Averages of Nevanlinna counting functions of holomorphic self-maps of the unit disk *

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Abstract

We give an integral representation of the Nevanlinna counting function N_φ of a holomorphic self-map φ of the unit disk D in terms of its boundary values φ^*. This representation enables us to explicitly compute the averages of N_φ over the circle and over the small disks around the origin. As a consequence, we give, for example, a computational proof of the well known sub-averaging property of N_φ.

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1 Introduction

We are only concerned with holomorphic self-maps φ of the unit disk D on the complex plane. The Nevanlinna counting function

$$N_\varphi(w) = \sum_{\varphi(z)=w} \log \frac{1}{|z|}$$

plays a very important role in the holomorphic change of variables by $w = \varphi(z)$ in the integral representation [St] and in the study of the composition operator $C_\varphi(f) = f \circ \varphi$. For example, C_φ is a compact operator on the

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Hardy space H^2 if and only if $N_\varphi(w) = o\left(\log \frac{1}{|w|}\right)$. See [Sh1, Sh2]. In this paper, we obtain a representation of N_φ in terms of the boundary values φ^* of φ by applying the Jensen's formula to $\frac{a-\varphi}{1-\overline{a}\varphi}$ in Proposition 2.1. It clarifies the behavior of N_φ more clearly, and enables us to compute the averages of N_φ over the circles and over the small disk around the origin as in Theorem 3.1. The usefulness of such representations is justified by giving a computational proof of the well known sub-averaging property of N_φ and by other consequences and the representation of the Nevanlinna counting functions of Rudin's orthogonal functions in Section 4.

2 Another representation of N_φ

For a holomorphic self-map φ and $a \in D$, the bounded function $\frac{a-\varphi}{1-\overline{a}\varphi}$ has the canonical factorization as follows:

$$\frac{a-\varphi(z)}{1-\overline{a}\varphi(z)} = B_a(z)S_a(z)F_a(z), \tag{2.1}$$

where B_a is the Blaschke product

$$B_a(z) = \prod_{\varphi(z_i)=a} \frac{|z_i|}{z_i}\frac{z_i - z}{1-\overline{z_i}z}, \text{ (multiplicities counted)} \tag{2.2}$$

S_a is the singular inner function

$$S_a(z) = \exp\left(-\int_{\partial D} \frac{\zeta+z}{\zeta-z} d\mu_a(\zeta)\right) \tag{2.3}$$

with the positive Borel measure μ_a singular with respect to the normalized Lebesgue measure $d\sigma$ on the boundary ∂D of D, and F_a is the outer function given by

$$F_a(z) = e^{i\gamma} \exp\left(\int_{\partial D} \frac{\zeta+z}{\zeta-z} \log\left|\frac{a-\varphi^*(\zeta)}{1-\overline{a}\varphi^*(\zeta)}\right| d\sigma(\zeta)\right), \tag{2.4}$$

γ real, with $\varphi^*(\zeta) = \lim_{r\nearrow 1} \varphi(r\zeta)$ which exists almost every $\zeta \in \partial D$. See [G] for the canonical factorization. Applying the Jensen's formula to (2.1), we have for $a \neq \varphi(0)$

$$\log\left|\frac{a-\varphi(0)}{1-\overline{a}\varphi(0)}\right| + \sum_{\substack{\varphi(z_i)=a \\ |z_i|\le r}} \log\frac{r}{|z_i|} \tag{2.5}$$
$$= \int_{\partial D} \log\left|\frac{a-\varphi(r\xi)}{1-\overline{a}\varphi(r\xi)}\right| d\sigma(\xi)$$
$$= \int_{\partial D} \log|B_a(r\xi)|\, d\sigma(\xi) - \mu_a(\partial D)$$
$$+ \int_{\partial D} \log\left|\frac{a-\varphi^*(\zeta)}{1-\overline{a}\varphi^*(\zeta)}\right| d\sigma(\zeta).$$

We applied the Fubini's theorem to interchange the order of integration for the last equality. Letting $r \nearrow 1$ on both sides, we obtain

$$\log\left|\frac{a-\varphi(0)}{1-\overline{a}\varphi(0)}\right| + N_\varphi(a) \tag{2.6}$$
$$= -\mu_a(\partial D) + \int_{\partial D} \log\left|\frac{a-\varphi^*(\zeta)}{1-\overline{a}\varphi^*(\zeta)}\right| d\sigma(\zeta).$$

We note that for the Blaschke product B_a,

$$\lim_{r\nearrow 1} \int_{\partial D} \log B_a(r\xi) d\sigma(\xi) = 0. \tag{2.7}$$

See [G]. The representation (2.6) is a refined version of Lemma 2 in [N] or of Littlewood's inequality [Sh1, p.187] and is the main part of the following proposition.

Proposition 2.1 *Let φ be a holomorphic self-map of D. Then*

(a)

$$N_\varphi(a) = \log\left|\frac{1-\overline{a}\varphi(0)}{a-\varphi(0)}\right|$$
$$+ \int_{\partial D} \log\left|\frac{a-\varphi^*(\zeta)}{1-\overline{a}\varphi^*(\zeta)}\right| d\sigma(\zeta) - \mu_a(\partial D) \tag{2.8}$$
$$= -\log|a-\varphi(0)|$$
$$+ \int_{\partial D} \log|a-\varphi^*(\zeta)|\, d\sigma(\zeta) - \mu_a(\partial D) \tag{2.9}$$

for $a \neq \varphi(0)$*, where* μ_a *is the singular measure associated with the singular factor of* $\dfrac{a-\varphi}{1-\overline{a}\varphi}$*. In particular,*

$$N_\varphi(a) \leq \log\left|\frac{1-\overline{a}\varphi(0)}{a-\varphi(0)}\right|. \tag{2.10}$$

(b) *If* φ *is an inner function, i.e.,* $|\varphi^*(\zeta)| = 1$ *a.e.* $\zeta \in D$*, then*

$$N_\varphi(a) = \log\left|\frac{1-\overline{a}\varphi(0)}{a-\varphi(0)}\right| - \mu_a(\partial D). \tag{2.11}$$

(c) $\mu_a(\partial D) = 0$ *for nearly all* $a \in D$*, i.e., for all* a *in* D *except for a set of Logarithmic capacity zero and*

$$\mu_a(\partial D) \leq \log\left|\frac{1-\overline{a}\varphi(0)}{a-\varphi(0)}\right|. \tag{2.12}$$

In particular, $\mu_a(\partial D) \to 0$ *as* $|a| \nearrow 1$.

We note that (2.8) is another form of (2.6). Since $\log|1-\overline{a}\varphi|$ is harmonic in D, we see that

$$\log|1-\overline{a}\varphi(0)| = \int_{\partial D} \log|1-\overline{a}\varphi^*(\zeta)|d\sigma(\zeta). \tag{2.13}$$

(2.9) follows from (2.8) and (2.13). Since $\left|\dfrac{a-\varphi^*}{1-\overline{a}\varphi^*}\right| \leq 1$ and $\mu_a(\partial D) \geq 0$, (2.10) follows from (2.8).

For the inner function φ, $\left|\dfrac{a-\varphi^*}{1-\overline{a}\varphi^*}\right| = 1$ and so (2.11) follows from (2.8). For the proof of (c), we recall the generalized version of Frostman's theorem by W. Rudin. See [F, R3, R4]. The theorem is more general but we state it only for our bounded self-map φ.

Theorem A (Rudin) *Let* φ *be a bounded self-map of* D*. Then the least harmonic majorant of* $\log|a-\varphi|$ *is given by the Poisson integral of* $\log|a-\varphi^*|$ *for nearly all* $a \in D$*.*

Proof of Proposition 2.1(c): From (2.1), the canonical factorization of $a - \varphi(z)$ has the form:

$$a - \varphi(z) = B_a(z)S_a(z)F_a(z)(1-\overline{a}\varphi(z)).$$

We note that the outer factor of $a - \varphi(z)$ has the form

$$
\begin{aligned}
&F_a(z)(1 - \overline{a}\varphi(z)) \\
&\qquad = e^{i\gamma} \exp\left(\int_{\partial D} \frac{\zeta + z}{\zeta - z} \log |a - \varphi^*(\zeta)| d\sigma(\zeta) \right).
\end{aligned}
$$

Therefore, the least harmonic majorant of $\log |a - \varphi|$ is given the Poisson integral of

$$
\log |a - \varphi^*(\zeta)| d\sigma(\zeta) - d\mu_a(\zeta).
$$

See Lemma 5.2 in [G] for example. Therefore, it follows from Theorem A that $\mu_a = 0$ for nearly all $a \in D$. Finally, (2.12) follows also from (2.6). This completes the proof of Proposition 2.1. □

3 Averages of N_φ

The representation (2.9) of N_φ of enables us to compute the averages of Nevanlinna counting function N_φ over the circles and disks around the origin as in the following theorem, which have very useful consequences. It is very amusing fact that the averages can be neatly represented by the boundary values as in the following theorem.

Theorem 3.1 *Let φ be a holomorphic self-map of D. Then*

(a)

$$
\begin{aligned}
&\int_{\partial D} N_\varphi(\rho\eta) d\sigma(\eta) \qquad (3.1) \\
&\qquad = -\log^+ \frac{|\varphi(0)|}{\rho} + \int_{\partial D} \log^+ \frac{|\varphi^*(\zeta)|}{\rho} d\sigma(\zeta) \\
&\qquad = -\log^+ \frac{|\varphi(0)|}{\rho} + \int_\rho^1 \frac{\sigma\{|\varphi^*| > t\}}{t} dt \\
&\qquad = -\log^+ \frac{|\varphi(0)|}{\rho} - \int_\rho^1 \log \frac{t}{\rho} d\alpha(t),
\end{aligned}
$$

where $\alpha(t) = \sigma\{|\varphi^*| > t\}$ is a nonincreasing function of t and $\log^+ x = \max(\log x, 0)$.

(b)

$$\frac{1}{R^2}\iint_{|a|<R} N_\varphi(a)dA(a) \tag{3.2}$$
$$= \log\frac{1}{|\varphi(0)|} + \int_{\partial D}\log|\varphi^*(\zeta)|d\sigma(\zeta)$$
$$+\frac{1}{2}\int_{|\varphi^*|<R}\left\{\left(\frac{|\varphi^*(\zeta)|}{R}\right)^2 - 1\right.$$
$$\left. - \log\left(\frac{|\varphi^*(\zeta)|}{R}\right)^2\right\} d\sigma(\zeta),$$

for $0 < R < |\varphi(0)|$. Here, $dA = 2\rho\ d\rho\ d\sigma(\eta)$ denotes the normalized area measure.

Proof. (a) We first note that $\mu_a(\partial D)$ does not contribute to the averages since it is zero nearly all $a \in \partial D$. Write $a = \rho\eta$ and integrate (2.9) with respect to $d\sigma(\eta)$. Apply Fubini's theorem to interchange the order of integrations and use the well known integral

$$\int_{\partial D}\log|\rho\eta - w|d\sigma(\eta) = \log^+\frac{|w|}{\rho},$$

to obtain

$$\int_{\partial D} N_\varphi(\rho\eta)d\sigma(\eta) = -\int_{\partial D}\log|\rho\eta - \varphi(0)|d\sigma(\eta)$$
$$+\int_{\partial D}\int_{\partial D}\log|\rho\eta - \varphi^*(\zeta)|d\sigma(\eta)d\sigma(\zeta)$$
$$= -\log^+\frac{|\varphi(0)|}{\rho} + \int_{\partial D}\log^+\frac{|\varphi^*(\zeta)|}{\rho}d\sigma(\zeta),$$

for $\rho \neq |\varphi(0)|$. This relation is true for $\rho = |\varphi(0)|$ by continuity. The second representation follows by applying Theorem 8.16 in [R2] and the third follows by applying integration by part to the second representation. See Exercise 17 on p.141 in [R1] for example.

(b) Let $0 < R < |\varphi(0)|$. We integrate (3.1) against $\dfrac{2\rho}{R^2}d\rho$. First, compute

$$\begin{aligned}
&\frac{1}{R^2}\int_0^R 2\rho \log^+ \frac{|\varphi(0)|}{\rho} d\rho \\
&= \frac{1}{R^2}\int_0^R 2\rho \log|\varphi(0)| d\rho + \frac{1}{R^2}\int_0^R 2\rho \log\frac{1}{\rho}\, d\rho \\
&= \log\frac{|\varphi(0)|}{R} + \frac{1}{2}.
\end{aligned} \tag{3.3}$$

Now, we compute

$$\begin{aligned}
&\frac{1}{R^2}\int_0^R \int_{\partial D} 2\rho \log^+ \frac{|\varphi^*(\zeta)|}{\rho}\, d\rho \\
=&\frac{1}{R^2}\int_{|\varphi^*|<R}\int_0^R 2\rho \log^+ \frac{|\varphi^*(\zeta)|}{\rho}\, d\rho\, d\sigma(\zeta) \\
+&\frac{1}{R^2}\int_{|\varphi^*|\geq R}\int_0^R 2\rho \log^+ \frac{|\varphi^*(\zeta)|}{\rho}\, d\rho\, d\sigma(\zeta) \\
=&\frac{1}{R^2}\int_{|\varphi^*|<R}\int_0^{|\varphi^*(\zeta)|} 2\rho \log \frac{|\varphi^*(\zeta)|}{\rho}\, d\rho\, d\sigma(\zeta) \\
+&\frac{1}{R^2}\int_{|\varphi^*|\geq R}\int_0^R 2\rho \log \frac{|\varphi^*(\zeta)|}{\rho}\, d\rho \\
=&\frac{1}{2R^2}\int_{|\varphi^*|<R} |\varphi^*|^2\, d\sigma(\zeta) + \int_{|\varphi^*|\geq R} \log|\varphi^*(\zeta)|\, d\sigma(\zeta) \\
+&\frac{1}{2}\sigma\{|\varphi^*|\geq R\} - \sigma\{|\varphi^*|\geq R\}\log R \\
=&\int_{\partial D} \log\frac{|\varphi^*(\zeta)|}{R}\, d\sigma(\zeta) + \frac{1}{2} - \\
&\frac{1}{2}\int_{|\varphi^*|<R}\left\{\left(\frac{|\varphi^*(\zeta)|}{R}\right)^2 - 1 - \log\left(\frac{|\varphi^*(\zeta)|}{R}\right)^2\right\} d\sigma(\zeta)
\end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we obtain (3.2). This completes the proof.

The usefulness of the averages of N_φ is seen in the following corollary. Especially, it gives a new computational proof of the sub-averaging property of N_φ.

Corollary 3.2 *Let φ be a holomorphic self-map of D. Then*

(a) *Sub-averaging property. If* $0 < R < |\varphi(0)|$, *then*

$$N_\varphi(0) \le \frac{1}{R^2} \iint_{|w|<R} N_\varphi(w)\,dA(w). \tag{3.5}$$

See [Sh1].

(b)

$$\lim_{\rho \nearrow 1} \frac{1}{\log \frac{1}{\rho}} \int_{\partial D} N_\varphi(\rho\eta)\,d\sigma(\eta) = \sigma\{|\varphi^*| = 1\}. \tag{3.6}$$

In particular, (i) $|\varphi^*| < 1$ *a.e. if and only if*

$$\lim_{\rho \nearrow 1} \frac{1}{\log \frac{1}{\rho}} \int_{\partial D} N_\varphi(\rho\eta)\,d\sigma(\eta) = 0 \tag{3.7}$$

and (ii) φ *is inner, i.e.,* $|\varphi^*| = 1$ *a.e. if and only if*

$$\lim_{\rho \nearrow 1} \frac{1}{\log \frac{1}{\rho}} \int_{\partial D} N_\varphi(\rho\eta)\,d\sigma(\eta) = 1. \tag{3.8}$$

(c) φ *is inner if and only if*

$$\int_{\partial D} N_\varphi(\rho\zeta)\,d\sigma(\zeta) = \log \frac{1}{\max(|\varphi(0)|,\ \rho)} \tag{3.9}$$

for all $0 < \rho < 1$.

Proof. For (a), it suffices to note that

$$N_\varphi(0) = \log \frac{1}{|\varphi(0)|} + \int_{\partial D} \log|\varphi^*|\,d\sigma - \mu_a(\partial D)$$

from (2.8) and the quantity

$$\left\{ \left(\frac{|\varphi^*(\zeta)|}{R} \right)^2 - 1 - \log \left(\frac{|\varphi^*(\zeta)|}{R} \right)^2 \right\}$$

in (3.2) is nonnegative since $x - 1 \ge \log x$ for $x > 0$.

For (b), we note that $\log^+ \dfrac{\varphi(0)}{\rho} = 0$ for ρ sufficiently close to 1 and that

$$\begin{aligned} &\int_{\partial D} \log^+ \frac{|\varphi^*(\zeta)|}{\rho} d\sigma(\zeta) \\ &= \int_{|\varphi^*|=1} \log \frac{1}{\rho} d\sigma(\zeta) + \int_{\rho<|\varphi^*|<1} \log \frac{|\varphi^*(\zeta)|}{\rho} d\sigma(\zeta) \\ &= \sigma\{|\varphi^*| = 1\} \log \frac{1}{\rho} + \int_{\rho<|\varphi^*|<1} \log \frac{|\varphi^*(\zeta)|}{\rho} d\sigma(\zeta). \end{aligned} \tag{3.10}$$

(3.6) now follows since the second integral in (3.10) is dominated by

$$\sigma\{\rho < |\varphi^*| < 1\} \log \frac{1}{\rho} = o\left(\log \frac{1}{\rho}\right), \text{ as } \rho \nearrow 1.$$

(c): If φ is inner and $\rho < |\varphi(0)|$, then (3.1) becomes

$$\begin{aligned} \int_{\partial D} N_\varphi(\rho\eta) d\sigma(\eta) &= -\log^+ \frac{|\varphi(0)|}{\rho} + \log \frac{1}{\rho} \\ &= \log \frac{1}{\max(\rho,\ |\varphi(0)|)}. \end{aligned}$$

If φ is not inner, then $\sigma\{|\varphi^*| < 1\} > 0$. For $\rho > |\varphi(0)|$, (3.1) can be written as

$$\begin{aligned} &\int_{\partial D} N_\varphi(\rho\eta) d\sigma(\eta) \\ &\quad = \int_{|\varphi^*|=1} \log \frac{1}{\rho} d\sigma + \int_{\rho<|\varphi^*|<1} \log \frac{|\varphi^*(\zeta)|}{\rho} d\sigma(\zeta) \\ &\quad = (1 - \sigma\{|\varphi^*| < 1\}) \log \frac{1}{\rho} + o\left(\log \frac{1}{\rho}\right) \\ &\quad \neq \log \frac{1}{|\varphi(0)|}, \text{ as } \rho \nearrow 1. \end{aligned}$$

4 Application

Among the holomorphic self-maps φ with $\|\varphi\|_\infty \equiv \sup\{|\varphi(z)| : z \in D\} = 1$, T. Nakazi characterized the Rudin's orthogonal functions, *i.e.*, holomorphic self-maps φ for which $\{\varphi^n,\ n = 0, 1, 2, \cdots\}$ is orthogonal with respect to the usual inner product on H^2 as follows. For more on Rudin's orthogonal functions, see [B, N].

Theorem B (Nakazi) *For a holomorphic self-map of D with $\|\varphi\|_\infty$, the following are equivalent:*

(a) $\{\varphi^n,\ n = 0, 1, 2, \cdots\}$ *is orthogonal in* H^2.

(b) $N_\varphi(z) = N_\varphi(|z|)$ *for nearly all z in D.*

(c) *There exists a positive Borel probability measure ν_0 on $[0,1]$ with $1 \in$ supp ν_0 such that*

$$N_\varphi(z) = \int_{|z|}^1 \log \frac{t}{|z|}\ d\nu_0(t)$$

for nearly all z in D.

We can describe the measure ν_0 more precisely in terms of φ^* as an application of Theorem 3.1 (a). For Rudin's orthogonal functions φ, $\varphi(0) = 0$ and $N_\varphi(z)$ is the same as the its radialization, *i.e.*, its average over circles around the origin,

$$N_\varphi(\rho\zeta) = \int_{\partial D} N_\varphi(\rho\zeta)\ d\sigma(\zeta) = -\int_\rho^1 \log \frac{t}{\rho}\ d\alpha(t)$$

for nearly all $z \in D$ by Theorem 3.1 (a). Therefore, the positive Borel measure in Theorem B(c) is given by $d\nu_0 = -d\alpha(t)$ where $\alpha(t) = \sigma\{|\varphi^*| > 0\}$ as a nonincreasing function. Clearly, $1 \in$ supp ν_0 since $\|\varphi\|_\infty = 1$. We note that the mass at 1 is given by $\nu(\{1\}) = \sigma\{|\varphi^*| = 1\}$.

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