On Random Discretization for Ito Diffusion Processes

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Abstract

Let X_t be 1-dimensional diffusion process satisfying some stochastic differential equation. We discretize the equation at stopping times and give an upper error bound for the Euler scheme w.r.t some moment of maximum step size. We give a random discretization example for uniform approximation of Brownian motion process and provide a method for the related random number generation. We compare the values of the uniform error bounds between fixed and adaptive random discretizations.

Key words: diffusion process, random discretization, uniform approximation, random number generation

1 Introduction

Let X_t be 1-dimensional diffusion process satisfying the following equation

$$X_t = a(t, X_t)dt + b(t, X_t)dW_t, \quad X_0 = 0, \quad t \in [0, 1] \tag{1}$$

where W_t is a standard 1 dimensional Brownian motion and the coefficient functions a and b are measurable functions satisfying

$$|a(t, x)| + |b(t, x)| \leq K(1 + |x|); \quad x \in \mathbb{R}, \quad t \in [0, 1]$$

and

$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq K|x - y|; \quad x, y \in \mathbb{R}^n, \quad t \in [0, 1]$$

for some constant K.

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Preprint submitted to Elsevier Science *20 July 2002*

In the strong approximation of stochastic differential equations, we choose discrete time steps and apply a numerical scheme with partial information of the driving Brownian motion i.e. discretely observed Brownian motion and some multiple stochastic integrals on each time subinterval. So, the time sampling is closely related to the quality of the approximation. Most commonly used time discretization is equidistant one. But if we control the step sizes according to the local errors adaptively, we may reduce the error amount or save some computational costs for a given allowed error bound - see [2] and [7]. Some general commentaries on random discretizations are found in [3] and the predictability of the discretization times is emphasized for the recursive computation i.e., the $n+1$ - th time node τ_{n+1} should be $\mathcal{A}_{\tau_n}$ measurable for a given filtration $\{\mathcal{A}_t\}_{t\geq 0}$. Also the deterministic upper bound for the maximum step size is imposed for the convergence of numerical scheme. In recent years, asymptotically optimal discretization methods are developed by N. Hofmann at al. for pathwise global errors and linearly interpolated numerical solutions - see [4], [5], and [8]. We review them briefly here. Let X_t be the solution of (1) and $\bar{Y}_t$ be the linearly interpolated numerical solution. In [5], the following global L_2-norm error

$$e(\bar{Y}) = \left[E\left(\int_0^1 |X_t - \bar{Y}_t|^2 dt \right) \right]^{1/2}$$

was considered and the asymptotically optimal time steps are suggested as

$$\tau_{n+1} = \tau_n + \min\left(\frac{h}{|b|(\tau_n, \bar{X}_{\tau_n})}, h^{2/3} \right), \qquad h > 0.$$

The convergence rate for this time discretization is also shown to be truly optimal among all approximations based on N measurable selection of observations of the driving Brownian motion. In [8], d-dimensional system of SDEs with m-noise was considered with a different error criterion, global uniform L^p-norm error,

$$e_p(\bar{X}) = (E||X - \bar{Y}||_\infty^p)^{1/p}, \qquad p \geq 1,$$

where

$$||X - \bar{Y}||_\infty = \max_{t\in[0,1]} \max_{1\leq i\leq d} |X_i(t) - \bar{Y}_i(t)|,$$

and with a different adaptive time selection. These discretization strategies follow both the predictability and the deterministic maximum step size restriction. In this article, we give a general error bound for Euler scheme for random discretization at stopping times w.r.t some moment of the maximum step size. We do not assume the predictability condition nor impose any deterministic upper bound for the step sizes. As an application example of random discretization, we suggest a method for the uniformly close reconstruction of Brownian motion process. We discuss about the related random number generation and compare the asymptotic values of the uniform errors.

2 An Error Bound for Random Discretizations

Consider the scalar SDE (1). Let $0 = \tau_0 < \cdots < \tau_{N+1} = 1$ be stopping times where N is also random and $N < \infty$ $w.p.1$. Then X_t in (1) is written as

$$X_{\tau_{n+1}} = \sum_{k=0}^{n\wedge N} \int_{\tau_k}^{\tau_{k+1}} a(s, X_s)ds + \sum_{k=0}^{n\wedge N} \int_{\tau_k}^{\tau_{k+1}} b(s, X_s)dW_s, \qquad 0 \le n \le N,$$

and the discrete Euler scheme is

$$Y_{\tau_{n+1}} = \sum_{k=0}^{n\wedge N} a(\tau_k, Y_k)\Delta\tau_k + \sum_{k=0}^{n\wedge N} b(\tau_k, Y_{\tau_k})\Delta W_{\tau_k}$$

where $\Delta\tau_k = \tau_{k+1} - \tau_k$ and $\Delta W_{\tau_k} = W_{\tau_{k+1}} - W_{\tau_k}$. We measure the error of the Euler scheme on these random times with the following maximum L^p-norm

$$\left[E\left(\max_{1\le k\le N+1} |X_{\tau_k} - Y_{\tau_k}|^p \right) \right]^{1/p}, \qquad p \ge 2.$$

For an unspecified positive constant K, we have

$$\begin{aligned} \max_{1\le n\le N+1} |X_{\tau_k} - Y_{\tau_k}|^p &\le K \left(\sum_{k=0}^{N} \int_{\tau_k}^{\tau_{k+1}} |a(s, X_s)|ds \right)^p + K \max_{1\le n\le N} \left| \sum_{k=0}^{n} \int_{\tau_k}^{\tau_{k+1}} b(s, X_s)dW_s \right|^p \\ &\le K(A_1 + A_2 + A_3 + B_1 + B_2 + B_3), \end{aligned}$$

where

$$\begin{aligned} A_1 &= \left(\sum_{k=0}^{N} \int_{\tau_k}^{\tau_{k+1}} |a(s, X_s) - a(\tau_k, X_s)|ds \right)^p, \\ B_1 &= K \max_{1\le n\le N} \left| \sum_{k=0}^{n} \int_{\tau_k}^{\tau_{k+1}} (b(s, X_s) - b(\tau_k, X_s))dW_s \right|^p, \end{aligned}$$

and let A_2 and A_3 be A_1 with the integrands replaced by

$$|a(\tau_k, X_s) - a(\tau_k, X_{\tau_s})| \quad \text{and} \quad |a(\tau_k, X_{\tau_k}) - a(\tau_k, Y_{\tau_k})|$$

respectively. B_2 and B_3 are defined similarly for B_1.

Lemma 1

$$||A_1||_2 + ||A_2||_2 + ||B_1||_2 + ||B_2||_2 \le KE\left(\max_{0\le k\le N} \Delta\tau_k^{2p} \right)^{1/4}.$$

Proof. Note that

$$\sum_{k=0}^{(n_t-1)\wedge N} \int_{\tau_k}^{\tau_{k+1}} b(\cdot,\cdot)dW_s + \int_{\tau_{n_t}}^{t} b(\cdot,\cdot)dW_s$$

is a continuous time martingale. Hence, by the Burkholder-Davis-Gundy's inequality,

$$\begin{aligned}
EB_1^2 &\le KE\left(\left[\sum_{k=0}^{N}\int_{\tau_k}^{\tau_{k+1}}(b(s,X_s)-b(\tau_k,X_s))^2 ds\right]^p\right) \\
&\le KE\left(\left[\sum_{k=0}^{N}\int_{\tau_k}^{\tau_{k+1}}(1+|X_s|)^2\Delta\tau_k ds\right]^p\right) \\
&\le KE\left[\sup_{0\le t\le 1}(1+|X_s|)^{2p}\cdot\max_{0\le k\le N}\Delta\tau_k^p\right] \\
&\le KE\left(\max_{0\le k\le N}\Delta\tau_k^{2p}\right)^{1/2}.
\end{aligned}$$

By using the B-D-G's inequality again, we have

$$\begin{aligned}
EB_2^2 &\le KE\left(\left[\sum_{k=0}^{N}\int_{\tau_k}^{\tau_{k+1}}(b(\tau_k,X_s)-b(\tau_k,X_{\tau_k}))^2 ds\right]^p\right) \\
&\le KE\left(\left[\sum_{k=0}^{N}\int_{\tau_k}^{\tau_{k+1}}\left(\int_{\tau_k}^{s}|a(t,X_t)|dt\right)^2 ds\right]^p\right) \\
&+ KE\left(\left[\sum_{k=0}^{N}\int_{\tau_k}^{\tau_{k+1}}\sup_{\tau_k\le s\le\tau_{k+1}}\left|\int_{\tau_k}^{s} b(t,X_t)dW_t\right|^2 ds\right]^p\right).
\end{aligned}$$

The first term is easily estimated by $KE\left(\max_{0\le k\le N}\Delta\tau_k^{4p}\right)^{1/2}$ and by the B-D-G's inequality, the second term is estimated as

$$\begin{aligned}
&E\left(\left[\sum_{k=0}^{N}\int_{\tau_k}^{\tau_{k+1}}\sup_{\tau_k\le s\le\tau_{k+1}}\left|\int_{\tau_k}^{s} b(t,X_t)dW_t\right|^2 ds\right]^p\right) \\
&\qquad\le E\left(\sup_{0\le k\le N}\sup_{\tau_k\le s\le\tau_{k+1}}\left|\int_{\tau_k}^{s} b(t,X_t)dW_t\right|^{2p}\right) \\
&\qquad= E\left(\sup_{0\le s\le 1}\left|\int_0^s b(t,X_t)I_{\tau_{n_s}\le t\le s}dW_t\right|^{2p}\right) \\
&\qquad\le KE\left(\left[\int_0^1 b^2(t,X_t)I_{\tau_N\le t\le 1}dt\right]^p\right) \\
&\qquad\le KE\left(\sup_{0\le t\le 1}|b(t,X_t)|^{2p}\cdot\max_{0\le k\le N}\Delta\tau_k^p\right)\le KE\left(\max_{0\le k\le N}\Delta\tau_k^{2p}\right)^{1/2}.
\end{aligned}$$

Similarly, we can estimate $||A_1||_2$ and $||A_2||_2$.

Let us consider a σ-field

$$\Psi = \sigma(N, \tau_1, \ldots, \tau_N).$$

Since

$$E(A_3|\Psi) \leq K \sum_{k=0}^{N} E\left(\max_{0\leq m\leq k} |X_{\tau_m} - Y_{\tau_m}|^p|\Psi\right) \Delta\tau_k,$$

by Gronwall's inequality, we have

$$E\left(\max_{1\leq k\leq N+1} |X_{\tau_k} - Y_{\tau_k}|^p|\Psi\right) \leq KE\left(A_1 + A_2 + B_1 + B_2 + B_3|\Psi\right).$$

Since

$$\begin{aligned}
E(B_3|\Psi) &\leq KE\left(\left[\sum_{k=0}^{N}(b(\tau_k, X_{\tau_k}) - b(\tau_k, Y_{\tau_k}))^2(\Delta W_{\tau_k})^2\right]^{p/2}|\Psi\right) \\
&\leq KE\left(\left[\sum_{k=0}^{N}(X_{\tau_k} - Y_{\tau_k})^2(\Delta W_{\tau_k})^2\frac{1}{N}\right]^{p/2} \cdot N^{p/2}|\Psi\right) \\
&\leq K\sum_{k=0}^{N} E\left(|X_{\tau_k} - Y_{\tau_k}|^p|\Delta W_{\tau_k}|^p|\Psi\right) \cdot \frac{1}{N} \cdot N^{p/2} \\
&\leq K\sum_{k=0}^{N} E\left(\max_{1\leq m\leq k}|X_{\tau_m} - Y_{\tau_m}|^p|\Psi\right) \cdot \frac{1}{N} \cdot \left(N^{p/2} \cdot \max_{0\leq k\leq N}|\Delta W_{\tau_k}|^p\right),
\end{aligned}$$

we apply the Gronwall's inequality again to have

$$\begin{aligned}
E\left(\max_{1\leq k\leq N+1} |X_{\tau_k} - Y_k|^p|\Psi\right) &\leq KE\left(A_1 + A_2 + B_1 + B_2|\Psi\right) \\
&\qquad \cdot \exp\left\{N^{p/2} \cdot \max_{0\leq k\leq N}|\Delta W_{\tau_k}|^p\right\}.
\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}
E\left(\max_{1\leq k\leq N+1} |X_{\tau_k} - Y_{\tau_k}|^p\right) &\leq K\left(||A_1||_2 + ||A_2||_2 + ||B_1||_2 + ||B_2||_2\right) \\
&\qquad \cdot E\exp\left\{2N^{p/2} \cdot \max_{0\leq k\leq N}|\Delta W_{\tau_k}|^p\right\}.
\end{aligned}$$

Hence, by applying Lemma 1, we obtain the desired result ;

Theorem 2 *Let N and stopping times $\{\tau_k\}_{1\leq k\leq N}$ satisfy*

$$E\exp\left\{2N^{p/2} \cdot \max_{0\leq k\leq N}|\Delta W_{\tau_k}|^p\right\} < K, \tag{2}$$

for some $K > 0$ and $p \geq 2$, then

$$E\left(\max_{1\leq k\leq N+1} |X_{\tau_k} - Y_{\tau_k}|^p\right) \leq KE\left(\max_{0\leq k\leq N} \Delta\tau_k^{2p}\right)^{1/4}.$$

3 Uniformly Close Reconstruction of Brownian Motion

For numerical simulation of Brownian motion, we use normal random numbers with small variance, add them one by one on discretization points and interpolate them linearly. But we don't know what will happen actually on each small time subinterval even though their probabilities are small. Here, we suggest a new method for uniformly close reconstruction of Brownian motion for which we need not worry about that small amount of probability. This is also a motivation for section 2.

We are concerned in the following simple processes

$$dX_t = adt + bdW_t, \qquad X_0 = 0, \tag{3}$$

where a and b are constants. We call the process X_t *Brownian motion with drift.*

For a small $h > 0$, let τ_h be the first hitting time of W_t to the levels $y = h$ or $y = -h$ and recursively we define

$$\tau_n := \inf\{t > \tau_{n-1} : W_t > W_{\tau_{n-1}} + h \text{ or } W_t < W_{\tau_{n-1}} - h\}, \tag{4}$$

$1 \leq n \leq N$ and

$$N := \sup_{n \geq 1}\{n : \tau_n < 1\}, \qquad \tau_{N+1} = 1.$$

Hence we obtain a sequence of level crossing stopping times $\{\tau_n\}_{1\leq n\leq N+1}$ before $t = 1$. On the random times $\{\tau_n\}_{1\leq n\leq N+1}$, we apply the Euler scheme

$$Y_{\tau_{n+1}} = \sum_{k=0}^{n\wedge N} a\Delta\tau_k + \sum_{k=0}^{n\wedge N} b\Delta W_{\tau_k} \tag{5}$$

and linearly interpolate the discrete points. This numerical process coincides with X_t on $\{\tau_n\}_{1\leq n\leq N+1}$ and

$$\sup_{t\in[0,1]} |X_t - Y_t| < 2bh \tag{6}$$

i.e., Y_t has a deterministic uniform error bound by our construction. For the stopping times $\{\tau_n\}_{1\leq n\leq N+1}$, N, the condition (2) becomes

$$Ee^{2h^2N} < K$$

for some K but we could not prove it because of the complicated density function of τ. But still Y_t converges to X_t due to (6) when $h \downarrow 0$. Note that when W_t fluctuates much, it crosses many levels and the discretization number increases and vice versa. Hence, our new method provides us a nice adaptive time discretization for the process (3) while the adaptive schemes in [5] and

[8] only suggest equidistant time discretization. Hence, our method is useful for the simple process (3) when just small number of sampling (in the average sense) is possible. The only problem we encounter is the random number generation for τ.

4 Random Number Generation for the First Hitting Times

For the implementation of the numerical scheme (5), we need to generate $\Delta\tau_k$'s and ΔW_{τ_k}'s. ΔW_{τ_k}'s are simply h times a Bernoulli sequence and $\tau_h = \Delta\tau_k$ is the first hitting time of a Brownian motion to the two boundaries $y = h$ and $y = -h$. Hence the density function of $\Delta\tau_k$ is given as

$$f_h(t) := \frac{2}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} (4n+1)h \exp\left\{-\frac{\{(4n+1)h\}^2}{2t}\right\}, \quad t > 0 \qquad (7)$$

- see [6]. The cumulative distribution function (CDF) is depicted in Figure 1 when $h = 1$. We will use the inverse CDF method. But when h is small, numerical error will be big. So we use normalization and rescaling instead. Recall that

$$Z_t = \frac{1}{h} W_{h^2 t}, \qquad t \geq 0$$

is also a standard Brownian motion. Hence, the first hitting time of W_t to the boundaries $\pm h$ is h^2 times that for Z_t to the boundaries ± 1. So, we only need to generate random numbers for $h = \pm 1$ and we denote $\tau = \tau_1$. We use the inverse CDF method for the boundary ± 1 and the Newton's method is a possible way to find the inverse values of the uniform random numbers i.e., for $F(t) := \int_0^t f(s)ds$, and for each uniform random number $u \in (0,1)$, we will find the zero of the equation, $\tilde{F}(t) := F(t) - u = 0$. Then for an appropriate initial guess $t_0 > 0$, we obtain a very close solution in just several steps as follows:

$$t^{(n+1)} = t^{(n)} - \frac{\tilde{F}(t^{(n)})}{\tilde{F}'(t^{(n)})}, \qquad n \geq 0.$$

But if we include many terms in our calculation from the infinite series (7), this method is found to be time consuming. So, we tried the following simplified algorithm in Matlab :

(1) Equally divide the horizontal axis time interval $[0, 6]$ into $6 \times n$ and find the values of the distribution function there by integrating the density function from 0 up to each time node. Let us denote the integrated values by $F(i),\ 1 \leq i \leq 6 \times n$.
(2) Generate M uniform random number u's and for each u, we assign $u \rightarrow i/n$ if $F(i) \leq u < F(i+1)$.

<< *Figure* 1 *may be located here* >>

<< *Table* 1 *may be located here* >>

We tested the newly generated random numbers for some moments of τ which is obtained from the exponential martingale $\exp\left(\lambda W_t - \lambda^2 t/2\right)$, $\lambda \in \mathbb{R}$ and related polynomial type martingales - see [1] for the first and second moment and the third moment is also obtained similarly using the polynomial martingale

$$W_t^6 - 15tW_t^4 + 45t^2W_t^2 - 15t^3, \quad t \geq 0,$$

which are for $h = 1$,

$$E\tau = 1, \quad E\tau^2 = 5/3, \quad \text{and} \quad E\tau^3 = 61/15.$$

We included ± 30 terms from the infinite series (7). We took $n = 1000$ and $M = 5000$ and 7000. See Table 1 for the test results.

5 A Comparison for Errors

For fixed equidistant discretization, usually, the uniform error for Brownian motion W_t is

$$E\left[\max_{0\leq t\leq 1} |W_t - Y_t|\right]$$

for the approximate process Y_t i.e., the average of the maximum error. But for our adaptive random discretization (4), the uniform error has the deterministic upper bound and not uniform just in the average sense. In this section, we compare the errors for these two discretizations i.e. the average sense uniform error for the equidistant discretization (AE) and the deterministic uniform error for the random discretization (4)(DR).

Since $E\tau_h = h^2$, $E\tau_h \cdot EN \approx 1$, and the deterministic error bound is $2h$ for standard Brownian motion, we have

$$2h \approx \frac{2}{\sqrt{EN}}. \tag{8}$$

We let $m = EN$ and compare the deterministic error $2h$ with the average error for m-equidistant discretization.

Let $Y^{(m)}(t)$ be the linearly interpolated Brownian motion for the equidistant nodes $\{i/m\}_{1\leq i\leq m}$ and $B_i,\ i = 1, 2, \ldots$ be independent Brownian bridges on the unit interval, then

$$E\Big[\max_{0\le t\le 1}|W_t - Y^{(m)}(t)|\Big] = \frac{1}{\sqrt{m}}\cdot E\Big[\max_{0\le l\le m}||B_l||_\infty\Big],$$

where $||B_l||_\infty = \max_{0\le s\le 1}|B_l(s)|$. According to Corollary 1 of [4], we have

$$\lim_{m\to\infty}\frac{1}{\sqrt{\ln m}}\cdot E\Big[\max_{0\le l\le m}||B_l||_\infty\Big] = \frac{1}{\sqrt{2}}$$

and hence

$$E\Big[\max_{0\le t\le 1}|W_t - Y^{(m)}(t)|\Big] \approx \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{\ln m}}{\sqrt{m}} \tag{9}$$

when m is large. In Table 2, we compare the two kinds of errors (DR) and (AE) by the values (8) and (9). When m increases, we can observe that DR becomes even smaller than AE.

<< Table 2 may be located here >>

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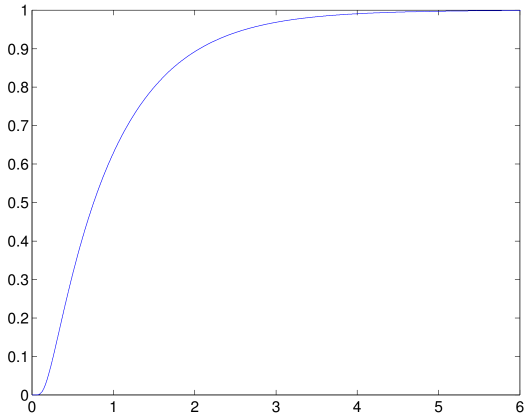

Fig. 1. The distribution function for τ, $h = 1$.

	Eτ	Eτ^2	Eτ^3
exact	1	1.6667	4.0667
M=5000	1.0052	1.6779	4.0505
M=7000	1.0129	1.6904	4.0557

Table 1
Test for the moments of τ

	$m = 10^2$	$m = 10^3$	$m = 10^4$	$m = 10^5$
DR	0.2000	0.0632	0.0200	0.0063
AE	0.1517	0.0588	0.0215	0.0076

Table 2
Comparison for the uniform errors