DUAL PROBLEMS FOR WEAK AND QUASI APPROXIMATION PROPERTIES

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It is shown that for the separable dual X^* of a Banach space X if X^* has the weak approximation property, then X has the metric quasi approximation property. Using this it is shown that for the separable dual X^* of a Banach space X the quasi approximation property and metric quasi approximation property are inherited from X^* to X and for a separable and reflexive Banach space X X having the weak approximation property, bounded weak approximation property, quasi approximation property, metric weak approximation property, and metric quasi approximation property are equivalent. Also it is shown that the weak approximation property, bounded weak approximation property, and quasi approximation property are not inherited from a Banach space X to X^*.

2000 *Mathematics Subject Classification.* Primary 46B28; Secondary 46B10

Key words and phrases. dual problem, approximation property, weak approximation property, quasi approximation property

1. Introduction and main results

Notation 1.1. Throughout this paper we use the following notations :

X : a Banach space
X^* : the dual space of X
w^* : the weak* topology on X^*
$\mathcal{B}(X)$: The collection of bounded linear operators on X.
$\mathcal{F}(X)$: The collection of bounded and finite rank linear operators on X.
$\mathcal{K}(X)$: The collection of compact operators on X.
$\mathcal{K}(X^*, w^*)$: The collection of compact and w^*-to-w^* continuous operators on X^*.
$\mathcal{K}(X, \lambda)$: The collection of compact operators T on X satisfying $\|T\| \leq \lambda$.
$\mathcal{K}(X^*, w^*, \lambda)$: The collection of compact and w^*-to-w^* continuous operators T on X^* satisfying $\|T\| \leq \lambda$.
Similarly we define $\mathcal{F}(X^*, w^*)$, $\mathcal{F}(X, \lambda)$, $\mathcal{F}(X^*, w^*, \lambda)$, $\mathcal{B}(X^*, w^*)$, $\mathcal{B}(X, \lambda)$, and $\mathcal{B}(X^*, w^*, \lambda)$.

We say that X has the *approximation property* (in short, AP) if for every compact $K \subset X$ and $\epsilon > 0$ there is a $T \in \mathcal{F}(X)$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$. Also we say that X has the λ-*bounded approximation property* (in short, λ-BAP) if for every compact $K \subset X$ and $\epsilon > 0$ there is a $T \in \mathcal{F}(X, \lambda)$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$, in particular, if $\lambda = 1$, then we say that X has the *metric approximation property* (in short, MAP). If X has the λ-bounded approximation property for some $\lambda > 0$, then we say that X has the *bounded approximation property* (in short, BAP). Recently Choi and Kim [CK] introduced weak versions of the approximation property. We say that X has the *weak approximation property* (in short, WAP) if for every $T \in \mathcal{K}(X)$, compact $K \subset X$, and $\epsilon > 0$ there is a $T_0 \in \mathcal{F}(X)$ such that $\|T_0x - Tx\| < \epsilon$ for all $x \in K$. Also we say that X has the *bounded weak approximation property* (in short, BWAP) if for every $T \in \mathcal{K}(X)$ there is a $\lambda_T > 0$ such that for every compact $K \subset X$ and $\epsilon > 0$ there is a $T_0 \in \mathcal{F}(X, \lambda_T)$ such that $\|T_0x - Tx\| < \epsilon$ for all $x \in K$. We say that X has the *quasi approximation property* (in short, QAP) if for every $T \in \mathcal{K}(X)$ and $\epsilon > 0$ there is a $T_0 \in \mathcal{F}(X)$ such that $\|T_0 - T\| < \epsilon$. We say that X has the *metric* *weak approximation property* (in short, MWAP) if for every $T \in \mathcal{K}(X,1)$, compact $K \subset X$, and $\epsilon > 0$ there is a $T_0 \in \mathcal{F}(X,1)$ such that $\|T_0x - Tx\| < \epsilon$ for all $x \in K$. We say that X has the *metric quasi approximation property* (in short, MQAP) if for every $T \in \mathcal{K}(X,1)$ and $\epsilon > 0$ there is a $T_0 \in \mathcal{F}(X,1)$ such that $\|T_0 - T\| < \epsilon$.

The purpose of this paper is to study inheritance from X (respectively, X^*) to X^* (respectively, X) (in short, dual problem) of above weak versions of the approximation property and some relations of the weak versions.

It is well known that the AP and λ-BAP are inherited from X^* to X (See Casazza [C]). In [CK] it was shown that the WAP and BWAP are inherited from X^* to X. For the MWAP we have the same result.

Theorem 1.2. *If X^* has the MWAP, then X has the MWAP. Hence, if X is reflexive, then X has the MWAP if and only if X^* has the MWAP.*

For the QAP and MQAP we need an additional assumption.

Theorem 1.3. *Suppose that X^* is separable. If X^* has the WAP, then X has the MQAP.*

From Theorem 1.3, (2.1), and (2.2) we have the following corollaries.

Corollary 1.4. *Suppose that X^* is separable. If X^* has the QAP, then X has the MQAP. In particular, if X^* has the MQAP, then X has the MQAP, and if X^* has the QAP, then X has the QAP.*

Corollary 1.5. *Suppose that X is a separable and reflexive Banach space. Then the following are equivalent.*
(a) *X has the WAP.*
(b) *X has the BWAP.*
(c) *X has the QAP.*
(d) *X has the MWAP.*
(e) *X has the MQAP.*

REMARK 1.6. Long ago Lindenstrauss and Tzafriri had the following question ([LT], p.37, Problem 1.e.9) :

If a Banach space X has the QAP, then does X have the AP ?

This question has not been solved yet. It is well-known [C] that for a reflexive Banach space X X has the AP if and only if X has the MAP. Hence if for a separable and reflexive Banach space X the above question had answer “Yes”, then X having the MAP, AP, MQAP, MWAP, QAP, BWAP, and WAP would be equivalent.

Now some parts of Corollary 1.5 need not the assumption of separability.

Corollary 1.7. *Suppose that X is a reflexive Banach space. Then the following are equivalent.*
(a) *X has the WAP.*
(b) *X has the BWAP.*
(c) *X has the MWAP.*

To prove Corollary 1.7 we need the following interesting result of Lindenstrauss [L1, Proposition 1].

Lemma 1.8. *Let X be a reflexive Banach space. If X_0 is a separable subspace of X, then there is a separable space Z satisfying $X_0 \subset Z \subset X$ such that there is a projection of norm 1 from X onto Z.*

Now we can prove Corollary 1.7.

Proof of Corollary 1.7. From (2.1) and (2.2) we only need to prove that (a) implies (c). Suppose that X has the WAP. Let $T \in \mathcal{K}(X, 1)$, compact $K \subset X$, and $\epsilon > 0$. Then the linear span $\langle T(B_X) \bigcup K \rangle$ of a relatively compact set $T(B_X) \bigcup K$ is a separable subspace of X, where B_X is the unit ball in X. By Lemma 1.8 there is a separable subspace Z of X such that $\langle T(B_X) \bigcup K \rangle \subset Z \subset X$ and there is a projection P of norm 1 from X onto Z. Since the WAP is inherited to complemented subspaces (See [CK, Theorem 4.1]), Z has the WAP. Since Z is separable and reflexive, by Corollary 1.5 Z has the MWAP. Now consider $PTI_Z \in \mathcal{K}(Z, 1)$, where I_Z is the inclusion from Z into X. Then there is a $T_0 \in \mathcal{F}(Z, 1)$ such that for all $x \in K$ $\|T_0 x - PTI_Z x\| < \epsilon$. Since $T(K) \subset \langle T(B_X) \rangle \subset Z$, for all $x \in K$

$$\|T_0 x - Tx\| < \epsilon.$$

Now $T_0 P \in \mathcal{F}(X, 1)$ and for all $x \in K$

$$\|T_0 P x - Tx\| = \|T_0 x - Tx\| < \epsilon.$$

Hence X has the MWAP. □

It is well known that the AP and BAP are not inherited from X to X^* (See [C]). For the WAP, BWAP, and QAP we have the same results.

Theorem 1.9. *There is a Banach space Y with a boundedly complete basis such that Y^* is separable and does not have the WAP. In particular Y has the WAP, BWAP, and QAP but Y^* does not have the WAP, BWAP, and QAP.*

Theorem 1.10. *There is a Banach space Z which has the AP but does not have the bounded compact approximation property such that Z^*, Z^{**}, ... are all separable and Z^* does not have the WAP. In particular Z has the WAP, BWAP, and QAP but Z^* does not have the WAP, BWAP, and QAP.*

2. Preliminaries and proofs of Theorem 1.9 and 1.10

At first, we introduce an topology on $\mathcal{B}(X)$, which is an important tool to study the approximation properties. For compact $K \subset X$, $\epsilon > 0$, and $T \in \mathcal{B}(X)$ we put

$$N(T, K, \epsilon) = \{R \in \mathcal{B}(X) : \sup_{x \in K} \|Rx - Tx\| < \epsilon\}.$$

Let $\mathcal{S}$ be the collection of all such $N(T, K, \epsilon)$'s. Now we denote by τ the topology on $\mathcal{B}(X)$ generated by $\mathcal{S}$. Grothendieck [G] initiated the study of the approximation properties and the relations between them. One important tool he used was the τ- topology. We can check that τ is a locally convex topology and for a net $(T_\alpha) \subset \mathcal{B}(X)$ and $T \in \mathcal{B}(X)$

$$T_\alpha \longrightarrow T \text{ in } (\mathcal{B}(X), \tau) \Longleftrightarrow \text{for each compact } K \subset X \ \sup_{x \in K} \|T_\alpha x - Tx\| \longrightarrow 0.$$

REMARK 2.1. From the definitions of the approximation properties and τ we see the following :
(a) X has the AP iff $I \in \overline{\mathcal{F}(X)}^\tau$, where I is the identity in $\mathcal{B}(X)$.
(b) X has the λ-BAP iff $I \in \overline{\mathcal{F}(X, \lambda)}^\tau$.
(c) X has the WAP iff $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}^\tau$.
(d) X has the BWAP iff for every $T \in \mathcal{K}(X)$ there is a $\lambda_T > 0$ such that $T \in \overline{\mathcal{F}(X, \lambda_T)}^\tau$.
(e) X has the QAP iff $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}$, where the closure is the operator norm closure.
(f) X has the MWAP iff $\mathcal{K}(X, 1) \subset \overline{\mathcal{F}(X, 1)}^\tau$.
(g) X has the MQAP iff $\mathcal{K}(X, 1) \subset \overline{\mathcal{F}(X, 1)}$.

In [CK] the following implications are shown :

$$BAP \Longrightarrow AP \Longrightarrow QAP \Longrightarrow BWAP \Longrightarrow WAP \tag{2.1}$$

Proposition 2.2 yields the following implications :

$$MAP \Longrightarrow MQAP \Longrightarrow MWAP$$
$$MQAP \Longrightarrow QAP \text{ and } MWAP \Longrightarrow BWAP \tag{2.2}$$

Proposition 2.2.
(a) *If X has the MAP, then X has the MQAP, and if X has the MQAP, then X has the MWAP.*
(b) *If X has the MQAP, then X has the QAP, and if X has the MWAP, then X has the BWAP.*

Proof. (a). Suppose that X has the MAP and let $T \in \mathcal{K}(X,1)$ and $\epsilon > 0$. Since $\overline{T(B_X)}$ is compact, there is a $T_0 \in \mathcal{F}(X,1)$ such that

$$\|T_0T - T\| = \sup_{x\in\overline{T(B_X)}} \|T_0x - x\| < \epsilon.$$

Since $T_0T \in \mathcal{F}(X,1)$, X has the MQAP. If X has the MQAP, then from Remark 2.1, (f) and (g) X has the MWAP.

(b). Note that X has the MQAP iff $\mathcal{K}(X,1) \subset \overline{\mathcal{F}(X,1)}$ iff $\lambda\mathcal{K}(X,1) \subset \lambda\overline{\mathcal{F}(X,1)}$ for each $\lambda > 0$ iff $\mathcal{K}(X,\lambda) \subset \overline{\mathcal{F}(X,\lambda)}$ for each $\lambda > 0$. Suppose that X has the MQAP and let $T \in \mathcal{K}(X)$. Then $T \in \mathcal{K}(X,\|T\|) \subset \overline{\mathcal{F}(X,\|T\|)} \subset \overline{\mathcal{F}(X)}$. Hence X has the QAP by Remark 2.1, (e). Other part is similar. $\square$

The following lemma can be found in [C, Proposition 1.3] which is due to Lindenstrauss [L2].

Lemma 2.3. *If V is a separable Banach space, then there is a separable Banach space W such that W^{**} has a boundedly complete basis, $W^{**}/W \cong V$, and $W^{***} \cong W^* \bigoplus V^*$.*

Now we can prove Theorem 1.9.

Proof of Theorem 1.9. Let V be the Willis space (See Willis [W]). Then V is a separable and reflexive Banach space and does not have the WAP [CK, Example 2.3]. By Lemma 2.3 there is a separable Banach space W such that W^{**} has a boundedly complete basis and $W^{***} \cong W^* \bigoplus V^*$. Let $Y = W^{**}$. Since W^* and V^* are separable, $Y^* = W^{***}$ is separable. Suppose that Y^* has the WAP. Since the WAP is inherited to complemented subspaces [CK, Theorem 4.1], V^* has the WAP. So V has the WAP. This is a contradiction, that is, Y^* should not have the WAP. Hence Y is a desired Banach space. Since a Banach apace having a basis has the AP, (2.1) shows other part of the theorem. $\square$

We say that X has the *compact approximation property* (in short, CAP) if for every compact $K \subset X$ and $\epsilon > 0$ there is a $T \in \mathcal{K}(X)$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$. Also we say that X has the λ*-bounded compact approximation property*(in short, λ-BCAP) if for every compact $K \subset X$ and $\epsilon > 0$ there is a $T \in \mathcal{K}(X,\lambda)$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$, in particular, if $\lambda = 1$, then we say that X has the *metric compact approximation property* (in short, MCAP). If X has the λ-bounded compact approximation property for some $\lambda > 0$, then we say that X has the *bounded compact approximation property* (in short, BCAP). From the definitions of the CAP, BCAP, and τ we see the following :

$$X \text{ has the CAP iff } I \in \overline{\mathcal{K}(X)}^{\tau}$$

and

$$X \text{ has the } \lambda\text{-BCAP iff } I \in \overline{\mathcal{K}(X,\lambda)}^{\tau}. \tag{2.3}$$

We need a lemma of Kim [Ki1].

Lemma 2.4. *Suppose that X^* is separable. Then X^* has the WAP if and only if X^* has the MWAP.*

We also need a lemma due to Casazza and Jarchow [CJ, Theorem 2.5].

Lemma 2.5. *There is a Banach space Z which has the AP but does not have the BCAP such that Z^*, Z^{**}, ... are all separable and have the MCAP.*

Now we can prove Theorem 1.10.

Proof of Theorem 1.10. Let Z be the Banach space of Lemma 2.5. Suppose that Z^* has the WAP. Then by Lemma 2.4 Z^* has the MWAP. From Remark 2.1,(f) and (2.3)

$$I \in \overline{\mathcal{F}(Z^*, 1)}^{\tau}.$$

, where I is the identity in $\mathcal{B}(Z^*)$. It follows that Z^* has the MAP. Thus Z has the MAP. Since the MAP implies the MCAP, this is a contradiction. Hence Z is a desired Banach space. (2.1) shows other part of the theorem. □

3. Proofs of Theorem 1.2 and 1.3

The following lemma is in [CK, Lemma 3.11] which is essentially due to Johnson [J, Lemma 1].

Lemma 3.1. *For each $\lambda > 0$ $\overline{\mathcal{F}(X^*, \lambda)}^{\tau} = \overline{\mathcal{F}(X^*, w^*, \lambda)}^{\tau}$.*

The following lemma comes from [Ki3].

Lemma 3.2. *If $\mathcal{C}$ is a bounded convex set in $\mathcal{B}(X)$, then $\overline{\mathcal{C}}^{\tau} = \overline{\mathcal{C}}^{wo}$ where wo means the weak operator topology on $\mathcal{B}(X)$.*

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. Suppose that X^* has the MWAP and let $T \in \mathcal{K}(X, 1)$. Then $T^* \in \mathcal{K}(X^*, 1)$ where T^* is the adjoint of T. Since X^* has the MWAP, By Remark 2.1,(f) and Lemma 3.1

$$T^* \in \overline{\mathcal{F}(X^*, w^*, 1)}^{\tau}.$$

Then there is a net (T_α) in $\mathcal{F}(X^*, w^*, 1)$ such that $T_\alpha \longrightarrow T^*$ in $(\mathcal{B}(X^*), \tau)$. Since each T_α is w^*-to-w^* continuous and $\|T_\alpha\| \le 1$, for each α there is a $S_\alpha \in \mathcal{F}(X, 1)$ such that $S_\alpha^* = T_\alpha$. So $S_\alpha^* \longrightarrow T^*$ in $(\mathcal{B}(X^*), \tau)$. In particular, for each $x \in X$ and $x^* \in X^*$

$$x^* S_\alpha x \longrightarrow x^* T x.$$

Thus $S_\alpha \longrightarrow T$ in $(\mathcal{B}(X), wo)$, where wo means the weak operator topology on $\mathcal{B}(X)$. By Lemma 3.2 $T \in \overline{co(\{S_\alpha\})}^{\tau} \subset \overline{\mathcal{F}(X, 1)}^{\tau}$. We have shown that $\mathcal{K}(X, 1) \subset \overline{\mathcal{F}(X, 1)}^{\tau}$. Hence X has the MWAP by Remark 2.1,(f). □

We need a result due to Kalton [K, Corollary 3].

Lemma 3.3. *Suppose that (T_n) is a sequence in $\mathcal{K}(X)$ and $T \in \mathcal{K}(X)$. If for each $x^* \in X^*$ and $x^{**} \in X^{**}$ $x^{**} T_n^* x^* \longrightarrow x^{**} T^* x^*$, then there is a sequence (S_n) of convex combinations of $\{T_n\}$ such that $\|S_n - T\| \longrightarrow 0$.*

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. Suppose that X^* has the WAP and let $T \in \mathcal{K}(X, 1)$. Then by Lemma 2.4 X^* has the MWAP since X^* is separable. Since $T^* \in \mathcal{K}(X^*, 1)$, by Lemma 3.1

$$T^* \in \overline{\mathcal{F}(X^*, w^*, 1)}^{\tau}.$$

If a Banach space Y is separable, then for each bounded subset $\mathcal{A}$ of $\mathcal{B}(Y)$ the relative τ-topology of $\mathcal{A}$ has a countable basic neighborhood [Ki2, Theorem 1.18]. Thus there is a sequence (T_n) in $\mathcal{F}(X^*, w^*, 1)$ such that

$$T_n \longrightarrow T^*$$

in $(\mathcal{B}(X^*), \tau)$. Since each T_n is w^*-to-w^* and $\|T_n\| \leq 1$, for each n there is a $S_n \in \mathcal{F}(X, 1)$ such that $S_n^* = T_n$. So $S_n^* \longrightarrow T^*$ in $(\mathcal{B}(X^*), \tau)$. In particular, for each $x^* \in X^*$ and $x^{**} \in X^{**}$

$$x^{**} S_n^* x^* \longrightarrow x^{**} T^* x^*.$$

By Lemma 3.3 there is a sequence (R_n) of convex combinations of $\{S_n\}$ such that

$$\|R_n - T\| \longrightarrow 0$$

and $(R_n) \subset \mathcal{F}(X, 1)$. This shows $T \in \overline{\mathcal{F}(X, 1)}$. We have shown that $\mathcal{K}(X, 1) \subset \overline{\mathcal{F}(X, 1)}$. Hence X has the MQAP by Remark 2.1,(g). □

4. Acknowledgement

The referee gives the author important comments on Corollary 1.7. The author deeply thanks the referee for his (or her) very helpful comments. The author also thanks Professor C. Choi for his kind comments on this paper.

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The author was supported by BK21 project.