CHANNELED SAMPLING IN TRANSLATION INVARIANT SUBSPACES

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Abstract. We develop single and two-channel sampling formula in the translation invariant subspaces in the multi resolution analysis $\{V_j\}$ of wavelet theory. First, we give a single channel sample formula in V_0, which extends results by G. G. Walter and W. Chen and S. Itoh. We then find necessary and sufficient conditions for two-channel sampling formula to hold in V_1.
Key words : channeled sampling, translation invariant subspace, wavelet

1. Introduction

The classical Whittaker-Shannon-Kotel'nikov(WSK) sampling theorem [4] states that if a signal $f(t)$ with finite energy is band-limited with the bandwidth π, then it can be completely reconstructed from its discrete values by the formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(t-n)}{\pi(t-n)}$$

which converges both in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$, which has been extended in many directions (e.g. [1], [6] and [8]). In 1992, G. G. Walter [9] developed a sampling theorem in wavelet subspaces, noticing that the sampling function $\sin \pi t/\pi t$ in the WSK theorem is a scaling function of a multi resolution analysis. He assumed that the scaling function $\phi(t)$ is a continuous real valued function with $\phi(t) = O(|t|^{-1-\epsilon})(\epsilon > 0)$ for $|t|$ large, which does not hold for $\sin \pi t/\pi t$. Following G. G. Walter's work, A. J. E. M. Janssen [5] used the Zak transform to generalize Walter's work. Later, W. Chen and S. Itoh [2] extended Walter's result by requiring only the condition $\{\phi(n)\} \in l^2$ on the scaling function without any decaying property. However, there were some gaps in the proof of the main result in [2].

In this work, we first re-examine the results in [2] and then extend it to single and double channel sampling formulas in the translation invariant subspaces of a multi resolution analysis.

2. Preliminaries

Definition 2.1. A function $\phi(t) \in L^2(\mathbb{R})$ is called a scaling function of a multi resolution analysis(MRA in short) $\{V_j\}$ if the closed subspaces V_j of $L^2(\mathbb{R})$,

$$V_j := \overline{\text{span}}\Big\{\phi(2^j t - k) : k \in \mathbb{Z}\Big\}, \quad j \in \mathbb{Z}$$

satisfy the following properties;

(i) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \cdots$;
(ii) $\overline{\bigcup V_j} = L^2(\mathbb{R})$;
(iii) $\bigcap V_j = \{0\}$;
(iv) $f(t) \in V_j$ if and only if $f(2t) \in V_{j+1}$;

(v) $\{\phi(t-k) : k \in \mathbb{Z}\}$ is a Riesz basis of V_0.

Then $\{\phi(2^j t-k) : k \in \mathbb{Z}\}$ becomes a Riesz basis of V_j for each j. The wavelet subspace W_j is defined to be the orthogonal complement of V_j in V_{j+1} so that

$$V_{j+1} = V_j \oplus W_j.$$

Then there exists a wavelet $\psi(t) \in L^2(\mathbb{R})$ that induces a Riesz basis $\{\psi(2^j t-k) : k \in \mathbb{Z}\}$ of W_j. Moreover, $\{\phi(2^j t-k), \psi(2^j t-k) : k \in \mathbb{Z}\}$ forms a Riesz basis of V_{j+1}.

For any $\phi(t) \in L^2(\mathbb{R})$, we let

$$\mathcal{F}(\phi)(\xi) = \hat{\phi}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(t) e^{-it\xi} dt \quad \text{and} \quad \mathcal{F}^{-1}(\hat{\phi})(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\phi}(\xi) e^{it\xi} d\xi$$

be the Fourier and inverse Fourier transforms of $\phi(t)$ and $\hat{\phi}(\xi)$ respectively. For a measurable function $f(t)$ on a set $X \subset \mathbb{R}$, we let

$$\|f(t)\|_0 := \sup_{|E|=0} \inf_{X \setminus E} |f(t)| \quad \text{and} \quad \|f(t)\|_\infty := \inf_{|E|=0} \sup_{X \setminus E} |f(t)|$$

be the *essential infimum* of $|f(t)|$ on X and the *essential supremum* of $|f(t)|$ on X respectively.

Proposition 2.1. [3] *Let $\phi(t) \in L^2(\mathbb{R})$. Then*

(i) $\{\phi(t-k) : k \in \mathbb{Z}\}$ is a Bessel sequence if and only if there is a constant $B > 0$ such that

$$\sum_k |\hat{\phi}(\xi + 2k\pi)|^2 \le B, \quad a.e. \ in \quad [0, 2\pi];$$

(ii) $\{\phi(t-k) : k \in \mathbb{Z}\}$ is a Riesz sequence if and only if there are constants $B \ge A > 0$ such that

$$A \le \sum_k |\hat{\phi}(\xi + 2k\pi)|^2 \le B, \quad a.e. \ in \quad [0, 2\pi].$$

We call A and B lower and upper Riesz bounds for a Riesz sequence $\{\phi(t-k) : k \in \mathbb{Z}\}$ respectively. For later use we give a corollary of Proposition 2.1.

Corollary 2.2. *Let $\phi(t) \in L^2(\mathbb{R})$, $M(\xi) \in L^\infty(\mathbb{R})$, and*

$$C(\phi)(t) := \mathcal{F}^{-1}(\hat{\phi} M)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\phi}(\xi) M(\xi) e^{it\xi} d\xi.$$

Then

(i) $\{C(\phi)(t-k) : k \in \mathbb{Z}\}$ is a Bessel sequence if $\{\phi(t-k) : k \in \mathbb{Z}\}$ is a Bessel sequence.

(ii) $\{C(\phi)(t-k) : k \in \mathbb{Z}\}$ is a Riesz sequence if $\{\phi(t-k) : k \in \mathbb{Z}\}$ is a Riesz sequence and $\|M(\xi)\|_0 > 0$.

Proof. (i): Let $\{\phi(t-k) : k \in \mathbb{Z}\}$ be a Bessel sequence with $\sum_k |\hat{\phi}(\xi + 2k\pi)|^2 \le B, \quad$ a.e. in $\quad [0, 2\pi]$. Then

$$\begin{aligned} \sum_k \left|\widehat{C(\phi)}(\xi + 2k\pi)\right|^2 &= \sum_k \left|\hat{\phi}(\xi + 2k\pi) M(\xi + 2k\pi)\right|^2 \\ &\le \sum_k \left|\hat{\phi}(\xi + 2k\pi)\right|^2 \left\|M(\xi)\right\|_\infty^2 \le B \left\|M(\xi)\right\|_\infty^2, \ \text{a.e. in} \ [0, 2\pi] \end{aligned}$$

so that $\{C(\phi)(t-k) : k \in \mathbb{Z}\}$ is a Bessel sequence by Proposition 2.1.

(ii): Let $\{\phi(t-k) : k \in \mathbb{Z}\}$ be a Riesz sequence with bounds A and B. Then, as in (i) we have

$$A\|M(\xi)\|_0^2 \le \sum_k \left|\widehat{C(\phi)}(\xi + 2k\pi)\right|^2 = \sum_k \left|\hat{\phi}(\xi + 2k\pi) M(\xi + 2k\pi)\right|^2 \le B\|M(\xi)\|_\infty^2$$

so that $\{C(\phi)(t-k) : k \in \mathbb{Z}\}$ is a Riesz sequence by Proposition 2.1. □

3. Single Channel Sampling in translation invariant Subspaces

In this section we give a single channel sampling in V_0, which extends results in G. G. Walter [9] and W. Chen and S. Itoh [2].

Lemma 3.1. [3] *Let $\phi(t) \in L^2(\mathbb{R})$ be such that $\{\phi(t-k) : k \in \mathbb{Z}\}$ is a Bessel sequence. Then, for any $\{c_k\} \in l^2, \sum_k c_k \phi(t-k)$ converges in $L^2(\mathbb{R})$ and*

$$\mathcal{F}\Big(\sum_k c_k \phi(t-k)\Big) = \sum_k \Big(c_k e^{-ik\xi}\hat{\phi}(\xi)\Big) = \Big(\sum_k c_k e^{-ik\xi}\Big)\hat{\phi}(\xi).$$

Let $\mathcal{F}^*$ be the discrete Fourier transform on l^p $(p = 1, 2)$ defined by $\mathcal{F}^*(\{c_k\})(\xi) := \sum_k c_k e^{-ik\xi}$. Then, $\mathcal{F}^*(\{c_k\})(\xi)$ belongs to $C[0, 2\pi]$ or $L^2[0, 2\pi]$ if $\{c_k\} \in l^1$ or l^2 respectively. We denote $\mathcal{F}^*(\{\phi(k)\})(\xi)$ by $\hat{\phi}^*(\xi)$ for $\phi(t) \in L^2(\mathbb{R})$ when $\phi(k)(k \in \mathbb{Z})$ are well defined.

Lemma 3.2. *If $\{a_k\}, \{b_k\} \in l^2$, and $\mathcal{F}^*(\{a_k\})(\xi) \in L^\infty[0, 2\pi]$, then $\big\{\sum_j a_j b_{k-j}\big\} \in l^2$ and*

$$\mathcal{F}^*(\{a_k\})(\xi)\mathcal{F}^*(\{b_k\})(\xi) = \mathcal{F}^*\Big(\Big\{\sum_j a_j b_{k-j}\Big\}\Big)(\xi).$$

Proof. Since $\mathcal{F}^*(\{a_k\})(\xi) \in L^\infty[0, 2\pi]$ and $\mathcal{F}^*(\{b_k\})(\xi) \in L^2[0, 2\pi]$, $\mathcal{F}^*(\{a_k\})(\xi)\mathcal{F}^*(\{b_k\})(\xi) \in L^2[0, 2\pi]$. Hence we can expand $\mathcal{F}^*(\{a_k\})(\xi)\mathcal{F}^*(\{b_k\})(\xi)$ into its Fourier series $\sum_n c_n e^{-in\xi}$ in $L^2[0, 2\pi]$, where

$$\begin{aligned} c_n &= \frac{1}{2\pi}\Big\langle \mathcal{F}^*(\{a_k\})(\xi)\mathcal{F}^*(\{b_k\})(\xi), e^{-in\xi}\Big\rangle_{L^2[0,2\pi]} \\ &= \frac{1}{2\pi}\Big\langle \sum_k a_k e^{-ik\xi}, \Big(\sum_k \overline{b_k} e^{ik\xi}\Big)e^{-in\xi}\Big\rangle_{L^2[0,2\pi]} \\ &= \frac{1}{2\pi}\Big\langle \sum_k a_k e^{-ik\xi}, \sum_k \overline{b_{n-k}} e^{-ik\xi}\Big\rangle_{L^2[0,2\pi]} = \sum_k a_k b_{n-k} \end{aligned}$$

by Parseval's identity. Hence the conclusion follows. □

Theorem 3.3. *Suppose that $\phi(t)$ is a scaling function for an MRA $\{V_j\}$ such that $\phi(n)$'s are well defined and $\{\phi(n)\} \in l^2$. Then, there exists $S(t) \in V_0$ such that $\{S(t-n) : n \in \mathbb{Z}\}$ is a Riesz basis of V_0 and*

$$f(t) = \sum_n f(n)S(t-n) \quad in \;\; L^2(\mathbb{R}), \quad f(t) \in V_0 \tag{3.1}$$

if and only if $0 < \|\hat{\phi}^*(\xi)\|_0 \le \|\hat{\phi}^*(\xi)\|_\infty < \infty$. *In this case, we have* $\hat{S}(\xi) = \dfrac{\hat{\phi}(\xi)}{\hat{\phi}^*(\xi)}$.

Proof. Assume $0 < \alpha := \|\hat{\phi}^*(\xi)\|_0 \le \beta := \|\hat{\phi}^*(\xi)\|_\infty < \infty$. Then $\frac{1}{|\hat{\phi}^*(\xi)|} \le \frac{1}{\alpha}$ a.e. in $[0, 2\pi]$ so that $\frac{1}{\hat{\phi}^*(\xi)} \in L^2[0, 2\pi]$. Let $\frac{1}{\hat{\phi}^*(\xi)} = \sum_k c_k e^{-ik\xi}$ be its Fourier series, where $\{c_k\} \in l^2$ and set $F(\xi) := \frac{\hat{\phi}(\xi)}{\hat{\phi}^*(\xi)}$. Then $F(\xi) \in L^2(\mathbb{R})$ and $F(\xi) = \big(\sum_k c_k e^{-ik\xi}\big)\hat{\phi}(\xi) = \sum_k \big(c_k e^{-ik\xi}\hat{\phi}(\xi)\big)$ by Lemma 3.1. Hence $S(t) := \mathcal{F}^{-1}(F)(t) = \sum_k c_k \phi(t-k) \in V_0$. Now, we show that $\{S(t-k) : k \in \mathbb{Z}\}$ is a Riesz sequence. Since $\hat{S}(\xi) = \frac{\hat{\phi}(\xi)}{\hat{\phi}^*(\xi)}$, we have

$$\frac{A_\phi}{\beta^2} \le \sum_k \big|\hat{S}(\xi + 2k\pi)\big|^2 = \frac{\sum_k \big|\hat{\phi}(\xi + 2k\pi)\big|^2}{|\hat{\phi}^*(\xi)|^2} \le \frac{B_\phi}{\alpha^2} \quad \text{a.e. in } [0, 2\pi]$$

where A_ϕ and B_ϕ are Riesz bounds for $\{\phi(t-k) : k \in \mathbb{Z}\}$. Hence $\{S(t-k) : k \in \mathbb{Z}\}$ is a Riesz sequence by Proposition 2.1 (*ii*).

For any $f(t) = \sum_k a_k \phi(t-k) \in V_0$ where $\{a_k\} \in l^2$, we have by Lemma 3.1,

$$\hat{f}(\xi) = \Big(\sum_k a_k e^{-ik\xi}\Big)\hat{\phi}(\xi) = \Big(\sum_k a_k e^{-ik\xi}\Big)\hat{\phi}^*(\xi)\hat{S}(\xi)$$

Since $\|\hat{\phi}^*(\xi)\|_\infty < \infty$,

$$\Big(\sum_k a_k e^{-ik\xi}\Big)\hat{\phi}^*(\xi) = \sum_n f(n) e^{-in\xi} \tag{3.2}$$

where $\big\{f(n) := \sum_k a_k \phi(n-k)\big\} \in l^2$ by Lemma 3.2. Hence

$$\hat{f}(\xi) = \Big(\sum_n f(n) e^{-in\xi}\Big)\hat{S}(\xi) = \sum_n \Big(f(n)e^{-in\xi}\hat{S}(\xi)\Big) \tag{3.3}$$

by Lemma 3.1 since $\{S(t-n) : n \in \mathbb{Z}\}$ is a Riesz sequence. Thus we have (3.1) by taking inverse Fourier transform on (3.3). Then $\overline{\text{span}}\{S(t-n) : n \in \mathbb{Z}\} = V_0$ so that $\{S(t-n) : n \in \mathbb{Z}\}$ is a Riesz basis of V_0.

Conversely assume that there exists $S(t) \in V_0$ such that $\{S(t-n) : n \in \mathbb{Z}\}$ is a Riesz basis of V_0 and (3.1) holds. In particular $\phi(t) = \sum_n \phi(n) S(t-n)$ so that

$$\hat{\phi}(\xi) = \sum_n \Big(\phi(n)e^{-in\xi}\hat{S}(\xi)\Big) = \Big(\sum_n \phi(n)e^{-in\xi}\Big)\hat{S}(\xi) = \hat{\phi}^*(\xi)\hat{S}(\xi). \tag{3.4}$$

Hence

$$\sum_k \big|\hat{\phi}(\xi + 2k\pi)\big|^2 = \big|\hat{\phi}^*(\xi)\big|^2 \sum_k \big|\hat{S}(\xi+2k\pi)\big|^2$$

so that

$$\frac{A_\phi}{B_S} \le \big|\hat{\phi}^*(\xi)\big|^2 \le \frac{B_\phi}{A_S} \quad \text{a.e. in } [0, 2\pi]$$

where (A_ϕ, B_ϕ) and (A_S, B_S) are Riesz bounds for $\{\phi(t-k) : k \in \mathbb{Z}\}$ and $\{S(t-k) : k \in \mathbb{Z}\}$ respectively. Thus $0 < \|\hat{\phi}^*(\xi)\|_0 \le \|\hat{\phi}^*(\xi)\|_\infty < \infty$. □

If $\{\phi(n)\} \in l^1$, then $\hat{\phi}^*(\xi) = \hat{\phi}^*(\xi + 2\pi) \in C[0, 2\pi]$ so that

$$\|\hat{\phi}^*(\xi)\|_0 = \min_{[0,2\pi]} |\hat{\phi}^*(\xi)| \quad \text{and} \quad \|\hat{\phi}^*(\xi)\|_\infty = \max_{[0,2\pi]} |\hat{\phi}^*(\xi)|.$$

Hence we have:

Corollary 3.4. *Suppose that $\phi(t)$ is a scaling function for an MRA $\{V_j\}$ such that $\phi(n)$'s are well defined and $\{\phi(n)\} \in l^1$. Then there exists $S(t) \in V_0$ such that $\{S(t-n) : n \in \mathbb{Z}\}$ is a Riesz basis of V_0 and (3.1) holds if and only if $\hat{\phi}^*(\xi) \neq 0$ in $[0, 2\pi]$.*

In [9], G. G. Walter requires that $\phi(t)$ is a continuous on $\mathbb{R}$ and $\phi(t) = O(|t|^{-1-\epsilon})(\epsilon > 0)$ for $|t|$ large. Then $\{\phi(n)\} \in l^1$ so that the results in [9] is a special case of Corollary 3.4. On the other hand, W. Chen and S. Itoh [2] claimed: under the same hypothesis as in Theorem 3.3, there exists $S(t) \in V_0$ with which (3.1) holds if and only if $\hat{\phi}^*(\xi)^{-1} \in L^2[0, 2\pi]$. However, there are some gaps in the arguments in [2]. In the proof of sufficiency for Theorem 1 in [2], $\big(\sum_k a_k e^{-ik\xi}\big)\hat{\phi}^*(\xi)$ belongs to $L^1[0, 2\pi]$ but not necessarily in $L^2[0, 2\pi]$ (unless $\|\hat{\phi}^*(\xi)\|_\infty < \infty$) so that $\{f(n)\} = \{\sum_k a_k \phi(n-k)\} \in l^\infty$ and the equation (3.2) becomes only a formal Fourier series expansion of a function in $L^1[0, 2\pi]$ (see Equation (15) in [2]). Even if $\hat{\phi}^*(\xi)^{-1} \in L^2[0, 2\pi]$ and $\|\hat{\phi}^*(\xi)\|_\infty < \infty$, (3.2) holds but (3.3) may not hold since $\{S(t-n) : n \in \mathbb{Z}\}$ is not a Bessel sequence unless $\|\hat{\phi}^*(\xi)\|_0 > 0$. Also, in the proof of necessity, we may not have (3.4) (see equation (17) in [2]) unless $\{S(t-n) : n \in \mathbb{Z}\}$ is a Riesz sequence.
We may extend Theorem 3.3 by the same reasoning to a single channel sampling as:

Theorem 3.5. *Let $M(\xi)$ be a measurable function on $\mathbb{R}$ such that $0 < \|M(\xi)\|_0 \leq \|M(\xi)\|_\infty < \infty$. Suppose that $\phi(t)$ is a scaling function for an MRA $\{V_j\}$ such that $C(\phi)(n)$'s are well defined and $\{C(\phi)(n)\} \in l^2$ where $C(\phi)(t) := \mathcal{F}^{-1}(\hat{\phi}M)(t)$. Then, there exists $S(t) \in V_0$ such that $\{S(t-n) : n \in \mathbb{Z}\}$ is a Riesz basis of V_0 and*

$$f(t) = \sum_n C(f)(n)S(t-n) \quad in \ \ L^2(\mathbb{R}), \quad f(t) \in V_0 \tag{3.5}$$

if and only if $0 < \|\widehat{C(\phi)^*}(\xi)\|_0 \leq \|\widehat{C(\phi)^*}(\xi)\|_\infty < \infty$. *In this case, we have* $\hat{S}(\xi) = \dfrac{\hat{\phi}(\xi)}{\widehat{C(\phi)^*}(\xi)}$.

Remark 3.1. If furthermore the scaling function $\phi(t)$ in Theorem 3.3 or Theorem 3.5 is piecewise continuous on $\mathbb{R}$ and $|\phi(t)| = O\big(|t|^{\frac{-1}{2}-\epsilon}\big)(\epsilon > 0)$ for $|t|$ large, then $\{\phi(n)\} \in l^2$ and V_0 becomes a reproducing kernel Hilbert space. Indeed, for any $f(t) = \sum_k a_k\phi(t-k) \in V_0$ where $\{a_k\} \in l^2$, we have

$$\begin{aligned} \big|f(t)\big| &\leq \sum_n |a_n|\big|\phi(t-n)\big| \leq \Big(\sum_n |a_n|^2\Big)^{\frac{1}{2}} \Big(\sum_n \big|\phi(t-n)\big|^2\Big)^{\frac{1}{2}} \\ &\leq \frac{\big\|f(t)\big\|_{L^2(\mathbb{R})}}{\sqrt{A}} \Big(\sum_n \big|\phi(t-n)\big|^2\Big)^{\frac{1}{2}}, \quad t \in \mathbb{R} \end{aligned}$$

where A is a lower Riesz bound for $\{\phi(t-k) : k \in \mathbb{Z}\}$. Since $\sum_n \big|\phi(t-n)\big|^2 < \infty$ for each t in $\mathbb{R}$, the point evaluation functional $l_t(f) = f(t)(t \in \mathbb{R})$ is bounded in V_0 so that V_0 is a reproducing kernel Hilbert space. Hence the sampling series (3.1) and (3.5) converge not only in $L^2(\mathbb{R})$ but also absolutely on $\mathbb{R}$.

Example 3.1. Shannon function $\phi(t) = \sin \pi t/\pi t$ is continuous on $\mathbb{R}$ and $\{\phi(n)\} = \{\delta_{n0}\} \in l^1$. Since $\hat{\phi}^*(\xi) = 1$ on $[0, 2\pi]$ but $|\phi(t)| = O(|t|^{-1})$ for $|t|$ large, the WSK sampling theorem is not covered by [2] or [9] but follows Corollary 3.4.

Example 3.2. Let $\phi(t)$ be the continuous scaling function considered by Chen and Itoh (Example 3 in [2]) such that

$$\hat{\phi}(\xi) = \begin{cases} -1, & -4\pi \leq \xi < -2\pi; \\ 1, & -2\pi \leq \xi < 0; \\ \xi^s, & 0 \leq \xi < 2\pi; \\ 0, & \text{otherwise} \end{cases}$$

with $0 < s < \frac{1}{2}$. Then we can easily see that $\phi(n) = O(\frac{1}{n})$ for $|n|$ large so that $\{\phi(n)\} \in l^2 \setminus l^1$. Even though $\hat{\phi}^*(\xi) = \xi^s$ on $[0, 2\pi]$ so that $\hat{\phi}^*(\xi) \in L^\infty[0, 2\pi]$ and $\hat{\phi}^*(\xi)^{-1} \in L^2[0, 2\pi]$, $\|\hat{\phi}^*(\xi)\|_0 = 0$ so that we can not expect a sampling formula from $\phi(t)$ suggested either by Theorem 1 in [2] or by Theorem 3.3.

Example 3.3. Let $M(\xi) = e^{-ia\xi}$ with $0 < a < 1$ so that $1 = \|e^{-ia\xi}\|_0 = \|e^{-ia\xi}\|_\infty$, and $\phi(t)$ a scaling function as in Remark 3.1. Then $C(\phi)(t) = \phi(t-a)$ and $\{\phi(n-a)\} \in l^2$ so that $Z_\phi(a, \xi) := \sum_n \phi(n-a)e^{-in\xi} \in L^2[0, 2\pi]$. Hence if $0 < \|Z_\phi(a, \xi)\|_0 \leq \|Z_\phi(a, \xi)\|_\infty < \infty$, then we obtain the shift-sampling $f(t) = \sum_n f(n-a)S(t-n)$.

4. Two-channel sampling in translation invariant subspaces

In this section we let $\phi(t)$ be a scaling function for an MRA $\{V_j\}$ and $\psi(t)$ the associated wavelet. Let $M_1(\xi)$ and $M_2(\xi)$ be in $L^\infty(\mathbb{R})$ and $C_i(f)(t) = \mathcal{F}^{-1}(\hat{f}M_i)(t)$ for $i = 1, 2$ and $f(t) \in L^2(\mathbb{R})$. Assume that $C_i(\phi)(n)$'s and $C_i(\psi)(n)$'s are well defined and $\{C_i(\phi)(n)\}$ and $\{C_i(\psi)(n)\}$ are in l^2. Let

$$A_{11}(\xi) := \sum C_1(\phi)(n)e^{-in\xi}; \quad A_{12}(\xi) := \sum C_2(\phi)(n)e^{-in\xi};$$
$$A_{21}(\xi) := \sum C_1(\psi)(n)e^{-in\xi}; \quad A_{22}(\xi) := \sum C_2(\psi)(n)e^{-in\xi},$$

and $A(\xi) := \left[A_{ij}(\xi)\right]_{i,j=1}^2$. Then $A_{ij}(\xi) \in L^2[0, 2\pi]$ and $A_{ij}(\xi) = A_{ij}(\xi + 2\pi)$. We always assume that $\|A_{ij}(\xi)\|_\infty < \infty$ for $i, j = 1, 2$ and $\det A(\xi) \neq 0$ a.e. in $[0, 2\pi]$. Set

$$A^{-1}(\xi) = B(\xi) := \left[B_{ij}(\xi)\right]_{i,j=1}^2.$$

Then $B(\xi) = B(\xi + 2\pi)$ is well defined a.e. in $\mathbb{R}$.

Lemma 4.1. *Let $\lambda_{1,B}(\xi)$ and $\lambda_{2,B}(\xi)$ be eigenvalues of $B(\xi)B(\xi)^*$ with $\lambda_{1,B}(\xi) \le \lambda_{2,B}(\xi)$. If $\|\det A(\xi)\|_0 > 0$, then*

$$0 < \|\lambda_{1,B}(\xi)\|_0 \le \|\lambda_{2,B}(\xi)\|_\infty < \infty.$$

Proof. Since $B(\xi)B(\xi)^*$ is nonsingular Hermitian a.e. in $[0, 2\pi]$,

$$0 < \lambda_{1,B}(\xi) \le \lambda_{2,B}(\xi) \quad \text{a.e. in } [0, 2\pi].$$

Since $A_{ij}(\xi) \in L^\infty[0, 2\pi]$ and $\|\det A(\xi)\|_0 > 0$, all entries of $B(\xi)$ and so $B(\xi)B(\xi)^*$ are also in $L^\infty[0, 2\pi]$ so that the characteristic equation of $B(\xi)B(\xi)^*$ is of the form

$$\lambda(\xi)^2 + f(\xi)\lambda(\xi) + g(\xi) = 0$$

where $f(\xi)$ and $g(\xi)$ are real-valued functions in $L^\infty[0, 2\pi]$. Hence, $0 < \|\lambda_{2,B}(\xi)\|_\infty < \infty$. Since

$$\lambda_{1,B}(\xi)\lambda_{2,B}(\xi) = \det[B(\xi)B(\xi)^*] = |\det A(\xi)|^{-2},$$

$$\|\det A(\xi)\|_\infty^{-2} \le \lambda_{1,B}(\xi)\lambda_{2,B}(\xi) \le \|\det A(\xi)\|_0^{-2} \quad \text{a.e. in } [0, 2\pi]$$

so that $\|\det A(\xi)\|_\infty^{-2}\|\lambda_{2,B}(\xi)\|_\infty^{-1} \le \lambda_{1,B}(\xi) \le \lambda_{2,B}(\xi) \le \|\lambda_{2,B}(\xi)\|_\infty$ a.e. in $[0, 2\pi]$. □

For any $\phi(t) \in L^2(\mathbb{R})$,

$$\|\phi\|_{L^2(\mathbb{R})}^2 = \|\hat{\phi}\|_{L^2(\mathbb{R})}^2 = \int_0^{2\pi} \sum_k \left|\hat{\phi}(\xi + 2k\pi)\right|^2 d\xi$$

so that $\{\hat{\phi}(\xi + 2k\pi)\}_{k\in\mathbb{Z}} \in l^2$ for a.e. in $[0, 2\pi]$.

Definition 4.1. For any $\phi(t)$ and $\psi(t)$ in $L^2(\mathbb{R})$, we call

$$G(\xi) := \begin{bmatrix} \sum_k |\hat{\phi}(\xi + 2k\pi)|^2 & \sum_k \hat{\phi}(\xi + 2k\pi)\overline{\hat{\psi}(\xi + 2k\pi)} \\ \overline{\sum_k \hat{\phi}(\xi + 2k\pi)\hat{\psi}(\xi + 2k\pi)} & \sum_k |\hat{\psi}(\xi + 2k\pi)|^2 \end{bmatrix}$$

the *Gramian* of $\{\phi, \psi\}$, which is well defined a.e. in $[0, 2\pi]$.

Then as a Hermitian matrix, $G(\xi)$ has real eigenvalues.

Theorem 4.2. [7] *Let $\lambda_{1,G}(\xi)$ and $\lambda_{2,G}(\xi)$ be eigenvalues of the Gramian $G(\xi)$ of $\{\phi, \psi\}$ such that $\lambda_{1,G}(\xi) \le \lambda_{2,G}(\xi)$. Then $\{\phi(t - k), \psi(t - k) : k \in \mathbb{Z}\}$ is a Riesz sequence if and only if there are constants $B \ge A > 0$ such that*

$$A \le \lambda_{1,G}(\xi) \le \lambda_{2,G}(\xi) \le B \quad \text{a.e. in } [0, 2\pi]. \tag{4.1}$$

Lemma 4.3. *Set* $\begin{bmatrix} F_1(\xi) \\ F_2(\xi) \end{bmatrix} := B(\xi) \begin{bmatrix} \hat{\phi}(\xi) \\ \hat{\psi}(\xi) \end{bmatrix}$ *on* $\mathbb{R}$. *If* $\|\det A(\xi)\|_0 > 0$, *then* $F_i(\xi) \in L^2(\mathbb{R})$, $S_i(t) := \mathcal{F}^{-1}(F_i)(t) \in V_1$ *for* $i = 1, 2$, *and* $\{S_i(t-n) : i = 1, 2$ *and* $n \in \mathbb{Z}\}$ *is a Riesz sequence.*

Proof. Since $B_{ij}(\xi) \in L^\infty(\mathbb{R})$, $F_i(\xi) = B_{i1}(\xi)\hat{\phi}(\xi) + B_{i2}(\xi)\hat{\psi}(\xi) \in L^2(\mathbb{R})$ for $i = 1, 2$. Since $B_{ij}(\xi) = B_{ij}(\xi + 2\pi) \in L^2[0, 2\pi]$, we may expand $B_{ij}(\xi)$ into its Fourier series $B_{ij}(\xi) = \sum_k b_{ij,k} e^{-ik\xi}$ where $\{b_{ij,k}\} \in l^2$. Then by Lemma 3.1,

$$\begin{aligned} F_i(\xi) &= \Big(\sum_k b_{i1,k} e^{-ik\xi}\Big)\hat{\phi}(\xi) + \Big(\sum_k b_{i2,k} e^{-ik\xi}\Big)\hat{\psi}(\xi) \\ &= \sum_k \Big(b_{i1,k} e^{-ik\xi}\hat{\phi}(\xi) + b_{i2,k} e^{-ik\xi}\hat{\psi}(\xi)\Big) \end{aligned}$$

so that

$$S_i(t) := \mathcal{F}^{-1}(F_i)(t) = \sum_k \Big(b_{i1,k}\phi(t-k) + b_{i2,k}\psi(t-k)\Big) \in V_1.$$

Let

$$S(\xi) := \begin{bmatrix} \sum_k |\hat{S}_1(\xi + 2k\pi)|^2 & \sum_k \hat{S}_1(\xi + 2k\pi)\overline{\hat{S}_2(\xi + 2k\pi)} \\ \overline{\sum_k \hat{S}_1(\xi + 2k\pi)\hat{S}_2(\xi + 2k\pi)} & \sum_k |\hat{S}_2(\xi + 2k\pi)|^2 \end{bmatrix}$$

be the Gramian of $\{S_1, S_2\}$ and $\lambda_{1,S}(\xi) \le \lambda_{2,S}(\xi)$ the eigenvalues of $S(\xi)$. Then we have by periodicity of $B(\xi)$,

$$S(\xi) = B(\xi)G(\xi)B(\xi)^*.$$

Let $U_S(\xi)$ and $U_G(\xi)$ be unitary matrices, which diagonalize $S(\xi)$ and $G(\xi)$ respectively, i.e.,

$$S(\xi) = U_S(\xi) \begin{bmatrix} \lambda_{1,S}(\xi) & 0 \\ 0 & \lambda_{2,S}(\xi) \end{bmatrix} U_S(\xi)^*$$

and

$$G(\xi) = U_G(\xi) \begin{bmatrix} \lambda_{1,G}(\xi) & 0 \\ 0 & \lambda_{2,G}(\xi) \end{bmatrix} U_G(\xi)^*.$$

Then

$$\begin{bmatrix} \lambda_{1,S}(\xi) & 0 \\ 0 & \lambda_{2,S}(\xi) \end{bmatrix} = R(\xi) \begin{bmatrix} \lambda_{1,G}(\xi) & 0 \\ 0 & \lambda_{2,G}(\xi) \end{bmatrix} R(\xi)^*$$

where

$$R(\xi) = U_S(\xi)^* B(\xi) U_G(\xi) := \begin{bmatrix} R_{11}(\xi) & R_{12}(\xi) \\ R_{21}(\xi) & R_{22}(\xi) \end{bmatrix}$$

so that

$$\lambda_{1,S}(\xi) = \lambda_{1,G}(\xi)|R_{11}(\xi)|^2 + \lambda_{2,G}(\xi)|R_{12}(\xi)|^2; \tag{4.2}$$

$$\lambda_{2,S}(\xi) = \lambda_{1,G}(\xi)|R_{21}(\xi)|^2 + \lambda_{2,G}(\xi)|R_{22}(\xi)|^2. \tag{4.3}$$

On the other hand,

$$\begin{aligned} R(\xi)R(\xi)^* &= U_S(\xi)^* B(\xi)B(\xi)^* U_S(\xi) \\ &= U_S(\xi)^* U_B(\xi) \begin{bmatrix} \lambda_{1,B}(\xi) & 0 \\ 0 & \lambda_{2,B}(\xi) \end{bmatrix} U_B(\xi)^* U_S(\xi), \end{aligned} \tag{4.4}$$

where $U_B(\xi)$ is the unitary matrix such that

$$B(\xi)B(\xi)^* = U_B(\xi) \begin{bmatrix} \lambda_{1,B}(\xi) & 0 \\ 0 & \lambda_{2,B}(\xi) \end{bmatrix} U_B(\xi)^*$$

with $\lambda_{1,B}(\xi) \leq \lambda_{2,B}(\xi)$. Set $U_S(\xi)^* U_B(\xi) = \left[D_{ij}(\xi)\right]_{i,j=1}^2$, which is also a unitary matrix. Then we have from diagonal entries of both sides of (4.4),

$$|R_{11}(\xi)|^2 + |R_{12}(\xi)|^2 = \lambda_{1,B}(\xi)|D_{11}(\xi)|^2 + \lambda_{2,B}(\xi)|D_{12}(\xi)|^2; \tag{4.5}$$

$$|R_{21}(\xi)|^2 + |R_{22}(\xi)|^2 = \lambda_{1,B}(\xi)|D_{21}(\xi)|^2 + \lambda_{2,B}(\xi)|D_{22}(\xi)|^2. \tag{4.6}$$

Then we have from (4.1), (4.2), (4.3), (4.5) and (4.6)

$$\lambda_{1,S}(\xi) \geq \lambda_{1,G}(\xi)\Big(|R_{11}(\xi)|^2 + |R_{12}(\xi)|^2\Big) \geq \lambda_{1,G}(\xi)\lambda_{1,B}(\xi) \quad \text{a.e. in } [0, 2\pi];$$

$$\lambda_{2,S}(\xi) \leq \lambda_{2,G}(\xi)\Big(|R_{21}(\xi)|^2 + |R_{22}(\xi)|^2\Big) \leq \lambda_{2,G}(\xi)\lambda_{2,B}(\xi) \quad \text{a.e. in } [0, 2\pi],$$

since $|D_{11}(\xi)|^2 + |D_{12}(\xi)|^2 = |D_{21}(\xi)|^2 + |D_{22}(\xi)|^2 = 1$ a.e. in $[0, 2\pi]$. Hence

$$0 < \|\lambda_{1,G}(\xi)\|_0 \|\lambda_{1,B}(\xi)\|_0 \leq \lambda_{1,S}(\xi) \leq \lambda_{2,S}(\xi) \leq \|\lambda_{2,G}(\xi)\|_\infty \|\lambda_{2,B}(\xi)\|_\infty < \infty \quad \text{a.e. in } [0, 2\pi]$$

by Lemma 4.1 so that $\{S_i(t-n) : i = 1, 2 \text{ and } n \in \mathbb{Z}\}$ is a Riesz sequence by Theorem 4.2 . □

Now we are ready to give the main result in this section.

Theorem 4.4. *Under the above setting, there exist* $S_i(t) \in V_1$ $(i = 1, 2)$ *such that* $\{S_i(t-n) : i = 1, 2 \text{ and } n \in \mathbb{Z}\}$ *is a Riesz basis of* V_1 *for which two-channel sampling formula*

$$f(t) = \sum_n C_1(f)(n) S_1(t-n) + \sum_n C_2(f)(n) S_2(t-n), \quad f \in V_1 \tag{4.7}$$

holds if and only if $\|\det A(\xi)\|_0 > 0$ *on* $[0, 2\pi]$. *In this case*

$$S_i(t) = \mathcal{F}^{-1}\big(B_{i1}(\xi)\hat{\phi}(\xi) + B_{i2}(\xi)\hat{\psi}(\xi)\big)(t) \quad \text{for } i = 1, 2. \tag{4.8}$$

Proof. Assume $\|\det A(\xi)\|_0 > 0$ on $[0, 2\pi]$ and define $S_i(t)$ by (4.8). Then $S_i(t) \in V_1$ $(i = 1, 2)$ and $\{S_i(t-n) : i = 1, 2 \text{ and } n \in \mathbb{Z}\}$ is a Riesz sequence by Lemma 4.3. For any $f(t) \in V_1$

$$f(t) = \sum_k c_{1,k}\phi(t-k) + \sum_k c_{2,k}\psi(t-k) \tag{4.9}$$

where $\{c_{i,k}\}_k \in l^2$ for $i = 1, 2$ since $\{\phi(t-k), \psi(t-k) : k \in \mathbb{Z}\}$ is a Riesz basis for V_1. Applying the bounded linear operator $C_i(\cdot)$ to (4.9) gives

$$C_i(f)(t) = \sum_k c_{1,k} C_i(\phi)(t-k) + \sum_k c_{2,k} C_i(\psi)(t-k). \tag{4.10}$$

On the other hand, we have by Lemma 3.1

$$\hat{f}(\xi) = \Big(\sum_k c_{1,k} e^{-ik\xi}\Big)\hat{\phi}(\xi) + \Big(\sum_k c_{2,k} e^{-ik\xi}\Big)\hat{\psi}(\xi)$$

Since $\begin{bmatrix} \hat{\phi}(\xi) \\ \hat{\psi}(\xi) \end{bmatrix} = A(\xi) \begin{bmatrix} \hat{S}_1(\xi) \\ \hat{S}_2(\xi) \end{bmatrix}$,

$$\begin{aligned} \hat{f}(\xi) &= \Big[\Big(\sum_k c_{1,k} e^{-ik\xi}\Big) A_{11}(\xi) + \Big(\sum_k c_{2,k} e^{-ik\xi}\Big) A_{21}(\xi)\Big]\hat{S}_1(\xi) \\ &\quad + \Big[\Big(\sum_k c_{1,k} e^{-ik\xi}\Big) A_{12}(\xi) + \Big(\sum_k c_{2,k} e^{-ik\xi}\Big) A_{22}(\xi)\Big]\hat{S}_2(\xi) \\ &= \sum_n \Big(\sum_k c_{1,k} C_1(\phi)(n-k) + \sum_k c_{2,k} C_1(\psi)(n-k)\Big) e^{-in\xi}\hat{S}_1(\xi) \\ &\quad + \sum_n \Big(\sum_k c_{1,k} C_2(\phi)(n-k) + \sum_k c_{2,k} C_2(\psi)(n-k)\Big) e^{-in\xi}\hat{S}_2(\xi) \\ &= \sum_n C_1(f)(n) e^{-in\xi}\hat{S}_1(\xi) + \sum_n C_2(f)(n) e^{-in\xi}\hat{S}_2(\xi) \end{aligned} \tag{4.11}$$

by (4.10), where $\{C_i(f)(n)\} \in l^2$ $(i = 1,2)$ by Lemma 3.2. Taking inverse Fourier transform on (4.11) gives (4.7), which implies $V_1 = \overline{\text{span}}\,\{S_i(t-n) : i = 1,2 \text{ and } n \in \mathbb{Z}\}$ so that $\{S_i(t-n) : i = 1,2 \text{ and } n \in \mathbb{Z}\}$ is a Riesz basis of V_1. Conversely assume that there exist $S_i(t) \in V_1$ $(i = 1,2)$ such that $\{S_i(t-n) : i = 1,2 \text{ and } n \in \mathbb{Z}\}$ is a Riesz basis of V_1 and (4.7) holds. In particular,

$$\begin{aligned}\phi(t) &= \sum_n C_1(\phi)(n)S_1(t-n) + \sum_n C_2(\phi)(n)S_2(t-n);\\ \psi(t) &= \sum_n C_1(\psi)(n)S_1(t-n) + \sum_n C_2(\psi)(n)S_2(t-n).\end{aligned}$$

By taking Fourier transform and using Lemma 3.1, we have

$$\begin{bmatrix} \hat{\phi}(\xi) \\ \hat{\psi}(\xi) \end{bmatrix} = A(\xi) \begin{bmatrix} \hat{S}_1(\xi) \\ \hat{S}_2(\xi) \end{bmatrix}.$$

We then have as in the proof of Lemma 4.3

$$G(\xi) = A(\xi)S(\xi)A(\xi)^*,$$

where $G(\xi)$ and $S(\xi)$ are Gramians of $\{\phi, \psi\}$ and $\{S_1, S_2\}$ respectively. Hence $\det G(\xi) = \det S(\xi)|\det A(\xi)|^2$ so that

$$|\det A(\xi)|^2 = \frac{\det G(\xi)}{\det S(\xi)} = \frac{\lambda_{1,G}(\xi)\lambda_{2,G}(\xi)}{\lambda_{1,S}(\xi)\lambda_{2,S}(\xi)} \geq \frac{\lambda_{1,G}(\xi)^2}{\lambda_{2,S}(\xi)^2} \quad \text{a.e. in } [0, 2\pi],$$

where $\lambda_{1,G}(\xi) \leq \lambda_{2,G}(\xi)$ and $\lambda_{1,S}(\xi) \leq \lambda_{2,S}(\xi)$ are eigenvalues of $G(\xi)$ and $S(\xi)$ respectively. Therefore,

$$|\det A(\xi)| \geq \frac{\lambda_{1,G}(\xi)}{\lambda_{2,S}(\xi)} \geq \frac{\|\lambda_{1,G}(\xi)\|_0}{\|\lambda_{2,S}(\xi)\|_\infty} \quad \text{a.e. in } [0, 2\pi]$$

so that $\|\det A(\xi)\|_0 > 0$ since both $\{\phi(t-n), \psi(t-n) : n \in \mathbb{Z}\}$ and $\{S_i(t-n) : i = 1,2 \text{ and } n \in \mathbb{Z}\}$ are Riesz sequences. □

Example 4.1. For Haar orthogonal system $\phi(t) = \chi_{[0,1)}(t)$ and $\psi(t) = \chi_{[0,\frac{1}{2})}(t) - \chi_{[\frac{1}{2},1)}(t)$. Let $M_1(\xi) = 1$ and $M_2(\xi) = e^{-ia\xi}$ with $0 < a \leq 1/2$. Then $C_1(\phi)(t) = \chi_{[0,1)}(t)$, $C_2(\phi)(t) = \chi_{[0,1)}(t-a)$, $C_1(\psi)(t) = \chi_{[0,\frac{1}{2})}(t) - \chi_{[\frac{1}{2},1)}(t)$ and $C_2(\psi)(t) = \chi_{[0,\frac{1}{2})}(t-a) - \chi_{[\frac{1}{2},1)}(t-a)$. Then

$$A(\xi) = \begin{bmatrix} 1 & e^{-i\xi} \\ 1 & -e^{-i\xi} \end{bmatrix}$$

so that $|\det A(\xi)| = 2$, which satisfies the condition of Theorem 4.4. Hence we have a sampling formula

$$f(t) = \sum_n f(n)S_1(t-n) + \sum_n f(n-a)S_2(t-n).$$

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