ON RELATIONS BETWEEN WEAK APPROXIMATION PROPERTIES AND THEIR INHERITANCES TO SUBSPACES

JU MYUNG KIM

Abstract. It is shown that for the separable dual X^* of a Banach space X, if X^* has the weak approximation property, then X^* has the metric weak approximation property. We introduce the properties W*D and MW*D for Banach spaces. Suppose that M is a closed subspace of a Banach space X such that $M^\perp$ is complemented in the dual space X^*, where $M^\perp = \{x^* \in X^* : x^*(m) = 0 \text{ for all } m \in M\}$. Then it is shown that if a Banach space X has the weak approximation property and W*D (respectively, metric weak approximation property and MW*D), then M has the weak approximation property (respectively, bounded weak approximation property).

1. Introduction and main results

A Banach space X is said to have the *approximation property* (in short, AP) if for every compact $K \subset X$ and $\epsilon > 0$, there is a bounded and finite rank operator T on X such that $\|Tx - x\| < \epsilon$ for all $x \in K$. Also a Banach space X is said to have the λ-*bounded approximation property* (in short, λ-BAP) if for every compact $K \subset X$ and $\epsilon > 0$, there is a bounded and finite rank operator T on X with $\|T\| \leq \lambda$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$. In particular, if $\lambda = 1$, then we say that X has the *metric approximation property* (in short, MAP). If X has the λ-bounded approximation property for some $\lambda > 0$, then we say that X has the *bounded approximation property* (in short, BAP). The AP, already appeared in Banach's book [1], is one of the fundamental properties in the Banach space theory. Grothendieck [4] initiated the investigation of the variants of the AP and Casazza [2] summarized various results on the AP. Recently Choi and Kim [3] introduced weak versions of the AP. A Banach space X is said to have the *weak approximation property* (in short, WAP) if for every compact operator T on X, compact $K \subset X$, and $\epsilon > 0$, there is a bounded and finite rank operator T_0 on X such that $\|T_0x - Tx\| < \epsilon$ for all $x \in K$. A Banach space X is said to have the *bounded weak approximation property* (in short, BWAP) if for every compact operator T on X, there is a $\lambda_T > 0$ such that for every compact $K \subset X$ and $\epsilon > 0$, there is a bounded and finite rank operator T_0 on X such that $\|T_0\| \leq \lambda_T$ and $\|T_0x - Tx\| < \epsilon$ for all $x \in K$. In the definition of the BWAP, if for every compact operator T on X with $\|T\| \leq 1$, $\lambda_T = 1$, then we say that X has the *metric weak approximation property* (in short, MWAP). A Banach space X is said to have the *quasi approximation property* (in short, QAP) if for every compact operator T

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on X and $\epsilon > 0$, there is a bounded and finite rank operator T_0 on X such that $\|T_0 - T\| < \epsilon$. In [3] the authors observed the following implications :

$$BAP \Longrightarrow AP \Longrightarrow QAP \Longrightarrow BWAP \Longrightarrow WAP.$$

To simplify above definitions, we need a topology on $\mathcal{B}(X)$, the Banach space of all bounded operators on a Banach space X. For compact $K \subset X$, $\epsilon > 0$, and $T \in \mathcal{B}(X)$, we put

$$N(T, K, \epsilon) = \{R \in \mathcal{B}(X) : \sup_{x \in K} \|Rx - Tx\| < \epsilon\}.$$

Let $\mathcal{S}$ be the collection of all such $N(T, K, \epsilon)$'s. We denote by τ the topology on $\mathcal{B}(X)$ generated by $\mathcal{S}$. It is easy to check that τ is a locally convex topology and for a net (T_α) and T in $\mathcal{B}(X)$,

$$T_\alpha \xrightarrow{\tau} T \Longleftrightarrow \text{for each compact } K \subset X \ \sup_{x \in K} \|T_\alpha x - Tx\| \longrightarrow 0.$$

Let X be a Banach space and $\lambda > 0$. Throughout this paper, we use the following notations :

T^* : The adjoint of an operator T.

w^* : The *weak** topology on the dual space X^* of X.

$\mathcal{F}(X)$: The collection of bounded and finite rank operators on X.

$\mathcal{K}(X)$: The collection of compact operators on X.

$\mathcal{K}(X^*, w^*)$: The collection of compact and w^*-to-w^* continuous operators on X^*.

$\mathcal{K}(X, \lambda)$: The collection of compact operators T on X satisfying $\|T\| \le \lambda$.

$\mathcal{K}(X^*, w^*, \lambda)$: The collection of compact and w^*-to-w^* continuous operators T on X^* satisfying $\|T\| \le \lambda$.

We similarly define $\mathcal{F}(X^*, w^*)$, $\mathcal{F}(X, \lambda)$, and $\mathcal{F}(X^*, w^*, \lambda)$.

Notice that every w^*-to-w^* continuous operator on X^* is an adjoint operator.

The next remark follows from the definitions.

Remark 1.1. Let X be a Banach space.

(a) X has the AP iff $I_X \in \overline{\mathcal{F}(X)}^\tau$, where I_X is the identity in $\mathcal{B}(X)$.

(b) X has the λ-BAP iff $I_X \in \overline{\mathcal{F}(X, \lambda)}^\tau$.

(c) X has the WAP iff $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}^\tau$.

(d) X has the BWAP iff for every $T \in \mathcal{K}(X)$, there is a $\lambda_T > 0$ such that $T \in \overline{\mathcal{F}(X, \lambda_T)}^\tau$.

(e) X has the MWAP iff $\mathcal{K}(X, 1) \subset \overline{\mathcal{F}(X, 1)}^\tau$.

(f) X has the QAP iff $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}$, where the closure is the operator norm closure.

Now we state main results in this paper.

Theorem 1.2. *Let X be a Banach space such that X^* is separable. If X^* has the WAP, then X^* has the MWAP.*

The proof of Theorem 1.2 is presented in Section 2. If a Banach space X has the MWAP, then X has the BWAP. In fact, if X has the MWAP, then for every $T \in \mathcal{K}(X)$,

$$T \in \mathcal{K}(X, \|T\|) = \|T\|\mathcal{K}(X, 1) \subset \|T\|\overline{\mathcal{F}(X, 1)}^\tau = \overline{\mathcal{F}(X, \|T\|)}^\tau.$$

Hence X has the BWAP. Since clearly the BWAP implies the WAP, we have the following corollary.

Corollary 1.3. Let X be a Banach space such that X^* is separable. Then the following are equivalent.
(a) X^* has the WAP.
(b) X^* has the BWAP.
(c) X^* has the MWAP.

In [3], by simple calculations it was shown that the WAP, BWAP, and QAP are inherited to the complemented subspaces. But their inheritances to general subspaces are much harder. Suppose that M is a closed subspace of a Banach space X and $M^\perp = \{x^* \in X^* : x^*(m) = 0 \text{ for all } m \in M\}$. It is a well known fact that if M is complemented in X, then $M^\perp$ is complemented in X^*. But the converse is false in general. In this paper we are concerned with the subspace M such that $M^\perp$ is complemented in X^*.

Theorem 1.4. *Suppose that M is a closed subspace of a Banach space X and $M^\perp$ is complemented in X^*.*
(a) *If X has the WAP and W*D, then M has the WAP.*
(b) *If X has the MWAP and MW*D, then $\mathcal{K}(M,1) \subset \overline{\mathcal{F}(M,\mu)}^{\tau}$ for some $\mu > 0$. In particular, M has the BWAP.*

In Section 3 we introduce the properties W*D and MW*D, and prove Theorem 1.4.

2. Proof of Theorem 1.2

By the locally convex space version of the Hahn-Banach theorem, we have the following lemma 2.1 (See Megginson [6, Corollary 2.2.20]).

Lemma 2.1. *Let X be a Banach space and $(\mathcal{B}(X), \mathcal{T})$ a locally convex space. Suppose that $\mathcal{Z}$ is a subspace of $\mathcal{B}(X)$ and $T \in \mathcal{B}(X)$. Then the following are equivalent.*
(a) *T belongs to the $\mathcal{T}$-closure of $\mathcal{Z}$.*
(b) *For every $f \in (\mathcal{B}(X), \mathcal{T})^*$ such that $f(S) = 0$ for all $S \in \mathcal{Z}$, we have $f(T) = 0$.*

The following lemma 2.2 is due to [6, Theorem 2.2.28]. A concrete proof is in [3, Lemma 3.8].

Lemma 2.2. *Let X be a Banach space and $(\mathcal{B}(X), \mathcal{T})$ a locally convex space. Suppose that $\mathcal{C}$ is a balanced convex set in $\mathcal{B}(X)$ and $T \in \mathcal{B}(X)$. Then the following are equivalent.*
(a) *T belongs to the $\mathcal{T}$-closure of $\mathcal{C}$.*
(b) *For every $f \in (\mathcal{B}(X), \mathcal{T})^*$ such that $|f(S)| \leq 1$ for all $S \in \mathcal{C}$, we have $|f(T)| \leq 1$.*

The following proposition is a result of Lemmas 2.1 and 2.2.

Proposition 2.3. Let X be a Banach space.
(a) X has the WAP if and only if for every $f \in (\mathcal{B}(X), \tau)^*$ such that $f(T) = 0$ for all $T \in \mathcal{F}(X)$, we have $f(T) = 0$ for all $T \in \mathcal{K}(X)$.
(b) X has the MWAP if and only if for every $f \in (\mathcal{B}(X), \tau)^*$ such that $|f(S)| \leq 1$ for all $S \in \mathcal{F}(X,1)$, we have $|f(T)| \leq 1$ for all $T \in \mathcal{K}(X,1)$.

For a Banach space X, we consider the subspace $\mathcal{Y}_X$ of $\mathcal{B}(X^*)$ given by

$$\begin{aligned}\mathcal{Y}_X = \{T \in \mathcal{B}(X^*) : \ &\text{there exist } (x_k)_{k=1}^m \subset X \text{ and } (x_k^*)_{k=1}^m \subset X^* \\ &\text{such that } Tx^* = \textstyle\sum_{k=1}^m x^*(x_k)x_k^* \text{ for } x^* \in X^*\}.\end{aligned} \tag{2.1}$$

To prove Theorem 1.2, we need two lemmas. The following lemma 2.4 is found in Lindenstrauss and Tzafriri [5]. A concrete proof is in [3, Lemma 3.5].

Lemma 2.4. *Let X be a Banach space and $\mathcal{Y}= \mathcal{Y}_X$ be as in (2.1). Then $\mathcal{Y}$ is τ-dense in $\mathcal{F}(X^*)$.*

The following lemma 2.5 is in the proof of [5, Theorem 1.e.15].

Lemma 2.5. *Suppose that X is a Banach space such that X^* is separable. Let $\mathcal{Y}= \mathcal{Y}_X$ be as in (2.1). Let $\varphi \in (\mathcal{B}(X^*), \tau)^*$ satisfying $|\varphi(T)| \le 1$ for $T \in \mathcal{Y}$, $\|T\| \le 1$, and $\epsilon > 0$. Then there is a $\psi_\epsilon \in (\mathcal{B}(X^*), \tau)^*$ such that $\psi_\epsilon(S) = \varphi(S)$ for every $S \in \mathcal{Y}$ and $|\psi_\epsilon(T)| \le 1 + \epsilon$ for every $T \in \mathcal{B}(X^*)$ with $\|T\| \le 1$.*

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. To apply Proposition 2.3(b), Assume $\varphi \in (\mathcal{B}(X^*), \tau)^*$ satisfying $|\varphi(S)| \le 1$ for $S \in \mathcal{F}(X^*, 1)$ and let $\epsilon > 0$. Then by Lemma 2.5 there is a $\psi_\epsilon \in (\mathcal{B}(X^*), \tau)^*$ such that $\psi_\epsilon(S) = \varphi(S)$ for every $S \in \mathcal{Y}$ and $|\psi_\epsilon(T)| \le 1 + \epsilon$ for every $T \in \mathcal{B}(X^*)$ with $\|T\| \le 1$. By Lemma 2.4 $\psi_\epsilon(S) = \varphi(S)$ for every $S \in \mathcal{F}(X^*)$. Since X^* has the WAP, $\psi_\epsilon(S) = \varphi(S)$ for every $S \in \mathcal{K}(X^*)$. In particular, $\psi_\epsilon(T) = \varphi(T)$ for every $T \in \mathcal{K}(X^*, 1)$. It follows that $|\varphi(T)| \le 1 + \epsilon$ for every $T \in \mathcal{K}(X^*, 1)$. Since ϵ was arbitrary, $|\varphi(T)| \le 1$ for every $T \in \mathcal{K}(X^*, 1)$. Hence X^* has the MWAP by Proposition 2.3(b). □

3. Proof of Theorem 1.4

We introduce two other topologies on $\mathcal{B}(X)$ or $\mathcal{B}(X^*)$, which are induced by subspaces of $\mathcal{B}(X)^\sharp$ or $\mathcal{B}(X^*)^\sharp$, the vector space of all linear functionals on $\mathcal{B}(X)$ or $\mathcal{B}(X^*)$.

Definition 3.1. Let $\mathcal{Z}_1$ be the space of all linear functionals φ on $\mathcal{B}(X)$ of the form

$$\varphi(T) = \sum_n x_n^*(Tx_n)$$

where $(x_n) \subset X$ and $(x_n^*) \subset X^*$ with $\sum_n \|x_n\|\|x_n^*\| < \infty$.

Let $\mathcal{Z}_2$ be the space of all linear functionals φ on $\mathcal{B}(X^*)$ of the form

$$\varphi(T) = \sum_n (Tx_n^*)x_n$$

where $(x_n) \subset X$ and $(x_n^*) \subset X^*$ with $\sum_n \|x_n\|\|x_n^*\| < \infty$.

Then the ν *topology* (in short, ν) on $\mathcal{B}(X)$ is the topology induced by $\mathcal{Z}_1$ and the *weak** *topology* (in short, *weak**) on $\mathcal{B}(X^*)$ is the topology induced by $\mathcal{Z}_2$.

From elementary facts about topologies induced by spaces of linear functionals on vector spaces, ν and *weak** are locally convex topologies. Also $(\mathcal{B}(X), \nu)^* = \mathcal{Z}_1$, $(\mathcal{B}(X^*), weak^*)^* = \mathcal{Z}_2$, and for a net (T_α) and T in $\mathcal{B}(X)$,

$$T_\alpha \xrightarrow{\nu} T \ \text{ iff } \ \sum_n x_n^*(T_\alpha x_n) \longrightarrow \sum_n x_n^*(Tx_n)$$

for each $(x_n) \subset X$ and $(x_n^*) \subset X^*$ with $\sum_n \|x_n\|\|x_n^*\| < \infty$; similarly, for a net (T_α) and T in $\mathcal{B}(X^*)$,

$$T_\alpha \overset{weak^*}{\longrightarrow} T \text{ iff } \sum_n (T_\alpha x_n^*) x_n \longrightarrow \sum_n (T x_n^*) x_n$$

for each $(x_n) \subset X$ and $(x_n^*) \subset X^*$ with $\sum_n \|x_n\|\|x_n^*\| < \infty$.

The name, the *weak** topology, comes from the fact that $\mathcal{B}(X^*)$ can be identified with $(X^* \hat{\otimes}_\pi X)^*$, the dual of the completed projective tensor product of X^* and X.

Definition 3.2. Let X be a Banach space.
(a) X is said to have the *weak* density* (in short, W*D) if $\mathcal{K}(X^*) \subset \overline{\mathcal{K}(X^*, w^*)}^{weak^*}$.
(b) X is said to have the *metric weak* density* (in short, MW*D) if $\mathcal{K}(X^*, 1) \subset \overline{\mathcal{K}(X^*, w^*, 1)}^{weak^*}$.

The following lemma comes from [4].

Lemma 3.3. *Let X be a Banach space. Then $(\mathcal{B}(X), \tau)^*$ consists of all functionals f of the form $f(T) = \sum_n x_n^*(T x_n)$, where $(x_n) \subset X$ and $(x_n^*) \subset X^*$ with $\sum_n \|x_n\|\|x_n^*\| < \infty$.*

Therefore $(\mathcal{B}(X), \nu)^* = (\mathcal{B}(X), \tau)^*$. Then Remark 1.1 can also be stated as the following.

Remark 3.4. Let X be a Banach space.
(a) X has the AP iff $I_X \in \overline{\mathcal{F}(X)}^\nu$.
(b) X has the λ-BAP iff $I_X \in \overline{\mathcal{F}(X, \lambda)}^\nu$.
(c) X has the WAP iff $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}^\nu$.
(d) X has the BWAP iff for every $T \in \mathcal{K}(X)$, there is a $\lambda_T > 0$ such that $T \in \overline{\mathcal{F}(X, \lambda_T)}^\nu$.
(e) X has the MWAP iff $\mathcal{K}(X, 1) \subset \overline{\mathcal{F}(X, 1)}^\nu$.

To prove Theorem 1.4, we need two more lemmas

Lemma 3.5. *Let X be a Banach space.*
(a) $\mathcal{F}(X^*) \subset \overline{\mathcal{F}(X^*, w^*)}^\tau \subset \overline{\mathcal{F}(X^*, w^*)}^{weak^*}$.
(b) $\mathcal{F}(X^*, \lambda) \subset \overline{\mathcal{F}(X^*, w^*, \lambda)}^\tau \subset \overline{\mathcal{F}(X^*, w^*, \lambda)}^{weak^*}$ *for each* $\lambda > 0$.

Lemma 3.5(a) is deduced from (b) which is found in [3].

Lemma 3.6. *Suppose that M is a closed subspace of a Banach space X and $M^\perp$ is complemented in X^*. Then there is a bounded operator U from M^* into X^* such that $(Um^*)m = m^*m$ for all $m \in M$ and $m^* \in M^*$.*

Proof. Since $M^\perp$ is complemented in X^*, there is a projection $P : X^* \longrightarrow M^\perp$ onto $M^\perp$. Define a map $U : M^* \longrightarrow X^*$ by

$$Um^* = x^* - Px^*$$

where x^* is any linear functional in X^* with $x^* = m^*$ on M. Since P is a projection on $M^\perp$, one easily checks that U is well-defined and

$$(Um^*)m = m^*m$$

for all $m^* \in M^*$ and $m \in M$. Of course, U is a bounded operator. □

Now we can prove Theorem 1.4.

Proof of Theorem 1.4. (a) Since X has the WAP, $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}^{\tau}$. It follows from this that $\mathcal{K}(X^*, w^*) \subset \overline{\mathcal{F}(X^*, w^*)}^{weak^*}$. Since X has the W*D, $\mathcal{K}(X^*) \subset \overline{\mathcal{F}(X^*, w^*)}^{weak^*}$. Now let $T \in \mathcal{K}(M)$, let U be the operator in Lemma 3.6 and I_M the inclusion from M into X. Then $UT^*I_M^* \in \mathcal{K}(X^*) \subset \overline{\mathcal{F}(X^*, w^*)}^{weak^*}$. Thus there is a net $(T_\alpha^*) \subset \mathcal{F}(X^*, w^*)$ such that $T_\alpha^* \overset{weak^*}{\longrightarrow} UT^*I_M^*$. That is,

$$\sum_n x_n^* T_\alpha x_n \longrightarrow \sum_n (UT^*I_M^* x_n^*) x_n$$

for every $(x_n) \subset X$ and $(x_n^*) \subset X^*$ satisfying $\sum_n \|x_n\|\|x_n^*\| < \infty$.

Consider a net $(I_M^* T_\alpha^* U) \subset \mathcal{F}(M^*)$ and assume that sequences $(m_n) \subset M$ and $(m_n^*) \subset M^*$ satisfy $\sum_n \|m_n\|\|m_n^*\| < \infty$. Then $(I_M(m_n)) \subset X, (U(m_n^*)) \subset X^*$, and $\sum_n \|I_M(m_n)\|\|U(m_n^*)\| \le \|U\| \sum_n \|m_n\|\|m_n^*\| < \infty$. Therefore we have

$$\begin{aligned}
\sum_n ((I_M^* T_\alpha^* U)(m_n^*))(m_n) &= \sum_n U(m_n^*) T_\alpha I_M(m_n) \\
&\longrightarrow \sum_n ((UT^*I_M^*)U(m_n^*))(I_M(m_n)) \\
&= \sum_n (UT^* m_n^*)(m_n) \\
&= \sum_n (T^* m_n^*)(m_n).
\end{aligned}$$

From this and Lemma 3.5(a) $T^* \in \overline{\mathcal{F}(M^*, w^*)}^{weak^*}$. It follows that $T \in \overline{\mathcal{F}(M)}^{\nu}$. Hence M has the WAP by Remark 3.4(c).

(b) Since X has the MWAP, $\mathcal{K}(X, 1) \subset \overline{\mathcal{F}(X, 1)}^{\tau}$. It follows from this that $\mathcal{K}(X^*, w^*, 1) \subset \overline{\mathcal{F}(X^*, w^*, 1)}^{weak^*}$. Since X has the MW*D, $\mathcal{K}(X^*, 1) \subset \overline{\mathcal{F}(X^*, w^*, 1)}^{weak^*}$. Now let $T \in \mathcal{K}(M, 1)$. Then $UT^*I_M^* \in \mathcal{K}(X^*, \|U\|) \subset \overline{\mathcal{F}(X^*, w^*, \|U\|)}^{weak^*}$. As in the proof of (a), applying Lemma 3.5(b) we can check $T^* \in \overline{\mathcal{F}(M^*, w^*, \|U\|^2)}^{weak^*}$. It follows from this that $T \in \overline{\mathcal{F}(M, \|U\|^2)}^{\nu} = \overline{\mathcal{F}(M, \|U\|^2)}^{\tau}$. Hence $\mathcal{K}(M, 1) \subset \overline{\mathcal{F}(M, \|U\|^2)}^{\tau}$. □

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Division of Applied Mathematics, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Korea
E-mail address: kjm21@kaist.ac.kr