SOME VIRTUES OF COMPACT ADJOINT OPERATORS AND APPROXIMATION PROPERTIES

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This paper is concerned with the space of all compact adjoint operators from dual spaces of Banach spaces into dual spaces of Banach spaces and approximation properties. For some topology on the space of all bounded linear operators from separable dual spaces of Banach spaces into dual spaces of Banach spaces, it is shown that if a bounded linear operator is approximated by a net of compact adjoint operators, then the operator can be approximated by a sequence of compact adjoint operators whose operator norms are less than or equal to the operator norm of the operator. Also we obtain applications of the theory and in particular, apply the theory to approximation properties.

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1. Introduction and the main theorem

In the Banach space theory, the approximation property, which already appeared in Banach's book [1], is one of the fundamental properties. Grothendieck [5] initiated the investigation of the variants of the approximation property and the relations between them. Many mathematicians have introduced and studied many other versions of the approximation property since that time. Casazza [2] summarized various results, which contain his ones, and introduced many open problems on the approximation property and its other versions. Recently Choi and Kim ([4], [9], and [10]) introduced and studied weak versions of the approximation property. In this paper, we establish a theorem on the space of all compact adjoint operators from dual spaces of Banach spaces into dual spaces of Banach spaces and apply the theorem to approximation properties.

In Section 2, we introduce weak versions of the approximation property and deduce some characterizations of the approximation property and its weak versions. Also we complete solutions of dual problems for the weak versions. In Section 3, a density property for the space of compact adjoint operators is established and in Section 4, we introduce a strong version of the compact approximation property.

Now we start by listing notations which are used throughout this paper.

Notation 1.1.

X, Y : Banach spaces
X^* : the dual space of X

w^* : the weak* topology on the dual space of a Banach space
T^* : the adjoint of a operator T
$\mathcal{B}(Y,X)$: The space of all bounded linear operators from Y into X.
$\mathcal{F}(Y,X)$: The space of all bounded and finite rank linear operators from Y into X.
$\mathcal{K}(Y,X)$: The space of all compact operators from Y into X.
$\mathcal{K}(X^*,Y^*,w^*)$: The space of all compact and w^*-to-w^* continuous operators from X^* into Y^*.
$\mathcal{K}(Y,X,\lambda)$: The collection of all compact operators T from Y into X satisfying $\|T\| \leq \lambda$.
$\mathcal{K}(X^*,Y^*,w^*,\lambda)$: The collection of all compact and w^*-to-w^* continuous operators T from X^* into Y^* satisfying $\|T\| \leq \lambda$.

Similarly we define $\mathcal{F}(X^*,Y^*,w^*)$, $\mathcal{F}(Y,X,\lambda)$, $\mathcal{F}(X^*,Y^*,w^*,\lambda)$, $\mathcal{B}(X^*,Y^*,w^*)$, $\mathcal{B}(Y,X,\lambda)$, and $\mathcal{B}(X^*,Y^*,w^*,\lambda)$. For convenience we denote $\mathcal{B}(X,X)$, $\cdots$ by $\mathcal{B}(X)$, $\cdots$.

Remark 1.2. Since a w^*-to-w^* continuous operator is the adjoint of an operator and an adjoint operator is w^*-to-w^* continuous, $\mathcal{K}(X^*,Y^*,w^*)$ is the space of all compact adjoint operators from X^* into Y^*.

We introduce a topology on $\mathcal{B}(X,Y)$, which is an important tool in this paper. For compact $K \subset X$, $\epsilon > 0$, and $T \in \mathcal{B}(X,Y)$, we put

$$N(T,K,\epsilon) = \{R \in \mathcal{B}(X,Y) : \sup_{x\in K} \|Rx - Tx\| < \epsilon\}.$$

Let $\mathcal{S}$ be the collection of all such $N(T,K,\epsilon)$'s. Now we denote by τ the topology on $\mathcal{B}(X,Y)$ generated by $\mathcal{S}$. It is easy to check that τ is a locally convex topology and for a net (T_α) and T in $\mathcal{B}(X,Y)$,

$$T_\alpha \xrightarrow{\tau} T \Longleftrightarrow \text{for each compact } K \subset X \ \sup_{x\in K} \|T_\alpha x - Tx\| \longrightarrow 0.$$

Now we state the main theorem in this paper.

Theorem 1.3. *Suppose that X^* is separable. Let $\mathcal{Y}$ be a subspace of $\mathcal{K}(X^*,Y^*,w^*)$ and let $T \in \mathcal{B}(X^*,Y^*)$. Then $T \in \overline{\mathcal{Y}}^\tau$ if and only if there is a sequence (T_n^*) in $\{S^* \in \mathcal{Y} : \|S^*\| \leq \|T\|\}$ such that $T_n^* \xrightarrow{\tau} T$.*

Theorem 1.3 is immediately proved by the following propositions 1.4 and 1.5.

Proposition 1.4. *Suppose that X is separable and $\mathcal{A}$ is a bounded set in $\mathcal{B}(X,Y)$. Then the τ-topology on $\mathcal{A}$ is metrizable.*

Kim [8, Theorem 1.18] showed Proposition 1.4 for $\mathcal{B}(X)$. Since the proofs are the same, we omit the proof of Proposition 1.4.

Proposition 1.5. *Suppose that X^* is separable. Let $\mathcal{Y}$ be a subspace of $\mathcal{K}(X^*,Y^*,w^*)$ and let $T \in \mathcal{B}(X^*,Y^*)$. Then $T \in \overline{\mathcal{Y}}^\tau$ if and only if $T \in \overline{\{S^* \in \mathcal{Y} : \|S^*\| \leq \|T\|\}}^\tau$.*

To show Proposition 1.5, we need the following two lemmas.

Lemma 1.6. *Let $(\mathcal{B}(X,Y),\mathcal{T})$ be a locally convex space. Suppose that $\mathcal{C}$ is a balanced convex set in $\mathcal{B}(X,Y)$ and $T \in \mathcal{B}(X,Y)$. Then $T \in \overline{\mathcal{C}}^\mathcal{T}$ if and only if for every $\varphi \in (\mathcal{B}(X,Y),\mathcal{T})^*$ such that $|\varphi(S)| \leq 1$ for all $S \in \mathcal{C}$, we have $|\varphi(T)| \leq 1$.*

Lemma 1.6 is essentially due to Megginson [12, Theorem 2.2.28] and a concrete proof is in [4, Lemma 3.8]. The following lemma 1.7 is proved in Section 5.

Lemma 1.7. *Suppose that X^* is separable and $\mathcal{Y}$ is a subspace of $\mathcal{K}(X^*, Y^*, w^*)$. If $\varphi \in (\mathcal{B}(X^*, Y^*), \tau)^*$ satisfying $|\varphi(S^*)| \leq 1$ for $S^* \in \{S^* \in \mathcal{Y} : \|S^*\| \leq 1\}$, and $\epsilon > 0$, then there is a $\psi_\epsilon \in (\mathcal{B}(X^*, Y^*), \tau)^*$ such that $\psi_\epsilon(R^*) = \varphi(R^*)$ for every $R^* \in \mathcal{Y}$ and $|\psi_\epsilon(U)| \leq 1 + \epsilon$ for every $U \in \mathcal{B}(X^*, Y^*, 1)$.*

Now we can prove Proposition 1.5.

Proof of Proposition 1.5. It only needs to prove the "only if" part. We may assume $T \neq 0$ and will show $T/\|T\| \in \overline{\{S^* \in \mathcal{Y} : \|S^*\| \leq 1\}}^\tau$. Let $\varphi \in (\mathcal{B}(X^*, Y^*), \tau)^*$ satisfying $|\varphi(S^*)| \leq 1$ for $S^* \in \{S^* \in \mathcal{Y} : \|S^*\| \leq 1\}$, and $\epsilon > 0$. To apply Lemma 1.6, we should show $|\varphi(T/\|T\|)| \leq 1$. Now by Lemma 1.7, there is a $\psi_\epsilon \in (\mathcal{B}(X^*, Y^*), \tau)^*$ such that $\psi_\epsilon(R^*) = \varphi(R^*)$ for every $R^* \in \mathcal{Y}$ and $|\psi_\epsilon(U)| \leq 1 + \epsilon$ for every $U \in \mathcal{B}(X^*, Y^*, 1)$. Since $T \in \overline{\mathcal{Y}}^\tau$, by continuity $\psi_\epsilon(T) = \varphi(T)$. So $\psi_\epsilon(T/\|T\|) = \varphi(T/\|T\|)$. Since $|\psi_\epsilon(U)| \leq 1 + \epsilon$ for every $U \in \mathcal{B}(X^*, Y^*, 1)$, $|\varphi(T/\|T\|)| \leq 1 + \epsilon$. Since ϵ is arbitrary, $|\varphi(T/\|T\|)| \leq 1$, which shows $T/\|T\| \in \overline{\{R \in \mathcal{Y} : \|R\| \leq 1\}}^\tau$ by Lemma 1.6. □

In particular, we are concerned with the following two corollaries of Theorem 1.3, which are mainly used in this paper.

Corollary 1.8. *Suppose that X^* is separable and let $T \in \mathcal{B}(X^*, Y^*)$. Then $T \in \overline{\mathcal{K}(X^*, Y^*, w^*)}^\tau$ if and only if there is a sequence (T_n^*) in $\mathcal{K}(X^*, Y^*, w^*, \|T\|)$ such that $T_n^* \xrightarrow{\tau} T$.*

Corollary 1.9. *Suppose that X^* is separable and let $T \in \mathcal{B}(X^*, Y^*)$. Then $T \in \overline{\mathcal{F}(X^*, Y^*, w^*)}^\tau$ if and only if there is a sequence (T_n^*) in $\mathcal{F}(X^*, Y^*, w^*, \|T\|)$ such that $T_n^* \xrightarrow{\tau} T$.*

2. The approximation property and its weak versions

In this section, we deduce some characterizations of the approximation property and its weak versions, and study inheritance from X (respectively X^*) to X^* (respectively X) (in short, dual problem) for the weak versions. First we introduce various approximation properties.

We say that X has the *approximation property* (in short, AP) if for every compact $K \subset X$ and $\epsilon > 0$, there is a $T \in \mathcal{F}(X)$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$. Also we say that X has the *λ-bounded approximation property* (in short, λ-BAP) if for every compact $K \subset X$ and $\epsilon > 0$, there is a $T \in \mathcal{F}(X, \lambda)$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$. In particular, if $\lambda = 1$, then we say that X has the *metric approximation property* (in short, MAP). If X has the λ-bounded approximation property for some $\lambda > 0$, then we say that X has the *bounded approximation property* (in short, BAP). Recently Choi and Kim ([4], [9], and [10]) introduced weak versions of the approximation property. We say that X has the *weak approximation property* (in short, WAP) if for every $T \in \mathcal{K}(X)$, compact $K \subset X$, and $\epsilon > 0$, there is a $T_0 \in \mathcal{F}(X)$ such that $\|T_0x - Tx\| < \epsilon$ for all $x \in K$. Also we say that X has the *bounded weak approximation property* (in short, BWAP) if for every $T \in \mathcal{K}(X)$, there is a $\lambda_T > 0$ such that for every compact $K \subset X$ and $\epsilon > 0$, there is a $T_0 \in \mathcal{F}(X, \lambda_T)$ such that $\|T_0x - Tx\| < \epsilon$ for all $x \in K$. We say that X has the *quasi approximation property* (in short, QAP) if for every $T \in \mathcal{K}(X)$ and $\epsilon > 0$, there is a $T_0 \in \mathcal{F}(X)$ such that $\|T_0 - T\| < \epsilon$. We say that X has the *metric weak approximation property* (in short, MWAP) if for every $T \in \mathcal{K}(X, 1)$, compact $K \subset X$, and $\epsilon > 0$, there is a $T_0 \in \mathcal{F}(X, 1)$ such that $\|T_0 x - Tx\| < \epsilon$ for all $x \in K$. We say that X has the *metric quasi approximation property* (in short, MQAP) if for every $T \in \mathcal{K}(X, 1)$ and $\epsilon > 0$, there is a $T_0 \in \mathcal{F}(X, 1)$ such that $\|T_0 - T\| < \epsilon$.

Remark 2.1. From the definitions of above properties and τ, we see the following :
(a) X has the AP iff $I_X \in \overline{\mathcal{F}(X)}^{\tau}$, where I_X is the identity in $\mathcal{B}(X)$.
(b) X has the λ-BAP iff $I_X \in \overline{\mathcal{F}(X, \lambda)}^{\tau}$.
(c) X has the WAP iff $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}^{\tau}$.
(d) X has the BWAP iff for every $T \in \mathcal{K}(X)$, there is a $\lambda_T > 0$ such that $T \in \overline{\mathcal{F}(X, \lambda_T)}^{\tau}$.
(e) X has the QAP iff $\mathcal{K}(X) = \overline{\mathcal{F}(X)}$, where the closure is the operator norm closure.
(f) X has the MWAP iff $\mathcal{K}(X, 1) \subset \overline{\mathcal{F}(X, 1)}^{\tau}$.
(g) X has the MQAP iff $\mathcal{K}(X, 1) = \overline{\mathcal{F}(X, 1)}$.

First we review the following results of Grothendieck [5].

Fact. (a) $(\mathcal{B}(X, Y), \tau)^*$ consists of all functionals f of the form $f(T) = \sum_n y_n^*(Tx_n)$, where $(x_n) \subset X$, $(y_n^*) \subset Y^*$, and $\sum_n \|x_n\| \|y_n^*\| < \infty$.
(b) X has the AP if and only if for every Banach space Y, $\mathcal{K}(Y, X) = \overline{\mathcal{F}(Y, X)}$.
(c) X^* has the AP if and only if for every Banach space Y, $\mathcal{K}(X, Y) = \overline{\mathcal{F}(X, Y)}$.

Now we have simple characterizations of AP and QAP.

Proposition 2.2.
(a) *X has the AP if and only if for every Banach space Y, $\mathcal{K}(Y, X, 1) = \overline{\mathcal{F}(Y, X, 1)}$.*
(b) *X^* has the AP if and only if for every Banach space Y, $\mathcal{K}(X, Y, 1) = \overline{\mathcal{F}(X, Y, 1)}$.*
(c) *X has the QAP if and only if X has the MQAP.*

Proof. We only show (a). The proofs of the others are the same. Now assume that X has the AP. let Y be a Banach space, $T \in \mathcal{K}(Y, X, 1)$, and $\epsilon > 0$. Choose $\delta > 0$ such that

$$\frac{\delta}{1+\delta} < \frac{\epsilon}{2}.$$

Since X has the AP, by **Fact**(b) there is a $S_0 \in \mathcal{F}(Y, X)$ such that

$$\|S_0 - T\| < \delta.$$

Then we observe $S_0 \in \mathcal{F}(Y, X, 1 + \delta)$. Put $T_0 = 1/(1 + \delta)S_0$. Then $T_0 \in \mathcal{F}(Y, X, 1)$ and we have

$$\|T_0 - T\| \leq \frac{1}{1+\delta}\|S_0 - T\| + \frac{\delta}{1+\delta}\|T\| < \epsilon.$$

Hence $T \in \overline{\mathcal{F}(Y, X, 1)}$ which completes the proof.

Suppose the converse. To use **Fact**(b), let Y be a Banach space and $T \in \mathcal{K}(Y, X)$. Then by the assumption we have

$$T \in \mathcal{K}(Y, X, \|T\|) = \|T\|\mathcal{K}(Y, X, 1) = \|T\|\overline{\mathcal{F}(Y, X, 1)} = \overline{\mathcal{F}(Y, X, \|T\|)} \subset \overline{\mathcal{F}(Y, X)}.$$

Hence $T \in \overline{\mathcal{F}(Y, X)}$ which completes the proof. $\square$

In [9], the author observed the following implications :

$$\begin{array}{ccccc} MAP & \Longrightarrow & MQAP \Longrightarrow MWAP & & \\ \Downarrow & & \Downarrow \qquad\qquad \Downarrow & & \\ BAP \Rightarrow AP \Rightarrow QAP & \Longrightarrow & BWAP \Longrightarrow WAP & & \end{array}$$

But by Proposition 2.2(c) the implications are simplified as the following :

$$MAP \Longrightarrow BAP \Longrightarrow AP \Longrightarrow MQAP \Longleftrightarrow QAP \Longrightarrow MWAP \Longrightarrow BWAP \Longrightarrow WAP \tag{2.1}$$

We now introduce another topology on $\mathcal{B}(X,Y)$, which is induced by a subspace of $\mathcal{B}(X,Y)^\sharp$, the vector space of all linear functionals on $\mathcal{B}(X,Y)$.

Definition 2.3. Let $\mathcal{Z}$ be the space of all linear functionals φ on $\mathcal{B}(X,Y)$ of the form

$$\varphi(T) = \sum_n y_n^*(Tx_n)$$

where $(x_n) \subset X$ and $(y_n^*) \subset Y^*$ with $\sum_n \|x_n\|\|y_n^*\| < \infty$.
Then the ν *topology* (in short, ν) on $\mathcal{B}(X,Y)$ is the topology induced by $\mathcal{Z}$.

From elementary facts about topologies induced by spaces of linear functionals on vector spaces, ν is a locally convex topology and $(\mathcal{B}(X,Y),\nu)^* = \mathcal{Z}$. Also for a net (T_α) and T in $\mathcal{B}(X,Y)$,

$$T_\alpha \xrightarrow{\nu} T \text{ iff } \sum_n y_n^*(T_\alpha x_n) \longrightarrow \sum_n y_n^*(Tx_n)$$

for each $(x_n) \subset X$ and $(y_n^*) \subset X^*$ with $\sum_n \|x_n\|\|x_n^*\| < \infty$.

From **Fact**(a) $(\mathcal{B}(X,Y),\nu)^* = (\mathcal{B}(X,Y),\tau)^*$. Hence from Remark 2.1 we have the following by an application of the separation theorem.

Remark 2.4.
(a) X has the AP iff $I_X \in \overline{\mathcal{F}(X)}^{\nu}$, where I_X is the identity in $\mathcal{B}(X)$.
(b) X has the λ-BAP iff $I_X \in \overline{\mathcal{F}(X,\lambda)}^{\nu}$.
(c) X has the WAP iff $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}^{\nu}$.
(d) X has the BWAP iff for every $T \in \mathcal{K}(X)$, there is a $\lambda_T > 0$ such that $T \in \overline{\mathcal{F}(X,\lambda_T)}^{\nu}$.
(f) X has the MWAP iff $\mathcal{K}(X,1) \subset \overline{\mathcal{F}(X,1)}^{\nu}$.

Now we state the main theorem in this section.

Theorem 2.5. *Suppose that X^* is separable. Then for every $T \in \mathcal{K}(Y,X)$, there is a net $(T_\alpha) \subset \mathcal{F}(Y,X)$ such that $T_\alpha^* \xrightarrow{\nu} T^*$ in $\mathcal{B}(X^*,Y^*)$ if and only if $\mathcal{K}(Y,X) \subset \overline{\mathcal{F}(Y,X)}$.*

We apply Theorem 2.5 to approximation properties before the proof is presented. From **Fact**(b) and the definition of the QAP, we have the following corollaries.

Corollary 2.6. *Suppose that X^* is separable. Then for every Banach space Y and $T \in \mathcal{K}(Y,X)$, there is a net $(T_\alpha) \subset \mathcal{F}(Y,X)$ such that $T_\alpha^* \xrightarrow{\nu} T^*$ in $\mathcal{B}(X^*,Y^*)$ if and only if X has the AP.*

Corollary 2.7. *Suppose that X^* is separable. Then for every $T \in \mathcal{K}(X)$, there is a net $(T_\alpha) \subset \mathcal{F}(X)$ such that $T_\alpha^* \xrightarrow{\nu} T^*$ in $\mathcal{B}(X^*)$ if and only if X has the QAP.*

Now to prove Theorem 2.5, we need the following lemma.

Lemma 2.8 (7, Corollary 3)**.** *Suppose that (T_n) is a sequence in $\mathcal{K}(Y,X)$ and $T \in \mathcal{K}(Y,X)$. If for each $x^* \in X^*$ and $y^{**} \in Y^{**}$ $y^{**}T_n^*x^* \longrightarrow y^{**}T^*x^*$, then there is a sequence (S_n) of convex combinations of $\{T_n\}$ such that $\|S_n - T\| \longrightarrow 0$.*

Now we can prove Theorem 2.5.

Proof of Theorem 2.5. Since the "if" part is clear, we only show the "only if" part. Now let $T \in \mathcal{K}(Y,X)$. Then by hypothesis there is a net $(T_\alpha) \subset \mathcal{F}(Y,X)$ such that $T_\alpha^* \xrightarrow{\nu} T^*$ in $\mathcal{B}(X^*,Y^*)$. It follows that

$$T^* \in \overline{\mathcal{F}(X^*,Y^*,w^*)}^{\nu} = \overline{\mathcal{F}(X^*,Y^*,w^*)}^{\tau}.$$

By a virtue of Corollary 1.9 there is a sequence (T_n^*) in $\mathcal{F}(X^*,Y^*,w^*)$ such that $T_n^* \xrightarrow{\tau} T^*$ in $\mathcal{B}(X^*,Y^*)$. In particular

$$y^{**}T_n^*x^* \longrightarrow y^{**}T^*x^*$$

for each $x^* \in X^*$ and $y^{**} \in Y^{**}$. By Lemma 2.8 there is a sequence (S_n) of convex combinations of $\{T_n\}$ such that $\|S_n - T\| \longrightarrow 0$. Since $(S_n) \subset \mathcal{F}(Y,X)$, $T \in \overline{\mathcal{F}(Y,X)}$. Hence $\mathcal{K}(Y,X) = \overline{\mathcal{F}(Y,X)}$. □

Now we consider dual problems for approximation properties. It is well known that the AP, BAP, and MAP are not inherited from X to X^* (See [2]). In [9], by the following propositions, it was shown that the WAP, BWAP, and QAP are not inherited from X to X^*.

Proposition 2.9. *There is a Banach space Y with a boundedly complete basis such that Y^* is separable and does not have the WAP.*

Proposition 2.10. *There is a Banach space Z which has the AP but does not have the bounded compact approximation property such that Z^*, Z^{**}, ... are all separable and Z^* does not have the WAP.*

Now from (2.1), Propositions 2.9 and 2.10, we have the following corollaries which show that the MWAP and MQAP are not inherited from X to X^*.

Corollary 2.11. *There is a Banach space Y with the MQAP such that Y^* is separable and does not have the WAP.*

Corollary 2.12. *There is a Banach space Z which has the MQAP but does not have the bounded compact approximation property such that Z^*, Z^{**}, ... are all separable and Z^* does not have the WAP.*

It is well known that the AP and λ-BAP are inherited from X^* to X (See [2]). In [4], it was shown that the WAP and BWAP are inherited from X^* to X. In [9], it was shown that MWAP are inherited from X^* to X. Now by the Principal of Local Reflexivity we have the following which is a result of [R, Proposition 5.55].

Theorem 2.13. *If X^* has the QAP(MQAP), then X has the QAP(MQAP).*

3. $\mathcal{K}(X^*, w^*)$ is not τ-dense in $\mathcal{K}(X^*)$ in general

For every Banach space X and Y, the following proposition holds, which is due to [4, Lemma 3.11].

Proposition 3.1. $\mathcal{F}(X^*, Y^*, \lambda) \subset \overline{\mathcal{F}(X^*, Y^*, w^*, \lambda)}^{\tau}$ *for each* $\lambda > 0$.

To prove Proposition 3.1, we need a result of Johnson [6].

Lemma 3.2 (6, Lemma 1)**.** *Let F be a finite-dimensional Banach space, $A \subset X^*$ a finite set, $S : X^* \longrightarrow F$ a bounded linear operator, and $\epsilon > 0$. Then there is a w^*-to-w^* continuous linear operator $T : X^* \longrightarrow F$ such that $Tx^* = Sx^*$ for all $x^* \in A$ and $\|T\| \leq \|S\| + \epsilon$.*

Proof of Proposition 3.1. Let $S \in \mathcal{F}(X^*, Y^*, \lambda)$, $K \subset X^*$ a compact set, and $\epsilon > 0$. Put $M = \sup_{x^* \in K} \|Sx^*\|$ and choose a $\delta > 0$ so that

$$(2\lambda + \delta)\delta < \frac{\epsilon}{2} \quad \text{and} \quad \frac{\delta M}{\lambda + \delta} < \frac{\epsilon}{2}.$$

Since K is compact, we have a finite set $A \subset K$ such that for each $x^* \in K$ there is $x_0^* \in A$ satisfying $\|x^* - x_0^*\| < \delta$.

Now by Lemma 3.2 we have a $T_1 \in \mathcal{F}(X^*, Y^*, w^*)$ such that $T_1 x^* = Sx^*$ for all $x^* \in A$ and $\|T_1\| \leq \|S\| + \delta$. Then one can check that

$$\sup_{x^* \in K} \|T_1 x^* - Sx^*\| < \frac{\epsilon}{2} \quad \text{and} \quad \|T_1\| \leq \lambda + \delta.$$

Put $T = \frac{\lambda}{\lambda+\delta} T_1$. Then we have

$$\sup_{x^* \in K} \|Tx^* - Sx^*\| \leq \frac{\lambda}{\lambda + \delta} \sup_{x^* \in K} \|T_1 x^* - Sx^*\| + \frac{\delta}{\lambda + \delta} \sup_{x^* \in K} \|Sx^*\| < \epsilon.$$

Since $T \in \mathcal{F}(X^*, Y^*, w^*, \lambda)$, we have a proof of the proposition. □

From Proposition 3.1, for every Banach space X and Y, we have the following.

Corollary 3.3. $\mathcal{F}(X^*, Y^*) \subset \overline{\mathcal{F}(X^*, Y^*, w^*)}^{\tau}$.

From Corollary 3.3, Corollary 1.9 can be restated by the following corollary.

Corollary 3.4. *Suppose that X^* is separable and let $T \in \mathcal{B}(X^*, Y^*)$. Then $T \in \overline{\mathcal{F}(X^*, Y^*)}^{\tau}$ if and only if there is a sequence (T_n^*) in $\mathcal{F}(X^*, Y^*, w^*, \|T\|)$ such that $T_n^* \xrightarrow{\tau} T$.*

Now the author is naturally led to the following question : For every Banach space X

$$\mathcal{K}(X^*) \subset \overline{\mathcal{K}(X^*, w^*)}^{\tau}?$$

In this section, it is shown that the question has negative answer by a known example.

Theorem 3.5. *There is a Banach space Z with the separable dual such that $\mathcal{K}(Z^*) \not\subset \overline{\mathcal{K}(Z^*, w^*)}^{\tau}$.*

We have the following application of Corollary 1.8 before the proof of Theorem 3.5 is presented.

Proposition 3.6. *Suppose that X^* is separable and $\lambda > 0$. Then $\mathcal{K}(X^*,Y^*) \subset \overline{\mathcal{K}(X^*,Y^*,w^*)}^{\tau}$ if and only if for every $T \in \mathcal{K}(X^*,Y^*,\lambda)$, there is a sequence $(T_n^*) \subset \mathcal{K}(X^*,Y^*,w^*,\lambda)$ such that $T_n^* \xrightarrow{\tau} T$.*

Proof. We only need to prove the " only if " part. Suppose that $\mathcal{K}(X^*,Y^*) \subset \overline{\mathcal{K}(X^*,Y^*,w^*)}^{\tau}$ and let $T \in \mathcal{K}(X^*,Y^*,\lambda)$. Then by a virtue of Corollary 1.8 there is a sequence (T_n^*) in $\mathcal{K}(X^*,Y^*,w^*,\|T\|) \subset \mathcal{K}(X^*,Y^*,w^*,\lambda)$ such that $T_n^* \xrightarrow{\tau} T$. □

We say that X has the *compact approximation property* (in short, CAP) if for every compact $K \subset X$ and $\epsilon > 0$, there is a $T \in \mathcal{K}(X)$ such that $\|Tx-x\| < \epsilon$ for all $x \in K$. Also we say that X has the λ-*bounded compact approximation property*(in short, λ-BCAP) if for every compact $K \subset X$ and $\epsilon > 0$, there is a $T \in \mathcal{K}(X,\lambda)$ such that $\|Tx-x\| < \epsilon$ for all $x \in K$. In particular, if $\lambda = 1$, then we say that X has the *metric compact approximation property* (in short, MCAP). If X has the λ-bounded compact approximation property for some $\lambda > 0$, then we say that X has the *bounded compact approximation property* (in short, BCAP).

Remark 3.7. As in Remark 2.4 we see the following :
(a) X has the CAP iff $I_X \in \overline{\mathcal{K}(X)}^{\nu}$.
(b) X has the λ-BCAP iff $I_X \in \overline{\mathcal{K}(X,\lambda)}^{\nu}$.

Now we prove Theorem 3.5.

Proof of Theorem 3.5. There is a Banach space Z which has the approximation property but does not have the BCAP such that Z^*, Z^{**}, ... are all separable and have the MCAP (See Casazza and Jarchow [3, Theorem 2.5]). We will show that the Banach space Z is a desired Banach space. Suppose that $\mathcal{K}(Z^*) \subset \overline{\mathcal{K}(Z^*,w^*)}^{\tau}$, and we will obtain a contradiction. Since Z^* is separable, by Proposition 3.6 $\mathcal{K}(Z^*,1) \subset \overline{\mathcal{K}(Z^*,w^*,1)}^{\tau} = \overline{\mathcal{K}(Z^*,w^*,1)}^{\nu}$. Since Z^* has the MCAP, $I_{Z^*} \in \overline{\mathcal{K}(Z^*,1)}^{\tau} \subset \overline{\mathcal{K}(Z^*,w^*,1)}^{\nu}$. It follows that $I_Z \in \overline{\mathcal{K}(Z,1)}^{\nu}$. Thus Z has the MCAP by Remark 3.7(b). Since Z does not have the BCAP, this is a contradiction. Hence $\mathcal{K}(Z^*) \not\subset \overline{\mathcal{K}(Z^*,w^*)}^{\tau}$. □

4. The strong compact approximation property

In this section, we introduce a strong version of the CAP and apply Corollary 1.8 to the strong version. Recall the ν topology.

Definition 4.1. We say that X has the *strong compact approximation property* (in short, SCAP) if there is a net (T_α) in $\mathcal{K}(X)$ such that $T_\alpha^* \xrightarrow{\nu} I_{X^*}$ in $\mathcal{B}(X^*)$. Also we say that X has the λ-*bounded strong compact approximation property* (in short, λ-BSCAP) if there is a net (T_α) in $\mathcal{K}(X,\lambda)$ such that $T_\alpha^* \xrightarrow{\nu} I_{X^*}$ in $\mathcal{B}(X^*)$. In particular, if $\lambda = 1$, then we say that X has the *metric strong compact approximation property* (in short, MSCAP). If X has the λ-bounded strong compact approximation property for some $\lambda > 0$, then we say that X has the *bounded strong compact approximation property* (in short, BSCAP).

Remark 4.2.
(a) From Remark 3.7, if X has the SCAP, then X and X^* have the CAP, and if X has the the λ-BSCAP, then X and X^* have the λ-BCAP. Also for a reflexive Banach space X,

X having the SCAP and CAP are equivalent, and X having the λ-BSCAP and λ-BCAP are equivalent.
(b) X has the SCAP iff $I_{X^*} \in \overline{\mathcal{K}(X^*, w^*)}^{\tau}$, and X has the λ-BSCAP iff $I_{X^*} \in \overline{\mathcal{K}(X^*, w^*, \lambda)}^{\tau}$.

From Corollary 1.8 and Remark 4.2(b), we have the following theorem.

Theorem 4.3. *Suppose that X^* is separable. Then X has the SCAP if and only if there is a sequence $(T_n^*) \subset \mathcal{K}(X^*, w^*, 1)$ such that $T_n^* \xrightarrow{\tau} I_{X^*}$. In particular X has the SCAP if and only if X has the MSCAP.*

Now we observe simple relations between the AP and SCAP.

Proposition 4.4.
(a) *X^* has the AP if and only if X has the SCAP and $\mathcal{K}(X^*, w^*) \subset \overline{\mathcal{F}(X^*, w^*)}^{\nu}$.*
(b) *X^* has the λ-BAP if and only if X has the λ-BSCAP and $\mathcal{K}(X^*, w^*, 1) \subset \overline{\mathcal{F}(X^*, w^*, 1)}^{\nu}$.*

Proof. (a) If X^* has the AP, then $I_{X^*} \in \overline{\mathcal{F}(X^*)}^{\tau} = \overline{\mathcal{F}(X^*, w^*)}^{\tau} \subset \overline{\mathcal{K}(X^*, w^*)}^{\tau}$. Hence X has the SCAP. Also since the AP implies the QAP, we have $\mathcal{K}(X^*, w^*) \subset \mathcal{K}(X^*) \subset \overline{\mathcal{F}(X^*)} \subset \overline{\mathcal{F}(X^*)}^{\tau} = \overline{\mathcal{F}(X^*, w^*)}^{\tau} = \overline{\mathcal{F}(X^*, w^*)}^{\nu}$. In fact, even if X has the QAP, $\mathcal{K}(X^*, w^*) \subset \overline{\mathcal{F}(X^*, w^*)}^{\nu}$. Suppose the converse. Then we have $I_{X^*} \in \overline{\mathcal{K}(X^*, w^*)}^{\nu} = \overline{\mathcal{F}(X^*, w^*)}^{\nu} = \overline{\mathcal{F}(X^*, w^*)}^{\tau}$. Hence X^* has the AP.

(b) If X^* has the λ-BAP, then $I_{X^*} \in \overline{\mathcal{F}(X^*, \lambda)}^{\tau} = \overline{\mathcal{F}(X^*, w^*, \lambda)}^{\tau} \subset \overline{\mathcal{K}(X^*, w^*, \lambda)}^{\tau}$. Hence X has the λ-BSCAP. Also since the AP implies the MQAP, we have $\mathcal{K}(X^*, w^*, 1) \subset \mathcal{K}(X^*, 1) \subset \overline{\mathcal{F}(X^*, 1)} \subset \overline{\mathcal{F}(X^*, 1)}^{\tau} = \overline{\mathcal{F}(X^*, w^*, 1)}^{\tau} = \overline{\mathcal{F}(X^*, w^*, 1)}^{\nu}$. In fact, even if X has the QAP, $\mathcal{K}(X^*, w^*, 1) \subset \overline{\mathcal{F}(X^*, w^*, 1)}^{\nu}$. Suppose the converse. Then we have $I_{X^*} \in \overline{\mathcal{K}(X^*, w^*, \lambda)}^{\nu} = \overline{\mathcal{F}(X^*, w^*, \lambda)}^{\nu} = \overline{\mathcal{F}(X^*, w^*, \lambda)}^{\tau}$. Hence X^* has the λ-BAP. $\square$

The following examples say that the converse of Remark 4.2(a) is false in general.

Example 4.5. 1) There is a Banach space Z which has the approximation property but does not have the BCAP such that Z^*, Z^{**}, ... are all separable and have the MCAP [3, Theorem 2.5]. The Banach space Z should not have the SCAP. In fact, if Z would have the SCAP, then by Theorem 4.3 Z must have the MSCAP. This gives a contradiction since Z does not have the BCAP.

2) Let W be the Willis space (See Willis [14]). Then W is separable and reflexive, has the MCAP but does not have the AP. Now there is a separable Banach space Z such that Z^{**} has a boundedly complete basis and $Z^{***} \cong Z^* \oplus W^*$ (See [2, Proposition 1.3]). Consider Z^{**}. Then Z^{**} has the MAP and Z^{***} has the BCAP. But Z^{**} should not have the SCAP. In fact, if Z^{**} would have the SCAP, then Z^{***} has the AP by Proposition 4.4(a). This contradicts fact that W does not have the AP.

5. Proof of Lemma 1.7

The proof of Lemma 1.7 is a generalization of the proof of [11, Theorem 1.e.15]. To prove Lemma 1.7, we need the following two results.

Lemma 5.1 (11, Lemma 1.e.16)**.** *Let X be a separable Banach space and $\epsilon > 0$. Then there is a sequence of functions $\{f_i\}_{i=1}^{\infty}$ on the unit ball B_X of X so that $x = \sum_{i=1}^{\infty} f_i(x)$,* *for every $x \in B_X$, each $f_i(x)$ is of the form $\sum_{j=1}^{\infty} \chi_{E_{i,j}}(x)x_{i,j}$, where $\{E_{i,j}\}_{j=1}^{\infty}$ are disjoint Borel sets of B_X, $\{x_{i,j}\}_{j=1}^{\infty} \subset B_X$ and $\sum_{i=1}^{\infty} \|f_i\|_\infty < 1+\epsilon$ where $\|f_i\|_\infty = \sup_x \|f_i(x)\| = \sup_j \|x_{i,j}\|$.*

Now let B_{X^*} and $B_{Y^{**}}$ be the unit ball of X^* and Y^{**}, respectively, given weak* topology. Then $B_{X^*} \times B_{Y^{**}}$ is a compact Hausdorff space. Let $C(B_{X^*} \times B_{Y^{**}})$ be the Banach space of all scalar valued continuous functions on $B_{X^*} \times B_{Y^{**}}$ and for $T^* \in \mathcal{B}(X^*, Y^*, w^*)$, we define g_{T^*} a function on $B_{X^*} \times B_{Y^{**}}$ by

$$g_{T^*}(x^*, y^{**}) = y^{**}T^*x^*.$$

Then we have :

Lemma 5.2. *Suppose that $\mathcal{Y}$ is a subspace of $\mathcal{K}(X^*, Y^*, w^*)$. Then the map $T^* \mapsto g_{T^*}$ defines a linear isometry from $\mathcal{Y}$ into $C(B_{X^*} \times B_{Y^{**}})$.*

Since the proof of Lemma 5.2 is the same as the proof of [7, Lemma 1], we omit the proof.

Now we prove Lemma 1.7.

Proof of Lemma 1.7. By Lemma 5.2, there is a subspace $\mathcal{Z}$ of $C(B_{X^*} \times B_{Y^{**}})$ such that $\mathcal{Y}$ is isometrically isomorphic to $\mathcal{Z}$. Let $\varphi \in (\mathcal{B}(X^*, Y^*), \tau)^*$ satisfying $|\varphi(S^*)| \le 1$ for $S^* \in \{S^* \in \mathcal{Y} : \|S^*\| \le 1\}$, and $\epsilon > 0$. Then by the Hahn-Banach, Riesz representation theorem, and Lemma 5.2, there is a Borel measure μ on $B_{X^*} \times B_{Y^{**}}$ with $\|\mu\| \le 1$ such that

$$\varphi(R^*) = \int_{B_{X^*} \times B_{Y^{**}}} y^{**}R^*x^* d\mu$$

for all $R^* \in \mathcal{Y}$. Now apply Lemma 5.1. Then for all $R^* \in \mathcal{Y}$

$$\begin{aligned}
\varphi(R^*) &= \int_{B_{X^*} \times B_{Y^{**}}} y^{**}R^*(\sum_{i=1}^{\infty} f_i(x^*)) d\mu \\
&= \sum_{i=1}^{\infty} \int_{B_{X^*} \times B_{Y^{**}}} y^{**}R^*(\sum_{j=1}^{\infty} \chi_{E_{i,j}}(x^*)x^*_{i,j}) d\mu \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{E_{i,j} \times B_{Y^{**}}} y^{**}R^*x^*_{i,j} d\mu \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} y^{**}_{i,j}R^*x^*_{i,j}
\end{aligned}$$

, where $y^{**}_{i,j}$ is the functional on Y^* defined by $y^{**}_{i,j}y^* = \int_{E_{i,j} \times B_{Y^{**}}} y^{**}y^* d\mu$. Since for every i, j, and $y^* \in B_{Y^*}$

$$|y^{**}_{i,j}y^*| \le \int_{E_{i,j} \times B_{Y^{**}}} |y^{**}y^*| d|\mu| \le |\mu|(E_{i,j} \times B_{Y^{**}})$$

, $\|y^{**}_{i,j}\| \le |\mu|(E_{i,j} \times B_{Y^{**}})$ for every i and j. Thus for every i, we have

$$\sum_{j=1}^{\infty} \|y^{**}_{i,j}\| \le \|\mu\| \le 1. \tag{5.1}$$

Also

$$\sum_{i=1}^{\infty} \sup_j \|x^*_{i,j}\| \le 1 + \epsilon. \tag{5.2}$$

Now recall in Section 2, **Fact**(a) and define the functional ψ_ϵ on $\mathcal{B}(X^*, Y^*)$ by

$$\psi_\epsilon(T) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} y^{**}_{i,j} T x^*_{i,j}.$$

Then $\psi_\epsilon(R^*) = \varphi(R^*)$ for every $R^* \in \mathcal{Y}$. From (5.1) and (5.2), $\psi_\epsilon \in (\mathcal{B}(X^*, Y^*), \tau)^*$ and

$$|\psi_\epsilon(T)| \le \|T\| \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \|x^*_{i,j}\| \|y^{**}_{i,j}\| \le (1+\epsilon)\|T\|$$

for every $T \in \mathcal{B}(X^*, Y^*)$. Hence ψ_ϵ is a desired functional. □

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