým had established the so-called Radon-Nikodým theorem, extending earlier work of J. Radon regarding Borel measures in Euclidean spaces. The establishment by A.N. Kolmogorov of the foundation of probability theory on a measure theoretic basis and the critical role played by the Radon-Nykodým theorem in understanding conditioning ensured absolute continuity of a permanent and central place in mathematical analysis. This note concerns the absolute continuity of regular Borel measures on compact Hausdorff spaces. Our main results build on earlier work of C.P. Niculescu; the ideas of Niculescu (see [2]) were used, for example, to broach the subject of weakly compact operators on $C(K)$-spaces and their relationship to absolute summing operators in ([1], Chapter 15).

2. Preliminaries

Let (Ω, Σ) be a measurable space, X a Banach space with unit ball B_X and dual space X^*, and $F : \Sigma \to X$ a countably additive vector measure. The *semivariation* of F is the set function $\|F\|(A) = \sup\{|x^*F|(A) : x^* \in B_{X^*}\}$, where $|x^*F|$ is the variation of the scalar measure x^*F. A vector measure $F : \Sigma \to X$ is *regular* if for any $E \in \Sigma$ and $\epsilon > 0$, there exist a compact set K and an open set O such that

Date: July 11, 2005.
2000 *Mathematics Subject Classification.* Primary: 28B05; Secondary: 28A15.
Key words and phrases. Vector measure, Rybakov control measure, Radon-Nikodým theorem.
Research of the first author is supported by BK 21 project.

$\|F\|(O \setminus K) < \epsilon$, where $K \subset E \subset O$. We say that a vector measure $F : \Sigma \to X$ is *absolutely continuous* with respect to another vector measure $G : \Sigma \to Y$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \Sigma$, $\|G\|(E) < \delta$ implies $\|F\|(E) < \epsilon$. We denote it by $F \ll G$.

Let X, Y and Z be Banach spaces. Suppose $T : X \to Y$ and $S : X \to Z$ are bounded linear operators. We say that T is *absolutely continuous* with respect to S, written $T \ll S$, if given $\epsilon > 0$, there is a $k > 0$ so that for any $x \in X$

$$\|Tx\| \leq k\|Sx\| + \epsilon\|x\|.$$

A measurable function $f : \Omega \to \mathbb{R}$ is *integrable* with respect to a vector measure $F : \Sigma \to X$ if

(1) f is x^*F integrable for every $x^* \in X^*$ and
(2) for every $A \in \Sigma$ there exists an element of X, denoted by $\int_A f dF$, such that

$$x^* \int_A f dF = \int_A f dx^* F$$

for every $x^* \in X^*$.

Identifying two functions if the set where they differ has null $\|F\|$ semivariation, we obtain a linear space of classes of functions which, when endowed with the norm

$$\|f\|_{L^1(F)} = \sup \left\{ \int_\Omega |f| d|x^*F| : x^* \in B_{X^*} \right\},$$

becomes a Banach space. We denote it by $L^1(F)$ (see [4]). A *Rybakov control measure* for $F : \Sigma \to X$ is a measure $\mu = |x^*F|$, such that $\mu(E) = 0$ if and only if $\|F\|(E) = 0$. The well-known Rybakov theorem says that any countably additive vector measure has a Rybakov control measure.(see [5], Theorem IX.2.2).

3. Absolute continuity of scalar-valued measures

Theorem 1. *Let Ω be a compact Hausdorff space with Borel σ-field Σ and let μ and ν be regular Borel probabilities on Ω. Then the following are equivalent:*

(a) $[C(\Omega) \hookrightarrow L^1(\mu)] \ll [C(\Omega) \hookrightarrow L^1(\nu)]$
(b) $\mu \ll \nu$
(c) $[B(\Sigma) \hookrightarrow L^1(\mu)] \ll [B(\Sigma) \hookrightarrow L^1(\nu)]$,

where $B(\Sigma)$ is the Banach space of all bounded Borel measurable functions equipped with the supremum norm.

Proof. $(a) \Rightarrow (b)$. Suppose μ is not absolutely continuous with respect to ν. Then there exists $E \in \Sigma$ and $\epsilon > 0$ such that $\mu(E) = 3\epsilon > 0$ and $\nu(E) = 0$. Using the regularity of μ and ν, choose a compact set $K \subset E$ so that $\mu(K) > 2\epsilon$ and a sequence of open sets (O_n) such that $O_n \supseteq E$ and $\nu(O_n \setminus E) = \nu(O_n) \to 0$ as $n \to \infty$. For the above $\epsilon > 0$ our assumption (a) gives a $k > 0$ such that

$$\int_\Omega |f| d\mu \leq k \int_\Omega |f| d\nu + \epsilon \|f\|_{C(\Omega)}$$

for all $f \in C(\Omega)$. For each n, the Urysohn lemma gives $f_n \in C(\Omega)$ such that $\chi_K \leq f_n \leq \chi_{O_n}$. Then

$$\begin{aligned} 2\epsilon < \mu(K) &\leq \int_\Omega f_n d\mu \\ &\leq k \int_\Omega |f_n| d\nu + \epsilon \|f_n\|_{C(\Omega)} \\ &\leq k\nu(O_n) + \epsilon \end{aligned}$$

for all n; a contradiction.

$(b) \Rightarrow (c)$. Assume $\mu \ll \nu$, then by the Radon-Nykodým theorem there exists $h \in L^1(\nu)$ such that $0 \leq h < \infty$ and $\mu(E) = \int_E h d\nu$ for all $E \in \Sigma$. Given $\epsilon > 0$ choose $k > 0$ such that

$$\int_{\{h>k\}} h d\nu < \epsilon.$$

Then, given $f \in B(\Sigma)$,

$$\begin{aligned} \|f\|_{L^1(\mu)} = \int_\Omega |f| d\mu &= \int_\Omega |f| h d\nu \\ &= \int_{\{h \leq k\}} |f| h d\nu + \int_{\{h>k\ \}} |f| h d\nu \\ &\leq k\|f\|_{L^1(\nu)} + \epsilon \|f\|_{B(\Sigma)}. \end{aligned}$$

$(c) \Rightarrow (a)$. This is clear. □

We now generalize the above result to vector valued functions.

Theorem 2. *Let $X \neq \{0\}$ be a Banach space, Ω a compact Hausdorff space with Borel σ-field Σ, and let μ and ν be regular Borel probabilities on Ω. Then the following are equivalent:*

(a) $[C(\Omega, X) \hookrightarrow L^1(\mu, X)] \ll [C(\Omega, X) \hookrightarrow L^1(\nu, X)]$
(b) $\mu \ll \nu$
(c) $[B(\Sigma, X) \hookrightarrow L^1(\mu, X)] \ll [B(\Sigma, X) \hookrightarrow L^1(\nu, X)]$,

where $f \in B(\Sigma, X)$ if and only if $f_n \to f$ uniformly on Ω for some sequence (f_n) of Borel simple functions $f_n : \Omega \to X$.

Proof. $(a) \Rightarrow (b)$. In view of Theorem 1, it suffices to show that (a) implies

(a^*) $[C(\Omega) \hookrightarrow L^1(\mu)] \ll [C(\Omega) \hookrightarrow L^1(\nu)]$.

Pick $x_0 \in X$ with $\|x_0\| = 1$. Given $\epsilon > 0$, by (a) there is a $k > 0$ such that

$$\int_\Omega \|\mathbf{f}\| d\mu \leq k \int_\Omega \|\mathbf{f}\| d\nu + \epsilon \|\mathbf{f}\|_{C(\Omega, X)} \tag{1}$$

for all $\mathbf{f} \in C(\Omega, X)$. For each $f \in C(\Omega)$, consider $f x_0 \in C(\Omega, X)$ in (1), in order to get

$$\int_\Omega |f| d\mu \leq k \int_\Omega |f| d\nu + \epsilon \|f\|_{C(\Omega)},$$

which proves (a^*).

$(b) \Rightarrow (c)$. The proof is similar to the proof of the previous theorem. In fact, assume $\mu \ll \nu$, then there is an $0 \leq h \in L^1(\nu)$ such that $\mu(E) = \int_E h d\nu$ for all

$E \in \Sigma$. Given $\epsilon > 0$ choose $k > 0$ such that

$$\int_{\{h>k\}} h d\nu < \epsilon.$$

Then given $\mathbf{f} \in B(\Sigma, X)$,

$$\begin{aligned}
\|\mathbf{f}\|_{L^1(\mu,X)} &= \int_\Omega \|\mathbf{f}\| d\mu \\
&= \int_{\{h\le k\}} \|\mathbf{f}\| h d\nu + \int_{\{h>k\}} \|\mathbf{f}\| h d\nu \\
&\le k\|\mathbf{f}\|_{L^1(\nu)} + \epsilon \|\mathbf{f}\|_{B(\Sigma)}.
\end{aligned}$$

$(c) \Rightarrow (a)$. This is clear.

□

4. Absolute continuity of vector measures

We can extend the result in the previous section to absolutely continuous vector measures.

Theorem 3. *Let Ω be a compact Hausdorff space with Borel σ-field Σ. Let X and Y be Banach spaces and let $F : \Sigma \to X$ and $G : \Sigma \to Y$ be countably additive regular vector measures. Then the following are equivalent:*

(a) $[C(\Omega) \hookrightarrow L^1(F)] \ll [C(\Omega) \hookrightarrow L^1(G)]$
(b) $F \ll G$
(c) $[B(\Sigma) \hookrightarrow L^1(F)] \ll [B(\Sigma) \hookrightarrow L^1(G)]$,

where $B(\Sigma)$ is the Banach space of all bounded Borel measurable scalar-valued functions on Ω.

Proof. (a)⇒(b). Suppose $F \ll G$ is false. Then there is a sequence (E_n) in Σ and $\epsilon > 0$ such that

$$\|F\|(E_n) > 3\epsilon \quad \text{and} \quad \|G\|(E_n) < \frac{1}{2n}$$

for all n. By the regularity of F and G there exist compact sets (K_n) and open sets (O_n) such that $K_n \subset E_n \subset O_n$ and

$$\|F\|(E_n \setminus K_n) < \epsilon \qquad \text{and} \qquad \|G\|(O_n \setminus E_n) < \frac{1}{2n}$$

for all n. By the Urysohn lemma, we can choose $f_n \in C(\Omega)$ such that

$$\chi_{K_n} \le f_n \le \chi_{O_n}.$$

We claim that $\|f_n\|_{L^1(F)} > 2\epsilon$ and $\|f_n\|_{L^1(G)} < \frac{1}{n}$ for all n. Indeed, notice that

$$\begin{aligned}
\|f_n\|_{L^1(F)} &= \sup\left\{ \int_\Omega |f_n|\, d|x^*F| : x^* \in B_{X^*} \right\} \\
&\ge \sup\left\{ |x^*F|(K_n) : x^* \in B_{X^*} \right\} \\
&= \|F\|(K_n) \\
&\ge \|F\|(E_n) - \|F\|(E_n \setminus K_n) \\
&> 3\epsilon - \epsilon
\end{aligned}$$

follows from the subadditivity of the semivariation. That $\|f_n\|_{L^1(G)} < \frac{1}{n}$ follows in a similar fashion. By (a) we have for the above $\epsilon > 0$ a constant $k > 0$ such that

$$\|f\|_{L^1(F)} \leq k\|f\|_{L^1(G)} + \epsilon\|f\|_{C(\Omega)}$$

for all $f \in C(\Omega)$. Then it follows from our claim that

$$2\epsilon < k \cdot \frac{1}{n} + \epsilon$$

for all n; a contradiction.

(b)⇒(c). Assume $F \ll G$. By the Rybakov theorem there is a $y_0^* \in Y^*$ such that $\|y_0^*\| \leq 1$ and $G \ll |y_0^* G|$. Put $\mu = |y_0^* G|$. Since we assume $F \ll G$ we have $F \ll \mu$. Hence, given $\epsilon > 0$ there is $\delta > 0$ such that $\|F\|(E) \leq \epsilon$ whenever $E \in \Sigma$ and $\mu(E) \leq \delta$. Put $k = \dfrac{\|F\|(\Omega)}{\delta}$. Then, for each $x^* \in B_{X^*}$, since $|x^* F| \ll \mu$, writing g_{x^*} for the Radon-Nikodým derivative of $|x^* F|$ with respect to μ, we have

$$\int_{\{g_{x^*} > k\}} g_{x^*}\, d\mu = |x^* F|(\{g_{x^*} > k\}) \leq \|F\|(\{g_{x^*} > k\}) \leq \epsilon$$

because

$$\mu(\{g_{x^*} > k\}) \leq \frac{1}{k}\int_\Omega g_{x^*}\, d\mu = \frac{1}{k}|x^* F|(\Omega) \leq \frac{\|F\|(\Omega)}{k} = \delta.$$

Hence, for any $f \in B(\Sigma)$ and $x^* \in B_{X^*}$, recalling that $\mu = |y_0^* G|$ with $\|y_0^*\| \leq 1$, we have

$$\begin{aligned}
\int_\Omega |f| d|x^* F| &= \int_\Omega |f| g_{x^*}\, d\mu \\
&= \int_{\{g_{x^*} \leq k\}} |f| g_{x^*}\, d\mu + \int_{\{g_{x^*} > k\ \}} |f| g_{x^*}\, d\mu \\
&\leq k \int_\Omega |f| d\mu + \|f\|_{B(\Sigma)} \int_{\{g_{x^*} > k\ \}} g_{x^*}\, d\mu \\
&\leq k\|f\|_{L^1(G)} + \epsilon\|f\|_{B(\Sigma)}.
\end{aligned}$$

Now, taking the supremum over $x^* \in B_{X^*}$, we obtain

$$\|f\|_{L^1(F)} \leq k\|f\|_{L^1(G)} + \epsilon\|f\|_{B(\Sigma)}$$

for $f \in B(\Sigma)$. This proves the implication.

$(c) \Rightarrow (a)$. This is obvious. □

Remark. It is interesting to observe that in Theorem 3 the implication $(a) \Rightarrow (c)$ can be proved directly if one applies Lusin's theorem and Tietze's extension theorem. Indeed, given $f \in B(\Sigma)$, Lusin's theorem gives a sequence (K_n) of compact subsets of Ω such that $f|_{K_n}$ is continuous and $\eta(K_n^c) < \frac{1}{n}$, $\mu(K_n^c) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Here $\eta = |x_0^* F|$ and $\mu = |y_0^* G|$ are Rybakov measures for F and G, respectively. Then using Tietze's extension theorem one obtains a sequence (f_n) in $C(\Omega)$ such that

$$\|f_n\|_{C(\Omega)} \leq \|f\|_{B(\Sigma)}$$

and

$$\int_\Omega |f_n - f| d\eta \leq \frac{2}{n}\|f\|_{B(\Sigma)}, \qquad \int_\Omega |f_n - f| d\mu \leq \frac{2}{n}\|f\|_{B(\Sigma)}$$

holds for all n. Hence, in order to prove $(a) \Rightarrow (c)$, it suffices to check that $\|f_n\|_{L^1(F)} \to \|f\|_{L^1(F)}$ and $\|f_n\|_{L^1(G)} \to \|f\|_{L^1(G)}$ as $n \to \infty$. Let $\epsilon > 0$ and

$\delta > 0$, and $k > 0$ be as in the proof of $(b) \Rightarrow (c)$ in Theorem 3. Also let g_{x^*} be the Radon-Nikodým derivative of $|x^*F|$ with respect to η. Then

$$\begin{aligned}\int_\Omega |f_n - f| d|x^*F| &= \int_{\{g_{x^*}>k\}} |f_n - f| g_{x^*} d\eta \\ &\leq k \int_\Omega |f_n - f| d\eta + 2\|f\|_{B(\Sigma)} \cdot \int_{\{g_{x^*}>k\}} g_{x^*} d\eta \\ &\leq 2\left(\frac{k}{n} + \epsilon\right) \|f\|_{B(\Sigma)}\end{aligned}$$

for all $x^* \in B_{X^*}$. Now, taking the supremum over x^*, then taking the limit as $n \to \infty$, one gets

$$\limsup_{n\to\infty} \|f_n - f\|_{L^1(F)} \leq 2\epsilon \|f\|_{B(\Sigma)}.$$

Since $\epsilon > 0$ was arbitrary one has $|\|f_n\|_{L^1(F)} - \|f\|_{L^1(F)}| \leq \|f_n - f\|_{L^1(F)} \to 0$ as $n \to \infty$.

Acknowledgement. The authors sincerely thank Joe Diestel for valuable comments and guidance with this paper.

References

[1] J. DIESTEL, H. JARCHOW AND A. TONGE, Absolutely summing operators, *Cambridge studies in advanced mathematics*, **43** (1995).
[2] C.P. NICULESCU, Absolute continuity in Banach space theory, *Rev. Roum. Math. Pures Appl.* **24** (1979), 113-422.
[3] F. AND M. RIESZ, Über die Randwerte einer analytischen Function. *Quatrième Congrè des Math. Scand. Stockholm* (1916), 27-44.
[4] G.P. CURBERA, Banach space properties of L^1 of a vector measure, *Proc. Am. Math. Soc.* **123**(12), 3796-3806.
[5] J. DIESTEL AND J.J. UHL, Jr., Vector measures, *Amer. Math. Soc. Surveys*, Providence, RI **15** (1977), 267-269.

Division of Applied Mathematics, KAIST, Taejon 305-701, Korea
Current address: Department of Mathematics, Kent State Universtity, Kent, OH 44242, USA
E-mail address: choi@math.kent.edu

Department of Mathematics, Kent State University, Kent, OH 44242, USA
E-mail address: mdekock@math.kent.edu