ON PARALLEL DISCRETE RANDOM NUMBER GENERATION AND WEAK APPROXIMATION OF SYSTEMS OF SDE'S

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Abstract. We suggest a simple parallel random number generation method for weak approximation of systems of SDE's by power series expansion expression. we also give statistical test results for the random numbers and numerical experiments for approximating SDE's.

1. Introduction

Let us consider a diffusion process X_t satisfying the following system of stochastic differential equations

$$dX_t^i = a^i(t, X_t)dt + \sum_{j=1}^{m} b^{ij}(t, X_t)dW_t^j, \qquad i = 1 \dots n \tag{1}$$

with $E(|X_0|^k) < \infty$ for $k = 1, 2, \cdots$ where W_t^j's are m independent Brownian motion and the coefficients a and b are Lipshitz continuous and with differentiable components a^i, b^{ij} such that

$$|a(x)| + |b(x)| < K(1 + |x|)$$

for some $K < \infty$ and $x \in \mathbb{R}^n$.

Numerical algorithms for weak approximation for such systems are well explained in [1] and [2]. But in practical implementations, we use Monte Carlo method and compute $\sum_{i=1}^{N} f(Y_t^{(i)})/N$, where $Y_t^{(i)}$ denotes ith implementation of numerical scheme Y_t. Weak approximations have an advantage that we can make use of simple discrete valued random variables with some moment conditions instead of normal variables. This saves some computation times. According to the simulation result in [3], CPU time for generating normal random numbers takes 1.2 times more than that for discrete valued random numbers but no gain in precision. For system (1), we need m 'independent' sequences of pseudo random numbers for weak order 1.0 Euler scheme and $m(m + 1)/2$ for weak order 2.0 numerical scheme. We can take m or $m(m + 1)/2$ different pseudo random number generators(PRNG's for short) in each case. There are a lot of PRNG's most of which are some variants of linear congruential generators and we may choose some of them for our purpose. But the independence between them is

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not guaranteed theoretically and moreover, when m is large, we may include some unreliable PRNG's. Hence, in many cases, we adopt methods generating independent streams of random numbers from a single uniform PRNG such as '$skipping\ ahead'$ and '$leapfrogging'$- see [4]. But these methods reduce the period of the generator and may not be appropriate in large scale Monte Carlo Simulations of systems (1) with many noises which often arise in physics and astronomy. In this Section, we suggest a very simple way of parallel discrete PRN generation using a single uniform PRNG by some integer base power series expansion of uniform random numbers. This method extends the period of the generator and also there is no risk of overlap. The independence between the parallel sequences only depends on the quality of the underlying uniform PRNG.

2. Weak Taylor Approximations

Weak order 1.0 Euler scheme : The weak order 1.0 Euler scheme for the system (1) has the form

$$Y^i_{n+1} = Y^i_n + a^i(t_n, Y_n) + \sum_{j=1}^{m} b^{i,j}(t_n, Y_n)\Delta W^j_n, \qquad Y_0 = X_0. \tag{2}$$

Here, we may substitute $\Delta\hat{W}^j$ for ΔW^j which satisfies the moment condition (**??**). We can choose $\Delta\hat{W}^j$ for example, random variables with distribution

$$P(\Delta\hat{W}^j = \pm\sqrt{\Delta}) = \frac{1}{2}.$$

Usually for $\Delta\hat{W}^j$, we generate a uniform random number U on $(0, 1)$ interval and compute $sign\,(U - 1/2)$ and multiply $\sqrt{\Delta}$. So, we need m 'independent' sequences of uniform PRN's.

Weak order 2.0 scheme : The order 2.0 weak Taylor scheme contains double stochastic integrals which are not easy to generate in general. Instead, the following $simplified\ order$ 2.0 $weak\ Taylor\ scheme$ is of practical use.

$$\begin{aligned} Y^i_{n+1} = Y^i_n + a^i\Delta + \frac{1}{2}L^0 a^i\Delta^2 + \sum_{j=1}^{m}\left\{\frac{1}{2}\Delta(L^0 b^{i,j} + L^j a^i)\right\}\Delta\hat{W}^j \\ + \frac{1}{2}\sum_{j_1,j_2=1}^{m} L^{j_1} b^{i,j_2}\left(\Delta\hat{W}^{j_1}\Delta\hat{W}^{j_2} + V_{j_1,j_2}\right). \end{aligned}$$

where $\Delta\hat{W}^j,\ j = 1\ldots m$ are independent random variables satisfying

$$P(\Delta\hat{W}^j = \pm\sqrt{3\Delta}) = \frac{1}{6}, \quad P(\Delta\hat{W}^j = 0) = \frac{2}{3} \tag{3}$$

and V_{j_1,j_2} are independent two-point distributed random variables such that

$$P(V_{j_1,j_2} = \pm\Delta) = \frac{1}{2}, \quad j_1 < j_2, \tag{4}$$

and

$$V_{j,j} = -\Delta \qquad V_{j_1,j_2} = -V_{j_2,j_1}\,.$$

So, we need $m + (m^2 - m)/2 = (m^2 + m)/2$ independent sequences of uniform PRN's for usually used method. We stress that this number grows rather rapidly when m is increased.

3. Algorithm

The following proposition is the motivation of this Section.

Proposition 1. *Let $M \in \mathbb{N} \setminus \{1\}$ and $\{M_i\}_{i\geq 1}$ be i.i.d. sequence of M-valued random variables such that $P(M_i = k) = 1/M$ for $k = 0, 1, \dots M-1$. Then*

$$\bar{U} := \sum_{i=1}^{\infty} M_i M^{-i}$$

has the uniform distribution on $(0,1)$.

Proof. Let $x \in (0,1)$ then it has the unique M-based expansion expression $x = \sum_{i=1}^{\infty} m_i M^{-i}$ and $P(\bar{U} = x) = P(M_1 = m_1, M_2 = m_2 \dots) = 0$. So, $\bar{U}$ does not have a point mass. Hence it is enough to show $P(\bar{U} \leq x) = x$ for $x = \sum_{i=1}^{n} m_i M^{-i}$

$$\begin{aligned} P(\bar{U} \leq x) &= P(0 \leq \bar{U} < m_1/M) + P(m_1/M \leq \bar{U} < m_1/M + m_2/M^2) \\ &\quad + \cdots + P(\sum_{i=1}^{n-1} m_i M^{-i} \leq \bar{U} < \sum_{i=1}^{n} m_i M^{-i}) \\ &= m_1/M + 1/M_2 \cdot m_2/M + \cdots 1/M^{n-1} \cdot m_n/M = x \end{aligned}$$

□

Hence, from one sample $x = \sum_{i=1}^{\infty} m_i M^{-i} \in (0,1),\ 0 \leq m_i \leq M-1$, we can extract a sequence of sample values $\{m_i\}_{i\geq 1}$ of $\{M_i\}_{i\geq 1}$ as follows:

$$m_i = \lfloor x \cdot M^i \rfloor \qquad (\mathrm{mod}\ M).$$

So, if we generate a sequence $\{x^j\}_{j\geq 1}$ from a uniform PRNG then we obtain the corresponding matrix of M-valued random numbers

$$\mathbb{M} = \begin{bmatrix} m_1^1 & m_1^2 & \cdots & m_1^n \\ m_2^1 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ m_k^1 & \cdots & \cdots & m_k^n \end{bmatrix}. \tag{5}$$

The k random numbers in the first column of $\mathbb{M}$ are as 'random' as rolling a dice in their nature and the quality of total random numbers in $\mathbb{M}$ depends only on the quality of the underlying uniform PRNG.

For the purpose of the weak Euler approximation, $M = 2,\ k = m$ and we assign $0 \to -1$, $1 \to 1$ and multiply $\sqrt{\Delta}$ for $\Delta\hat{W}$. In this case, rows up to $k = 53$

in (5) were available for weak approximation of scalar SDE in Matlab. For weak 2.0 Taylor scheme, $M = 6$, $k = m(m+1)/2$ and we need the following simple additional works. Since we need 3-valued random variables with distribution (3), we assign

$$0 \to -1, \quad 1 \to 1, \quad \{2,3,4,5\} \to 0$$

and multiply $\sqrt{3\Delta}$ for $\Delta\hat{W}$. We make m such sequence of random numbers and with remaining $(m^2 - m)/2$ sequences, we assign

$$\{0,1,2\} \to -1 \quad \{3,4,5\} \to 1$$

and multiply Δ for V_{j_1,j_2} in (4).

4. Some Statistical Tests

4.1. **Test for independence between rows of $\mathbb{M}$:** Let us run N columns of k-vector as in (5). Then there appear M^k different vectors $[m_1\ m_2\ \ldots\ m_k]$ where $m_k = 1, 2, \ldots, M$ with same probability $1/M^k$ and we count the number X_i whenever each column vector falls into each category. Then the statistic

$$Y = \sum_{i=1}^{M^k} \frac{(X_i - N/M^k)^2}{N/M^k}$$

is asymptotically distributed as χ^2 with $M^k - 1$ degree of freedom. The null hypothesis of independence is rejected if the computed value Y exceeds $\chi^2_{M^k-1,\alpha}$. But examining this test a little more, it is equivalent with the χ^2 goodness of fit test for the underlying uniform PRNG. Each vector above corresponds to a small subinterval with length $1/M^k$ in $(0,1)$. Since most widely used uniform PRNG's pass the χ^2 goodness of fit test - see [6], they already passed the above χ^2 test of independence.

4.2. **Run test for independence of each row :** As in [8], we can perform a run test to test independence of each row. We count consecutive numbers of m_i instead of runs-up and runs-down. For $M = 2$, we count up to 6 each consecutive numbers as in [8]. Consecutive sequences longer than 6 is lumped together. For these 14 categories, we perform a χ^2-test for 13 degrees of freedom.

4.3. **χ^2 goodness of fit test for each row :** Let us divide each row of $\mathbb{M}$ into vectors with l elements. Then there are M^l categories. We take l such that $M^l \sim 1000$ and $N = 6000 * l$ as recommended in [7] and we compute the following statistic for each row

$$Y = \sum_{i=1}^{M^l} \frac{(X_i - 6000/M^l)^2}{6000/M^l}.$$

Remark 1. The above tests are all standard ones and the test results for the newly generated random numbers seem to be just as good as the usually used method. Maybe it is because the underlying PRNG is already good enough. We might be satisfied with the longer period of the new method for a while until new statistical evidences appear.

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