JACOBI-STIRLING NUMBERS, JACOBI POLYNOMIALS, AND THE LEFT-DEFINITE ANALYSIS OF THE CLASSICAL JACOBI DIFFERENTIAL EXPRESSION

W. N. EVERITT, K. H. KWON, L. L. LITTLEJOHN, AND G. J. YOON

Abstract. We develop the left-definite analysis associated with the self-adjoint operator $A_k^{(\alpha,\beta)}$ in the Hilbert space $L^2_{\alpha,\beta}(-1,1) := L^2((-1,1); w_{\alpha,\beta}(t))$, where $w_{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$, generated from the classical second-order Jacobi differential equation

$$\ell_{\alpha,\beta,k}[y](t) = \frac{1}{w_{\alpha,\beta}(t)}\left(\left(-(1-t)^{\alpha+1}(1+t)^{\beta+1}y'(t)\right)' + k(1-t)^\alpha(1+t)^\beta y(t)\right) \quad (t \in (-1,1)),$$

that has the Jacobi polynomials $\{P_m^{(\alpha,\beta)}\}_{m=0}^\infty$ as eigenfunctions; here, $\alpha, \beta > -1$ and k is a fixed, non-negative constant. More specifically, for each $n \in \mathbb{N}$, we explicitly determine the unique left-definite Hilbert-Sobolev space $W_{n,k}^{(\alpha,\beta)}(-1,1)$ associated with $(L^2_{\alpha,\beta}(-1,1), A_k^{(\alpha,\beta)})$. Moreover, for each $n \in \mathbb{N}$, we determine the corresponding unique left-definite self-adjoint operator $B_{n,k}^{(\alpha,\beta)}$ in $W_{n,k}^{(\alpha,\beta)}(-1,1)$ and characterize its domain in terms of another left-definite space. The key to determining these spaces and inner products is in finding the explicit Lagrangian symmetric form of the integral composite powers of $\ell_{\alpha,\beta,k}[\cdot]$. In turn, the key to determining these powers is a remarkable new identity involving a double sequence of numbers which we call *Jacobi-Stirling numbers.*

1. Introduction

In a recent paper [16], Littlejohn and Wellman developed a general abstract left-definite theory for self-adjoint, bounded below operators A in a Hilbert space $(H, (\cdot,\cdot))$. More specifically, they construct a continuum of unique Hilbert spaces $\{(W_r, (\cdot,\cdot)_r)\}_{r>0}$ and, for each $r > 0$, a unique self-adjoint restriction B_r of A in W_r. The Hilbert space W_r is called the r^{th} *left-definite Hilbert space* associated with the pair (H, A) and the operator B_r is called the r^{th} *left-definite operator* associated with (H, A); further details of these constructions, spaces, and operators is given in Section 2 below. Left-definite theory (the terminology *left-definite* is due to Schäfke and Schneider (who used the German *Links-definit*) [23] in 1965) has its roots in the classic treatise of Weyl [28] on the theory of formally symmetric second-order differential expressions. We remark, however, that even though our motivation for the general left-definite theory developed in [16] arose through our interest in certain self-adjoint *differential* operators, the theory developed in [16] can be applied to any strictly positive, self-adjoint operator in a Hilbert space.

In this paper, we apply this left-definite theory to the self-adjoint Jacobi differential operator $A_k^{(\alpha,\beta)}$, generated by the classical second-order Lagrangian symmetrizable (see [15]) Jacobi differential expression

$$(1.1)\qquad \ell_{\alpha,\beta,k}[y](t) := \frac{1}{w_{\alpha,\beta}(t)}\left(\left(-(1-t)^{\alpha+1}(1+t)^{\beta+1})y'(t)\right)' + k(1-t)^\alpha(1+t)^\beta y(t)\right)$$
$$= -(1-t^2)y'' + (\alpha-\beta+(\alpha+\beta+2)t)y'(t) + ky(t) \quad (t \in (-1,1)),$$

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where

$$w_{\alpha,\beta}(t) := (1-t)^{\alpha}(1+t)^{\beta} \quad (t \in (-1,1)), \tag{1.2}$$

which has the Jacobi polynomials $\{P_r^{(\alpha,\beta)}\}_{r=0}^{\infty}$ as eigenfunctions. Throughout this paper, we assume $\alpha, \beta > -1$ and k is a fixed, non-negative constant. The right-definite setting for this Jacobi differential expression is the Hilbert space $L^2((-1,1); w_{\alpha,\beta}(t)) := L^2_{\alpha,\beta}(-1,1)$, defined by

$$L^2_{\alpha,\beta}(-1,1) := \{f : (-1,1) \to \mathbb{C} \mid f \text{ is Lebesgue measurable and } \int_{-1}^{1} |f(t)|^2 \, w_{\alpha,\beta}(t)dt < \infty\}, \tag{1.3}$$

with inner product

$$(f,g)_{\alpha,\beta} := \int_{-1}^{1} f(t)\overline{g}(t) w_{\alpha,\beta}(t)dt \quad (f,g \in L^2_{\alpha,\beta}(-1,1)). \tag{1.4}$$

When $\alpha = \beta = 0$, (1.1) is the *Legendre* differential expression which, for later purposes, we list as

$$\ell_{0,0,k}[y](t) := -\left((1-t^2)y'(t)\right)' + ky(t) \quad (t \in (-1,1)). \tag{1.5}$$

Historically, it was Titchmarsh (see [25] and [26]) who first studied, in detail, the analytic properties of $\ell_{0,0,k}[\cdot]$ in the right-definite setting $L^2(-1,1) := L^2_{0,0}(-1,1)$. In particular, he showed that the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ are eigenfunctions of a certain self-adjoint operator, which we denote by $A_k^{(0,0)}$, generated by the singular differential expression $\ell_{0,0,k}[\cdot]$. Another detailed reference concerning the Legendre expression, and its properties, is [7]; for a comprehensive treatment of the Jacobi expression (1.1) in the setting $L^2_{\alpha,\beta}(-1,1)$, see the thesis of Onyango-Otieno in [18].

This paper may be seen as a continuation of the results obtained for the Legendre differential expression in [10] and the related papers [3] and [8]. In [10], the authors develop the left-definite theory of the operator $A_k^{(0,0)}$ in the Hilbert space $L^2(-1,1)$, generated by (1.5), and having the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ as eigenfunctions. In [3], a new characterization of the domains $\mathcal{D}(A_k^{(0,0)})$ and $\mathcal{D}(B_{1,k}^{(0,0)})$, where $B_{1,k}^{(0,0)}$ is the first left-definite operator associated with $(L^2(-1,1), A_k^{(0,0)})$, are given as well as a new proof of the Everitt-Marić result [11]. In [8], the authors obtain further new characterizations of the domain $\mathcal{D}(A_k^{(0,0)})$ of $A_k^{(0,0)}$, including a different proof of the one given in [3].

We remark that, even though the theory obtained in [16] guarantees the existence of a *continuum* of left-definite spaces $\{W_{r,k}^{(\alpha,\beta)}(-1,1)\}_{r>0}$ and left-definite operators $\{B_{r,k}^{(\alpha,\beta)}\}_{r>0}$ associated with the pair $(L^2_{\alpha,\beta}(-1,1), A_k^{(\alpha,\beta)})$, we can only effectively determine these spaces and operators in this general Jacobi case when r is a positive integer; see Remark 2.1 in Section 2. The key to obtaining these explicit characterizations of $\{W_{r,k}^{(\alpha,\beta)}(-1,1)\}_{r\in\mathbb{N}}$ and $\{B_{r,k}^{(\alpha,\beta)}\}_{r\in\mathbb{N}}$ is in obtaining the Lagrangian symmetrizable form of each integral power $\ell^r_{\alpha,\beta,k}[\cdot]$ of the Jacobi differential expression $\ell_{\alpha,\beta,k}[\cdot]$. In turn, the key to obtaining these integral powers is a remarkable, and yet somewhat mysterious, combinatorial identity involving a function that can be viewed as a sort of generating function for these integral powers of $\ell_{\alpha,\beta,k}[\cdot]$. In our discussion of the combinatorics of these integral powers of $\ell_{\alpha,\beta,k}[\cdot]$, we introduce a double sequence $\{P^{(\alpha,\beta)}S_n^{(j)}\}$ of real numbers that we call *Jacobi-Stirling numbers*; these numbers, as we will see, share similar properties with the classical Stirling numbers of the second kind $\{S_n^{(j)}\}$. Furthermore, these Jacobi-Stirling numbers generalize the Legendre-Stirling numbers, whose properties are developed in [10] and [13].

The contents of this paper are as follows. In Section 2, we state some of the main left-definite results developed in [16]. In Section 3, we review key properties of the Jacobi differential equation, the Jacobi polynomials, and the right-definite self-adjoint operator $A_k^{(\alpha,\beta)}$, generated by the second-order Jacobi expression (1.1), having the Jacobi polynomials as eigenfunctions; explicit in this review is a discussion of the Glazman-Krein-Naimark theory (see [2] and [19]). In Section 4, we determine the Lagrangian symmetrizable form of each integral composite power of the second-order Jacobi expression $\ell_{\alpha,\beta,k}[\cdot]$ (see Theorem 4.2) in terms of the Jacobi-Stirling numbers and we discuss a remarkable identity involving these numbers (see Theorem 4.1 and Remark 4.2). Lastly, in Section 5, we establish the left-definite theory associated with the pair $(L^2_{\alpha,\beta}(-1,1), A_k^{(\alpha,\beta)})$. Specifically, we determine explicitly

(a) the sequence $\{W_{n,k}^{(\alpha,\beta)}(-1,1)\}_{n=1}^{\infty}$ of left-definite spaces associated with $(L^2_{\alpha,\beta}(-1,1), A_k^{(\alpha,\beta)})$ and we show that the Jacobi polynomials $\{P_m^{(\alpha,\beta)}\}_{m=0}^{\infty}$ form a complete *orthogonal* set in each of these Hilbert-Sobolev spaces;
(b) the sequence of left-definite self-adjoint operators $\{B_{n,k}^{(\alpha,\beta)}\}_{n=1}^{\infty}$, as well as their explicit domains $\{\mathcal{D}(B_{n,k}^{(\alpha,\beta)})\}_{n=1}^{\infty}$, associated with $(L^2_{\alpha,\beta}(-1,1), A_k^{(\alpha,\beta)})$. Furthermore, we show that $\{P_m^{(\alpha,\beta)}\}_{m=0}^{\infty}$ is a complete set of eigenfunctions for each of these operators $B_{n,k}^{(\alpha,\beta)}$;
(c) the domains $\mathcal{D}((A_k^{(\alpha,\beta)})^n)$, for each $n \in \mathbb{N}$, of the composite power $(A_k^{(\alpha,\beta)})^n$ of $A_k^{(\alpha,\beta)}$.

These results culminate in Theorem 5.4.

Throughout this paper, $\mathbb{N}$ will denote the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, while $\mathbb{R}$ and $\mathbb{C}$ will denote, respectively, the real and complex number fields. The term AC will denote absolute continuity; for an open interval $I \subset \mathbb{R}$, the notation $AC_{\text{loc}}(I)$ will denote those functions $f : I \to \mathbb{C}$ that are absolutely continuous on all compact subintervals of an interval $I \subset \mathbb{R}$. For $n \in \mathbb{N}$, the space $AC_{\text{loc}}^{(n)}(I)$ denotes the set of all functions f satisfying $f^{(j)} \in AC_{\text{loc}}(I)$ for $j = 0, 1, \ldots, n$. The space of all polynomials $p : \mathbb{R} \to \mathbb{C}$ will be denoted by $\mathcal{P}$. If A is a linear operator, $\mathcal{D}(A)$ will denote its domain. Lastly, a word is in order regarding displayed, bracketed information. For example,

$$f(t) \text{ has property } P \quad (t \in I),$$

and

$$g_m \text{ has property } Q \quad (m \in \mathbb{N}_0)$$

mean, respectively, that f has property P *for all* $t \in I$ and g_m has property Q *for all* $m \in \mathbb{N}_0$. Further notations are introduced as needed throughout the paper.

2. Left-definite Hilbert spaces and left-definite operators

Let V denote a vector space (over the complex field $\mathbb{C}$) and suppose that $(\cdot,\cdot)$ is an inner product with norm $\|\cdot\|$ generated from $(\cdot,\cdot)$ such that $H = (V, (\cdot,\cdot))$ is a Hilbert space. Suppose V_r (the subscripts will be made clear shortly) is a linear manifold (vector subspace) of V and let $(\cdot,\cdot)_r$ and $\|\cdot\|_r$ denote an inner product and associated norm, respectively, over V_r (quite possibly different from $(\cdot,\cdot)$ and $\|\cdot\|$). We denote the resulting inner product space by $W_r = (V_r, (\cdot,\cdot)_r)$.

Throughout this section, we assume that $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by kI, for some $k > 0$; that is,

$$(Ax, x) \geq k(x, x) \quad (x \in \mathcal{D}(A)).$$

It follows that A^r, for each $r > 0$, is a self-adjoint operator that is bounded below in H by $k^r I$.

We now define an r^{th} left-definite space associated with (H, A).

Definition 2.1. *Let $r > 0$ and suppose V_r is a linear manifold of the Hilbert space $H = (H, (\cdot,\cdot))$ and $(\cdot,\cdot)_r$ is an inner product on V_r. Let $W_r = (V_r, (\cdot,\cdot)_r)$. We say that W_r is an* $\mathbf{r}^{th}$ ***left-definite space*** *associated with the pair (H, A) if each of the following conditions hold:*

(1) *W_r is a Hilbert space,*
(2) *$\mathcal{D}(A^r)$ is a linear manifold of V_r,*
(3) *$\mathcal{D}(A^r)$ is dense in W_r,*

(4) $(x,x)_r \geq k^r\,(x,x) \quad (x \in V_r)$, *and*
(5) $(x,y)_r = (A^r x, y) \quad (x \in \mathcal{D}(A^r),\ y \in V_r)$.

It is not clear, from the definition, if such a self-adjoint operator A generates a left-definite space for a given $r > 0$. However, in [16], the authors prove the following theorem; the Hilbert space spectral theorem (see [22, Chapter 13]) plays a prominent role in establishing this result.

Theorem 2.1. (*see* [16, Theorems 3.1 and 3.4]) *Suppose* $A : \mathcal{D}(A) \subset H \to H$ *is a self-adjoint operator that is bounded below by* kI, *for some* $k > 0$. *Let* $r > 0$. *Define* $W_r = (V_r, (\cdot,\cdot)_r)$ *by*

$$V_r = \mathcal{D}(A^{r/2}), \tag{2.1}$$

and

$$(x,y)_r = (A^{r/2}x, A^{r/2}y) \quad (x, y \in V_r).$$

Then W_r *is a left-definite space associated with the pair* (H, A). *Moreover, suppose* $W_r' := (V_r', (\cdot,\cdot)_r')$ *is another* r^{th} *left-definite space associated with the pair* (H, A). *Then* $V_r = V_r'$ *and* $(x,y)_r = (x,y)_r'$ *for all* $x, y \in V_r = V_r'$; *i.e.* $W_r = W_r'$. *That is to say,* $W_r = (V_r, (\cdot,\cdot)_r)$ *is the* <u>*unique*</u> *left-definite space associated with* (H, A). *Moreover,*

- (a) *if* A *is bounded, then, for each* $r > 0$,
  - (i) $V = V_r$;
  - (ii) *the inner products* $(\cdot,\cdot)$ and $(\cdot,\cdot)_r$ *are equivalent*;
- (b) *if* A *is unbounded, then*
  - (i) V_r *is a proper subspace of* V;
  - (ii) V_s *is a proper subspace of* V_r *whenever* $0 < r < s$;
  - (iii) *the inner products* $(\cdot,\cdot)$ *and* $(\cdot,\cdot)_s$ *are not equivalent for any* $s > 0$;
  - (iv) *the inner products* $(\cdot,\cdot)_r$ *and* $(\cdot,\cdot)_s$ *are not equivalent for any* $r, s > 0$, $r \neq s$.

Remark 2.1. Although all five conditions in Definition 2.1 are necessary in the proof of Theorem 2.1, the most important property, in a sense, is the one given in (5). Indeed, this property, which we call the **Dirichlet identity** for A^r, asserts that the r^{th} left-definite inner product is generated from the r^{th} power of A; see (3.15) in the next section. In particular, if A is generated from a Lagrangian symmetrizable differential expression $\ell[\cdot]$, we see that the form of the r^{th} power of A is then determined by the r^{th} power of $\ell[\cdot]$. Practically speaking, in this case, it is possible to obtain these powers only when r is a positive integer. However, we refer the reader to [16] where an example is discussed in which the entire continuum of left-definite spaces is explicitly obtained.

Definition 2.2. *For* $r > 0$, *let* $W_r = (V_r, (\cdot,\cdot)_r)$ *denote the* r^{th} *left-definite space associated with* (H, A). *If there exists a self-adjoint operator* $B_r : \mathcal{D}(B_r) \subset W_r \to W_r$ *that is a restriction of* A, *that is,*

$$B_r f = Af \quad (f \in \mathcal{D}(B_r) \subset \mathcal{D}(A)),$$

we call such an operator an $\mathbf{r^{th}}$ ***left-definite operator associated with*** (H, A).

Again, it is not immediately clear that such an operator B_r exists for a given $r > 0$; in fact, however, as the next theorem shows, B_r exists and is unique.

Theorem 2.2. (*see* [16, Theorems 3.2 and 3.4]) *Suppose* A *is a self-adjoint operator in a Hilbert space* H *that is bounded below by* kI, *for some* $k > 0$. *For any* $r > 0$, *let* $W_r = (V_r, (\cdot,\cdot)_r)$ *be the* r^{th} *left-definite space associated with* (H, A). *Then there exists a unique left-definite operator* B_r *in* W_r *associated with* (H, A); *in fact,*

$$\mathcal{D}(B_r) = V_{r+2} \subset \mathcal{D}(A).$$

Furthermore,

- (a) *if* A *is bounded, then, for each* $r > 0$, $A = B_r$.

(b) *if A is unbounded, then*
  (i) *$\mathcal{D}(B_r)$ is a proper subspace of $\mathcal{D}(A)$ for each $r > 0$;*
  (ii) *$\mathcal{D}(B_s)$ is a proper subspace of $\mathcal{D}(B_r)$ whenever $0 < r < s$.*

The last theorem that we state in this section shows that the point spectrum, continuous spectrum, and resolvent set of A and each of its associated left-definite operators B_r $(r > 0)$ are identical.

Theorem 2.3. (*see* [16, Theorem 3.6]) *For each $r > 0$, let B_r denote the r^{th} left-definite operator associated with the self-adjoint operator A that is bounded below by kI, where $k > 0$. Then*

(a) *the point spectra of A and B_r coincide; i.e. $\sigma_p(B_r) = \sigma_p(A)$;*
(b) *the continuous spectra of A and B_r coincide; i.e. $\sigma_c(B_r) = \sigma_c(A)$;*
(c) *the resolvent sets of A and B_r are equal; i.e. $\rho(B_r) = \rho(A)$.*

We refer the reader to [16] for other theorems, and examples, associated with the general left-definite theory of self-adjoint operators A that are bounded below.

3. Jacobi polynomials and a discussion of the right-definite analysis of the classical Jacobi differential expression

We remind the reader that $\alpha, \beta > -1$ and the parameter k in (1.1) is a fixed, non-negative constant (later, particularly in Section 5, k will be a fixed, positive constant - the specific use of this parameter k is to shift the spectrum $\sigma(A_k^{(\alpha,\beta)})$ of the self-adjoint operator $A_k^{(\alpha,\beta)}$ (see below for a discussion of this operator) to a subset of the *positive* real numbers). At this point, it is convenient to introduce the following definitions, generalizing those in (1.3) and (1.4). For $j \in \mathbb{N}_0$, let

$$L^2_{\alpha+j,\beta+j}(-1,1) := \{f : (-1,1) \to \mathbb{C} \mid f \text{ is Lebesgue measurable and } \int_{-1}^{1} |f(t)|^2 \, w_{\alpha+j,\beta+j}(t) < \infty\}, \tag{3.1}$$

where $w_{\alpha,\beta}(t)$ is defined in (1.2). Of course, each $L^2_{\alpha+j,\beta+j}(-1,1)$ is a Hilbert space with inner product

$$\begin{aligned}(f,g)_{\alpha+j,\beta+j} :&= \int_{-1}^{1} f(t)\overline{g}(t) w_{\alpha+j,\beta+j}(t)dt \\ &= \int_{-1}^{1} f(t)\overline{g}(t)(1-t)^{\alpha+j}(1+t)^{\beta+j}dt \quad (f,g \in L^2_{\alpha+j,\beta+j}(-1,1)),\end{aligned} \tag{3.2}$$

and associated norm

$$\|f\|_{\alpha+j,\beta+j} := (f,f)^{1/2}_{\alpha+j,\beta+j} \quad (f \in L^2_{\alpha+j,\beta+j}(-1,1)). \tag{3.3}$$

A simple, but useful (see Corollary 5.1) observation, is the following:

$$f \in L^2_{\alpha+j,\beta+j}(-1,1) \Leftrightarrow (1-t^2)^{j/2} f \in L^2_{\alpha,\beta}(-1,1). \tag{3.4}$$

It is well known that, with

$$\lambda_{r,k}^{(\alpha,\beta)} := r(r+\alpha+\beta+1) + k \quad (r \in \mathbb{N}_0), \tag{3.5}$$

the Jacobi equation

$$\ell_{\alpha,\beta,k}[y](t) = \lambda_{r,k}^{(\alpha,\beta)} y(t) \quad (t \in (-1,1)),$$

where $\ell_{\alpha,\beta,k}[\cdot]$ is defined in (1.1), or equivalently, the well-known classical form of the Jacobi equation

$$(1-t^2)y''(t) + (\beta - \alpha - (\alpha+\beta+2)t)y'(t) + r(r+\alpha+\beta+1)y(t) = 0,$$

has a polynomial solution $P_r^{(\alpha,\beta)}(t)$ of degree r, called the r^{th} *Jacobi polynomial*. In particular, with the r^{th} Jacobi polynomial $P_r^{(\alpha,\beta)}(t)$ defined by

$$\begin{aligned} P_r^{(\alpha,\beta)}(t) &= k_r^{(\alpha,\beta)} \sum_{j=0}^{r} \frac{(1+\alpha)_r(1+\alpha+\beta)_{r+j}}{j!(r-j)!(1+\alpha)_j(1+\alpha+\beta)_r} \left(\frac{1-x}{2}\right)^j \\ &= k_r^{(\alpha,\beta)} \ {}_2F_1(-r, 1+\alpha+\beta+r; 1+\alpha; \frac{1-x}{2}) \quad (r \in \mathbb{N}_0), \end{aligned} \tag{3.6}$$

where

$$k_r^{(\alpha,\beta)} := \frac{(r!)^{1/2}(1+\alpha+\beta+2r)^{1/2}(\Gamma(\alpha+\beta+r+1))^{1/2}}{2^{(\alpha+\beta+1)/2}(\Gamma(\alpha+r+1))^{1/2}(\Gamma(\beta+r+1))^{1/2}},$$

it is the case that $\{P_r^{(\alpha,\beta)}\}_{r=0}^{\infty}$ forms a complete *orthonormal* set in $L^2_{\alpha,\beta}(-1,1)$; that is to say,

$$(P_r^{(\alpha,\beta)}, P_n^{(\alpha,\beta)})_{\alpha,\beta} = \delta_{r,n} \quad (r, n \in \mathbb{N}_0), \tag{3.7}$$

where $\delta_{r,n}$ is the Kronecker delta symbol. In particular, for each $j \in \mathbb{N}_0$, the Jacobi polynomials $\{P_r^{(\alpha+j,\beta+j)}\}_{r=0}^{\infty}$ form a complete orthonormal set in the Hilbert space $L^2_{\alpha+j,\beta+j}(-1,1)$. When $\alpha = \beta = 0$, these polynomials are called *Legendre polynomials*; if $\alpha = \beta = -1/2$, they are called *Chebychev polynomials of the first kind* and when $\alpha = \beta = 1/2$, they are called *Chebychev polynomials of the second kind*. In general, if $\alpha = \beta$, the Jacobi polynomials are often called *Gegenbauer* or *ultraspherical polynomials*. For details and various properties of the Jacobi polynomials, we recommend the classic treatises of [20, Chapter 16] and [24, Chapter IV].

The derivatives of the Jacobi polynomials satisfy the identity

$$\frac{d^j P_r^{(\alpha,\beta)}(t)}{dt^j} = c^{(\alpha,\beta)}(r,j) P_{r-j}^{(\alpha+j,\beta+j)}(t) \quad (r, j \in \mathbb{N}_0), \tag{3.8}$$

where

$$c^{(\alpha,\beta)}(r,j) := \frac{(r!)^{1/2}(\Gamma(\alpha+\beta+r+1+j))^{1/2}}{((r-j)!)^{1/2}(\Gamma(\alpha+\beta+r+1))^{1/2}} \quad (j = 0, 1, \ldots, r);$$

¿From (3.2) and (3.8), we see that

$$\begin{aligned} &\int_{-1}^{1} \frac{d^j (P_r^{(\alpha,\beta)}(t))}{dt^j} \frac{d^j (P_n^{(\alpha,\beta)}(t))}{dt^j} w_{\alpha+j,\beta+j}(t) dt \\ =&\frac{r!\Gamma(\alpha+\beta+r+1+j)}{(r-j)!\Gamma(\alpha+\beta+r+1)} \delta_{r,n} \quad (r, n, j \in \mathbb{N}_0). \end{aligned} \tag{3.9}$$

We now turn our attention to discuss some operator-theoretic properties of the Jacobi differential expression $\ell_{\alpha,\beta,k}[\cdot]$; we recommend [18] for further details on this analysis as well as the classic texts [2] and [19] for a general discussion of self-adjoint operators generated from Lagrangian symmetric differential expressions.

The maximal domain $\Delta_k^{(\alpha,\beta)}$ of $\ell_{\alpha,\beta,k}[\cdot]$ in $L^2_{\alpha,\beta}(-1,1)$ is defined to be

$$\Delta_k^{(\alpha,\beta)} := \{f \in L^2_{\alpha,\beta}(-1,1) \mid f, f' \in AC_{loc}(-1,1);\ (1/w_{\alpha,\beta})\ell_{\alpha,\beta,k}[f] \in L^2_{\alpha,\beta}(-1,1)\}. \tag{3.10}$$

Since $\Delta_k^{(\alpha,\beta)}$ contains $\mathcal{P}$, the space of polynomials, we see that $\Delta_k^{(\alpha,\beta)}$ is a dense vector subspace of $L^2_{\alpha,\beta}(-1,1)$. The *maximal operator* $T_{\max,k}^{(\alpha,\beta)}$, generated by $\ell_{\alpha,\beta,k}[\cdot]$ in $L^2_{\alpha,\beta}(-1,1)$ is defined to be

$$\begin{aligned} \mathcal{D}(T_{\max,k}^{(\alpha,\beta)}) &:= \Delta_k^{(\alpha,\beta)} \\ T_{\max,k}^{(\alpha,\beta)}(f) &:= \ell_{\alpha,\beta,k}[f]. \end{aligned}$$

The *minimal operator* $T_{\min,k}^{(\alpha,\beta)}$ is then defined as $T_{\min,k}^{(\alpha,\beta)} = (T_{\max,k}^{(\alpha,\beta)})^*$, the Hilbert space adjoint of $T_{\max,k}^{(\alpha,\beta)}$. This operator $T_{\min,k}^{(\alpha,\beta)}$ is closed, symmetric, and satisfies $(T_{\min,k}^{(\alpha,\beta)})^* = T_{\max,k}^{(\alpha,\beta)}$. Furthermore, the deficiency index $d(T_{\min,k}^{(\alpha,\beta)})$ of $T_{\min,k}^{(\alpha,\beta)}$ is given by

$$d(T_{\min,k}^{(\alpha,\beta)}) = \begin{cases} (0,0) & \text{if } \alpha,\beta \geq 1 \\ (1,1) & \text{if } \alpha \geq 1 \text{ and } \beta \in (-1,1) \text{ \underline{or} } \beta \geq 1 \text{ and } \alpha \in (-1,1) \\ (2,2) & \text{if } \alpha,\beta \in (-1,1). \end{cases} \tag{3.11}$$

Consequently, by the well-known von-Neumann theory of self-adjoint extensions of symmetric operators [6, Chapter XII], $T_{\min,k}^{(\alpha,\beta)}$ has self-adjoint extensions in $L^2_{\alpha,\beta}(-1,1)$ for all $\alpha,\beta > -1$; in fact, when $\alpha,\beta \geq 1$, there is a unique self-adjoint extension in $L^2_{\alpha,\beta}(-1,1)$. The values of the deficiency index in (3.11) can be seen from the fact that the singular endpoints $t = \pm 1$ of $\ell_{\alpha,\beta,k}[\cdot]$ satisfy the following limit-point/limit-circle criteria:

(i) $t = 1$ is in the limit-point case in $L^2_{\alpha,\beta}(-1,1)$ if $\alpha \geq 1$; if $-1 < \alpha < 0$, $t = 1$ is in the regular case and if $0 \leq \alpha < 1$, $t = 1$ is in the limit-circle, non-oscillatory case in $L^2_{\alpha,\beta}(-1,1)$;
(ii) $t = -1$ is in the limit-point case in $L^2_{\alpha,\beta}(-1,1)$ if $\beta \geq 1$; if $-1 < \beta < 0$, $t = -1$ is in the regular case and if $0 \leq \beta < 1$, $t = -1$ is in the limit-circle, non-oscillatory case in $L^2_{\alpha,\beta}(-1,1)$;

for more information on the terminology for these singular point classifications, we refer the reader to the documentation and references in [4].

¿From the Glazman-Krein-Naimark theory (see [2] and [19]), the operator $A_k^{(\alpha,\beta)} : \mathcal{D}(A_k^{(\alpha,\beta)}) \subset L^2_{\alpha,\beta}(-1,1) \to L^2_{\alpha,\beta}(-1,1)$ defined by

$$A_k^{(\alpha,\beta)} f = \ell_{\alpha,\beta,k}[f] \tag{3.12}$$

for $f \in \mathcal{D}(A_k^{(\alpha,\beta)})$, where

$$\mathcal{D}(A_k^{(\alpha,\beta)}) := \begin{cases} \Delta_k^{(\alpha,\beta)} & \text{if } \alpha,\beta \geq 1 \\ \{f \in \Delta_k^{(\alpha,\beta)} \mid \lim\limits_{t\to 1}(1-t)^{\alpha+1} f'(t) = 0\} & \text{if } |\alpha| < 1 \text{ and } \beta \geq 1 \\ \{f \in \Delta_k^{(\alpha,\beta)} \mid \lim\limits_{t\to -1}(1+t)^{\beta+1} f'(t) = 0\} & \text{if } |\beta| < 1 \text{ and } \alpha \geq 1 \\ \{f \in \Delta_k^{(\alpha,\beta)} \mid \lim\limits_{t\to \pm 1}(1-t)^{\alpha+1}(1+t)^{\beta+1} f'(t) = 0\} & \text{if } -1 < \alpha,\beta < 1, \end{cases} \tag{3.13}$$

is self-adjoint in $L^2_{\alpha,\beta}(-1,1)$; see also the comprehensive thesis [18] of Onyango-Otieno for further information on this self-adjoint operator. We note that $A_k^{(\alpha,\beta)}$ is the so-called *Friedrich's extension* except when $-1 < \alpha < 0$ or $-1 < \beta < 0$. The Jacobi polynomials $\{P_r^{(\alpha,\beta)}\}_{r=0}^{\infty}$ are a (complete) set of eigenfunctions of $A_k^{(\alpha,\beta)}$ in $L^2_{\alpha,\beta}(-1,1)$ and the spectrum of $A_k^{(\alpha,\beta)}$ is given by

$$\sigma(A_k^{(\alpha,\beta)}) = \{\lambda_{r,k}^{(\alpha,\beta)} \mid r \in \mathbb{N}_0\},$$

where $\lambda_{r,k}^{(\alpha,\beta)}$ is defined in (3.5). In particular, we see that

$$\sigma(A_k^{(\alpha,\beta)}) \subset [k,\infty),$$

from which it follows (see [22, Chapter 13]) that $A_k^{(\alpha,\beta)}$ is bounded below by kI in $L^2_{\alpha,\beta}(-1,1)$; that is to say,

$$(A_k^{(\alpha,\beta)} f, f)_{\alpha,\beta} \geq k(f,f)_{\alpha,\beta} \quad (f \in \mathcal{D}(A_k^{(\alpha,\beta)})). \tag{3.14}$$

Consequently, the left-definite theory discussed in Section 2 can be applied to this self-adjoint operator.

For $f, g \in \mathcal{D}(A_k^{(\alpha,\beta)})$, we also have the well-known, and classical, *Dirichlet identity* for $A_k^{(\alpha,\beta)}$:

$$(3.15)\qquad \begin{aligned}(A_k^{(\alpha,\beta)} f, g)_{\alpha,\beta} &= \int_{-1}^{1} \ell_{\alpha,\beta,k}[f](t)\overline{g}(t)(1-t)^{\alpha}(1+t)^{\beta}dt \\ &= \int_{-1}^{1} \left\{(1-t)^{\alpha+1}(1+t)^{\beta+1}f'(t)\overline{g}'(t) + k(1-t)^{\alpha}(1+t)^{\beta}f(t)\overline{g}(t)\right\} dt;\end{aligned}$$

we give another proof of this identity in Section 5 (see Remark 5.1). Furthermore, when $k > 0$, notice that the right-hand side of (3.15) satisfies the conditions of an inner product. Consequently, we define the inner product $(\cdot,\cdot)_{1,k}^{(\alpha,\beta)}$ on $\mathcal{D}(A_k^{(\alpha,\beta)}) \times \mathcal{D}(A_k^{(\alpha,\beta)})$ by

$$(3.16)\qquad \begin{aligned}&(f, g)_{1,k}^{(\alpha,\beta)} \\ &:= \int_{-1}^{1} \left\{(1-t)^{\alpha+1}(1+t)^{\beta+1}f'(t)\overline{g}'(t) + k(1-t)^{\alpha}(1+t)^{\beta}f(t)\overline{g}(t)\right\} dt \quad (f, g \in \mathcal{D}(A_k^{(\alpha,\beta)}));\end{aligned}$$

later in this paper, we extend this inner product to the set $V_1^{(\alpha,\beta)} \times V_1^{(\alpha,\beta)}$, where $V_1^{(\alpha,\beta)}$ is a certain vector space of functions (specifically, the *first* left-definite space) properly containing $\mathcal{D}(A_k^{(\alpha,\beta)})$. In the literature, this inner product $(\cdot,\cdot)_{1,k}^{(\alpha,\beta)}$ is called the *first left-definite inner product* associated with $(L^2_{\alpha,\beta}(-1,1), A_k^{(\alpha,\beta)})$. Notice that the weights in this inner product are precisely the terms in the Lagrangian symmetrizable differential expression $\ell_{\alpha,\beta,k}[\cdot]$; see (1.1) and Remark 2.1.

4. Jacobi-Stirling numbers and powers of the Jacobi differential expression

We now turn our attention to the explicit construction of the sequence of left-definite inner products $(\cdot,\cdot)_{n,k}^{(\alpha,\beta)}$ $(n \in \mathbb{N})$ associated with the pair $(L^2_{\alpha,\beta}(-1,1), A_k^{(\alpha,\beta)})$, where $A_k^{(\alpha,\beta)}$ is the self-adjoint Jacobi differential operator defined in (3.12) and (3.13). As discussed in Remark 2.1, these inner products are generated from the integral composite powers $\ell^n_{\alpha,\beta,k}[\cdot]$ $(n \in \mathbb{N})$ of the Jacobi differential expression $\ell_{\alpha,\beta,k}[\cdot]$, given inductively by

$$\ell^1_{\alpha,\beta,k}[y] = \ell_{\alpha,\beta,k}[y],\ \ell^2_{\alpha,\beta,k}[y] = \ell_{\alpha,\beta,k}(\ell_{\alpha,\beta,k}[y]), \ldots, \ell^n_{\alpha,\beta,k}[y] = \ell_{\alpha,\beta,k}\left(\ell^{n-1}_{\alpha,\beta,k}[y]\right) \quad (n \in \mathbb{N}).$$

One of the keys to the explicit determination of these integral powers of $\ell_{\alpha,\beta,k}[\cdot]$ are two double sequences of non-negative numbers, $\{P^{(\alpha,\beta)}S_n^{(j)}\}$ and $\{c_j^{(\alpha,\beta)}(n,k)\}_{j=0}^n$, which are both defined in the following theorem; connections between these numbers and the powers of the Jacobi differential expression $\ell_{\alpha,\beta,k}[\cdot]$ will be made in Theorem 4.2 below.

Theorem 4.1. *Suppose $k \geq 0$ and $n \in \mathbb{N}$. For each $m \in \mathbb{N}_0$, the recurrence relations*

$$(4.1)\qquad (m(m+\alpha+\beta+1)+k)^n = \sum_{j=0}^{n} c_j^{(\alpha,\beta)}(n,k)\frac{m!\Gamma(\alpha+\beta+m+1+j)}{(m-j)!\Gamma(\alpha+\beta+m+1)}$$

have a unique solution $(c_0^{(\alpha,\beta)}(n,k), c_0^{(\alpha,\beta)}(n,k), \ldots, c_n^{(\alpha,\beta)}(n,k))$, where each $c_j^{(\alpha,\beta)}(n,k)$ is independent of m, given explicitly by

$$(4.2)\qquad c_0^{(\alpha,\beta)}(n,k) := \begin{cases} 0 & \text{if } k = 0 \\ k^n & \text{if } k > 0,\end{cases}$$

and, for $j \in \{1, 2, \ldots, n\}$,

$$(4.3)\qquad c_j^{(\alpha,\beta)}(n,k) := \begin{cases} P^{(\alpha,\beta)}S_n^{(j)} & \text{if } k = 0 \\ \sum_{s=0}^{n-j}\binom{n}{s}P^{(\alpha,\beta)}S_{n-s}^{(j)}k^s & \text{if } k > 0\end{cases} \qquad (j \in \{1, \ldots, n\}),$$

where each $P^{(\alpha,\beta)}S_n^{(j)}$ *is positive and given by*

$$P^{(\alpha,\beta)}S_n^{(j)} = \sum_{r=0}^{j}(-1)^{r+j}\frac{\Gamma(\alpha+\beta+r+1)\Gamma(\alpha+\beta+2r+2)\,[r(r+\alpha+\beta+1)]^n}{r!(j-r)!\Gamma(\alpha+\beta+2r+1)\Gamma(\alpha+\beta+j+r+2)} \tag{4.4}$$

for each $n \in \mathbb{N}$ *and* $j \in \{1, 2, \ldots, n\}$. *Moreover,* $P^{(\alpha,\beta)}S_n^{(j)}$ *is the coefficient of* t^{n-j} *in the Taylor series expansion of*

$$f_j^{(\alpha,\beta)}(t) := \prod_{r=1}^{j}\frac{1}{1-r(r+\alpha+\beta+1)t} \qquad \left(|t| < \frac{1}{j(j+\alpha+\beta+1)}\right). \tag{4.5}$$

Proof. Fix $k \geq 0$ and $n \in \mathbb{N}$; let $m \in \mathbb{N}_0$. Written out, the identity in (4.1) becomes

$$\begin{aligned}(m(m+\alpha+\beta+1)+k)^n = {} & c_0^{(\alpha,\beta)}(n,k) + c_1^{(\alpha,\beta)}(n,k)m(\alpha+\beta+m+1) \\ & + c_2^{(\alpha,\beta)}(n,k)m(m-1)(\alpha+\beta+m+2)(\alpha+\beta+m+1) + \cdots \\ & + c_n^{(\alpha,\beta)}(n,k)P(m,n)P(\alpha+\beta+m+n,n),\end{aligned} \tag{4.6}$$

where $P(r,q) = r(r-1)\ldots(r-q+1)$ for $r \in \mathbb{R}$ and $q \in \mathbb{N}$. If $m = 0$, (4.6) immediately yields $c_0^{(\alpha,\beta)}(n,k) = k^n$, which establishes (4.2). Similarly, when $m = 1$ and $m = 2$, we readily obtain from (4.6) the values of $c_1^{(\alpha,\beta)}(n,k)$ and $c_2^{(\alpha,\beta)}(n,k)$:

$$c_1^{(\alpha,\beta)}(n,k) = \frac{(\alpha+\beta+2+k)^n - k^n}{\alpha+\beta+2},$$

and

$$c_2^{(\alpha,\beta)}(n,k) = \frac{(\alpha+\beta+2)(2\alpha+2\beta+6+k)^n - 2(\alpha+\beta+3)(\alpha+\beta+2+k)^n + (\alpha+\beta+4)k^n}{2(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)}.$$

In general, it is not difficult to see that each $c_j^{(\alpha,\beta)}(n,k)$ is unique, independent of m, and given by

$$\begin{aligned}c_j^{(\alpha,\beta)}(n,k) &= \sum_{r=0}^{j}(-1)^{r+j}\frac{\Gamma(\alpha+\beta+r+1)\Gamma(\alpha+\beta+2r+2)}{r!(j-r)!\Gamma(\alpha+\beta+2r+1)\Gamma(\alpha+\beta+r+j+2)}\,(r(r+\alpha+\beta+1)+k)^n \\ &= \sum_{r=0}^{j}\sum_{s=0}^{n}\binom{n}{s}(-1)^{r+j}\frac{\Gamma(\alpha+\beta+r+1)\Gamma(\alpha+\beta+2r+2)\,(r(r+\alpha+\beta+1)^{n-s}}{r!(j-r)!\Gamma(\alpha+\beta+2r+1)\Gamma(\alpha+\beta+r+j+2)}k^s \\ &= \sum_{s=0}^{n}\left(\sum_{r=0}^{j}\frac{(-1)^{r+j}\Gamma(\alpha+\beta+r+1)\Gamma(\alpha+\beta+2r+2)\,(r(r+\alpha+\beta+1)^{n-s}}{r!(j-r)!\Gamma(\alpha+\beta+2r+1)\Gamma(\alpha+\beta+r+j+2)}\right)\binom{n}{s}k^s \\ &= \sum_{s=0}^{n}\binom{n}{s}P^{(\alpha,\beta)}S_{n-s}^{(j)}k^s. \qquad (4.7)\end{aligned}$$

This establishes (4.4) but not the identity in (4.3) (the sum in (4.3) has upper limit $n-j$, not n as in (4.7)); we return to prove (4.3) after we prove the identity in (4.5). To prove the positivity of each $P^{(\alpha,\beta)}S_n^{(j)}$ and that $f_j^{(\alpha,\beta)}(t)$, defined in (4.5), generates the numbers $\{P^{(\alpha,\beta)}S_n^{(j)}\}$, let $j \in \mathbb{N}$ and decompose

$$\prod_{m=1}^{j}\frac{t}{1-m(m+\alpha+\beta+1)t}$$

into partial fractions; that is, write

$$\begin{aligned}\frac{t^j}{(1-(\alpha+\beta+2)t)(1-(2\alpha+2\beta+6)t)\cdots(1-j(j+\alpha+\beta+1)t)} \\ = \sum_{m=1}^{j}\frac{A_m}{1-m(m+\alpha+\beta+1)t} \quad (|t| < \frac{1}{1-j(j+\alpha+\beta+1)}).\end{aligned} \tag{4.8}$$

Consequently,

$$\begin{aligned} t^j &= A_1(1-(2\alpha+2\beta+6)t)(1-(3\alpha+3\beta+12)t)\cdots(1-j(j+\alpha+\beta+1)t) \\ &+ A_2(1-(\alpha+\beta+2)t)(1-(3\alpha+3\beta+12)t)\cdots(1-j(j+\alpha+\beta+1)t) \\ &+\dots \\ &+ A_j(1-(\alpha+\beta+2)t)(1-(2\alpha+2\beta+6)t)\cdots(1-(j-1)(j+\alpha+\beta)t).\end{aligned} \tag{4.9}$$

If we let $t = \dfrac{1}{\alpha+\beta+2}$ in (4.9), we find that

$$A_1 = \frac{(-1)^{j+1}\Gamma(\alpha+\beta+2)\Gamma(\alpha+\beta+4)}{1!(j-1)!\Gamma(\alpha+\beta+3)\Gamma(\alpha+\beta+j+3)}.$$

Similarly, letting $t = \dfrac{1}{2\alpha+2\beta+6}$ in (4.9), we obtain

$$A_2 = \frac{(-1)^{j}\Gamma(\alpha+\beta+3)\Gamma(\alpha+\beta+6)}{2!(j-2)!\Gamma(\alpha+\beta+5)\Gamma(\alpha+\beta+j+4)}.$$

In general, setting $t = \dfrac{1}{m(m+\alpha+\beta+1)}$ yields

$$A_m = \frac{(-1)^{m+j}\Gamma(\alpha+\beta+m+1)\Gamma(\alpha+\beta+2m+2)}{m!(j-m)!\Gamma(\alpha+\beta+2m+1)\Gamma(\alpha+\beta+j+m+2)} \quad (1 \le m \le j).$$

Returning to (4.8), we find that

$$\begin{aligned} & t^j \prod_{m=1}^{j}\frac{1}{1-m(m+\alpha+\beta+1)t} \\ &= \prod_{m=1}^{j}\frac{t}{1-m(m+\alpha+\beta+1)t} \\ &= \sum_{m=1}^{j}\frac{A_m}{1-m(m+\alpha+\beta+1)t} \\ &= \sum_{m=1}^{j}\sum_{n=0}^{\infty} A_m(m(m+\alpha+\beta+1))^n t^n \quad \left(|t| < \frac{1}{j(j+\alpha+\beta+1)}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=1}^{j} A_m(m(m+\alpha+\beta+1))^n\right) t^n \quad \left(|t| < \frac{1}{j(j+\alpha+\beta+1)}\right).\end{aligned}$$

Hence

$$
\begin{aligned}
&\prod_{m=1}^{j}\frac{1}{1-m(m+\alpha+\beta+1)t}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=1}^{j}A_m(m(m+\alpha+\beta+1))^n\right)t^{n-j}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=1}^{j}\frac{(-1)^{m+j}\Gamma(\alpha+\beta+m+1)\Gamma(\alpha+\beta+2m+2)(m(m+\alpha+\beta+1))^n}{m!(j-m)!\Gamma(\alpha+\beta+2m+1)\Gamma(\alpha+\beta+j+m+2)}\right)t^{n-j} \qquad (4.10)\\
&=\sum_{n=0}^{\infty}P^{(\alpha,\beta)}S_n^{(j)}t^{n-j}.
\end{aligned}
$$

¿From this identity, it follows that

$$P^{(\alpha,\beta)}S_n^{(j)}=0 \quad (j\in\mathbb{N};\; n=0,1,\ldots,j-1); \tag{4.11}$$

consequently, from (4.3), we now see that

$$c_j^{(\alpha,\beta)}(n,k)=\sum_{s=0}^{n-j}\binom{n}{s}P^{(\alpha,\beta)}S_{n-s}^{(j)}k^s \quad (j=1,2,\ldots,n),$$

as claimed in (4.3). Lastly, since the coefficient of each term of the Taylor (geometric) series of

$$\frac{1}{1-m(m+\alpha+\beta+1)t} \quad (|t|<\frac{1}{m(m+\alpha+\beta+1)};1\le m\le j)$$

is positive and each $P^{(\alpha,\beta)}S_n^{(j)}$ $(j=1,2,\ldots,n)$ is a certain Cauchy product of these positive coefficients, it is clear that each $P^{(\alpha,\beta)}S_n^{(j)}$ is positive when $j,n\in\mathbb{N}$ with $n\ge j$. In turn, we see that $c_j^{(\alpha,\beta)}(n,k)>0$ for $j\in\{1,2,\ldots,n\}$ and $c_0^{(\alpha,\beta)}(n,k)=k^n\ge 0$. This completes the proof of the theorem. □

Definition 4.1. *Let $n,j\in\mathbb{N}_0$. If $n\in\mathbb{N}$ and $1\le j\le n$, we call the number $P^{(\alpha,\beta)}S_n^{(j)}$, defined in* (4.4)*, the **Jacobi-Stirling number of order** $(\mathbf{n},\mathbf{j})$ **associated with** (α,β). We extend this definition by defining these numbers to be $P^{(\alpha,\beta)}S_0^{(0)}=1, P^{(\alpha,\beta)}S_n^{(j)}=0$ if $j\in\mathbb{N}$ and $0\le n\le j-1$ (see* (4.11)*) , and $P^{(\alpha,\beta)}S_n^{(0)}=0$ for $n\in\mathbb{N}$. In short, we refer to $P^{(\alpha,\beta)}S_n^{(j)}$ as the **Jacobi-Stirling number of order** $(\mathbf{n},\mathbf{j})$, or simply as a **Jacobi-Stirling number**.*

Observe, from (4.4), that the Jacobi-Stirling numbers $\{P^{(\alpha,\beta)}S_n^{(j)}\}$ are symmetric in α and β; that is,

$$P^{(\alpha,\beta)}S_n^{(j)}=P^{(\beta,\alpha)}S_n^{(j)} \quad (n,j\in\mathbb{N}_0).$$

Remark 4.1. Why associate the name "Stirling" to this double sequence of real numbers? In [16], Littlejohn and Wellman show that the classical Stirling numbers of the second kind $S_n^{(j)}$ (see, for example, [1, pp. 824-825] and [5, Chapter V]) appear as the coefficients of the terms in the Lagrangian symmetrizable form of the n^{th} power of the classical second-order Laguerre differential expression

$$\ell_{Lag}[y](t)=t^{-\alpha}\exp(t)(-t^{\alpha+1}\exp(-t)y'(t))' \quad (t\in(0,\infty)); \tag{4.12}$$

indeed, they prove that

$$\ell_{Lag}^n[y](t)=t^{-\alpha}\exp(t)\sum_{j=1}^{n}(-1)^jS_n^{(j)}\left(t^{\alpha+j}\exp(-t)y^{(j)}(t)\right)^{(j)} \quad (n\in\mathbb{N}). \tag{4.13}$$

This seems to be a new application of these important combinatorial numbers (a similar, previously known, result states that the Stirling numbers of the second kind appear as the coefficients of the operator identity

$$((t-a)\frac{d}{dt})^n = \sum_{j=1}^{n} S_n^{(j)}(t-a)^j \frac{d^j}{dt^j} \quad (a \in \mathbb{C};\ n \in \mathbb{N});$$

see [21]). Similarly, the authors in [9] show that the Stirling numbers of the second kind also appear in the powers of the Lagrangian symmetrizable form of the classical Hermite differential expression

$$\ell_H[y](t) = \exp(t^2)\left(\left(-(\exp(-t^2)y'(t)\right)' + k\exp(-t^2)y(t)\right) \quad (t \in (-\infty,\infty)).$$

Moreover, in a recent paper [10], the authors develop the left-definite theory for the classical Legendre differential expression defined in (1.5). In doing so, they discovered a new sequence of numbers $\{PS_n^{(j)}\}$ that they call **Legendre-Stirling numbers**. In fact, a key result in their left-definite analysis is the explicit determination of the n^{th} composite power of the Legendre differential expression, namely

$$\ell_{0,0,0}^n[y](t) = \sum_{j=0}^{n}(-1)^j PS_n^{(j)}\left((1-t^2)^j y^{(j)}(t)\right)^{(j)} \quad (n \in \mathbb{N}).$$

Moreover, in regards to the notation of this paper, these Legendre-Stirling numbers are explicitly given by

$$PS_n^{(j)} = P^{(0,0)}S_n^{(j)}.$$

In a subsequent paper [13], Gawronski and Littlejohn show that the Legendre-Stirling numbers have many properties in common with the classical Stirling numbers of the second kind. Indeed, the Legendre-Stirling numbers have a vertical, horizontal, and rational generating function, and they satisfy vertical and triangular recurrence relations similar to the Stirling numbers of the second kind; see [5, Chapter V] for notation and a compendium of results for the classical Stirling numbers of the second kind. We discuss some of these properties below.

We list a few Jacobi-Stirling numbers $P^{(\alpha,\beta)}S_n^{(j)}$ in the following table.

j/n	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$j=0$	1	0	0	0	0	0
$j=1$	0	1	$\alpha+\beta+2$	$(\alpha+\beta+2)^2$	$(\alpha+\beta+2)^3$	$(\alpha+\beta+2)^4$
$j=2$	0	0	1	$3\alpha+3\beta+8$	$7\alpha^2+38\alpha+7\beta^2+$ $38\beta+14\alpha\beta+52$	$320+344\alpha+124\alpha^2+15\alpha^3+$ $344\beta+248\alpha\beta+45\alpha^2\beta+$ $124\beta^2+45\alpha\beta^2+15\beta^3$
$j=3$	0	0	0	1	$6\alpha+6\beta+20$	$25\alpha^2+25\beta^2+170\alpha+$ $170\beta+50\alpha\beta+292$
$j=4$	0	0	0	0	1	$10\alpha+10\beta+40$
$j=5$	0	0	0	0	0	1

Table 1: A List of Jacobi-Stirling Numbers (for example, $P^{(\alpha,\beta)}S_5^{(3)} = 25\alpha^2+25\beta^2+170\alpha+170\beta+50\alpha\beta+292$)

The numbers in the j^{th} row demonstrate the identity in (4.5). For example, reading along the row beginning with $j = 2$, we see that

$$\prod_{r=1}^{2} \frac{1}{1 - r(r + \alpha + \beta + 1)t}$$
$$= 1 + (3\alpha + 3\beta + 8)t + (7\alpha^2 + 38\alpha + 7\beta^2 + 38\alpha + 14\alpha\beta + 52)t^2$$
$$+ (320 + 344\alpha + 124\alpha^2 + 15\alpha^3 + 344\beta + 248\alpha\beta + 45\alpha^2\beta + 124\beta^2 + 45\alpha\beta^2 + 15\beta^3)t^3$$
$$+ \ldots .$$

However, the main interest in these numbers from the point of view of this manuscript is seen in the *columns*; indeed, the numbers in the n^{th} column of Table 1 are precisely the coefficients of the n^{th} power of the Jacobi differential expression $\ell_{\alpha,\beta,0}[\cdot]$; see Theorem 4.2, Corollary 4.1, and the examples considered below.

¿From a purely combinatorial point of view, we note that there are several interesting properties of these Jacobi-Stirling numbers; we mention some of these properties now but defer proofs and an extensive study of these numbers to a future paper. For example, we note that these numbers satisfy the following *triangular recurrence relation*:

$$P^{(\alpha,\beta)}S_n^{(j)} = P^{(\alpha,\beta)}S_{n-1}^{(j-1)} + j(j + \alpha + \beta + 1)P^{(\alpha,\beta)}S_{n-1}^{(j)} \quad (n, j \in \mathbb{N})$$
$$P^{(\alpha,\beta)}S_n^{(0)} = P^{(\alpha,\beta)}S_0^{(j)} = 0 \quad (n, j \in \mathbb{N})$$
$$P^{(\alpha,\beta)}S_0^{(0)} = 1;$$

see [5, Chapter V] where the reader will find that the Stirling numbers of the second kind satisfy a similar-looking recurrence relation. The Jacobi-Stirling numbers also satisfy the following identity

$$x^n = \sum_{j=0}^{n} P^{(\alpha,\beta)}S_n^{(j)} \langle x \rangle_j^{(\alpha,\beta)} \quad (n \in \mathbb{N}_0), \tag{4.14}$$

where $\langle x \rangle_j^{(\alpha,\beta)}$ is a *generalized falling factorial* defined, for $x \in \mathbb{C}$, by

$$\langle x \rangle_j^{(\alpha,\beta)} := \begin{cases} 1 & \text{if } j = 0 \\ \prod_{r=0}^{j-1}(x - r(r + \alpha + \beta + 1)) & \text{if } j \in \mathbb{N}. \end{cases} \tag{4.15}$$

Notice the remarkable similarity of (4.14) with the well-known identity for the Stirling numbers of the second kind:

$$x^n = \sum_{j=0}^{n} S_n^{(j)}(x)_j \quad (n \in \mathbb{N}_0), \tag{4.16}$$

where $(x)_j$ is the falling factorial (see [21]) defined, for any $x \in \mathbb{C}$, by

$$(x)_j := \begin{cases} 1 & \text{if } j = 0 \\ \prod_{r=0}^{j-1}(x - r) & \text{if } j \in \mathbb{N}. \end{cases}$$

In several texts, the identity in (4.16) is used to *define* the Stirling numbers of the second kind $\{S_n^{(j)}\}$. By 'inverting' (4.16), we obtain the Stirling numbers of the first kind $\{s_n^{(j)}\}$ as the solutions of the equation

$$(x)_n = \sum_{j=0}^{n} s_n^{(j)} x^j \quad (n \in \mathbb{N}_0).$$

Consequently, using this strategy along with (4.14), we obtain the analogue of the Stirling numbers of the first kind $\{P^{(\alpha,\beta)}s_n^{(j)}\}$ for the Jacobi-Stirling numbers. as the solutions of the equation

$$\langle x\rangle_n^{(\alpha,\beta)} = \sum_{j=0}^{n} P^{(\alpha,\beta)}s_n^{(j)}x^j.$$

Consequently, it may be more natural to call the numbers $\{P^{(\alpha,\beta)}S_n^{(j)}\}$ the "Jacobi-Stirling numbers of the second kind". Indeed, we see that there is an associated sequence of numbers $\{P^{(\alpha,\beta)}s_n^{(j)}\}$, the analog of the Stirling numbers of the first kind, which we could call the "Jacobi-Stirling numbers of the first kind". However, we resist this temptation on the grounds of notational inconvenience. Indeed, as mentioned in Section 3, the Jacobi polynomials $\{P_n^{(-1/2,-1/2)}\}$ are often called the "Chebychev polynomials of the first kind"; consequently, it might be natural to call the numbers $\{P^{(-1/2,-1/2)}S_n^{(j)}\}$ the "Chebychev-Stirling numbers of the first kind of the second kind" and their combinatorial counterparts $\{P^{(-1/2,-1/2)}s_n^{(j)}\}$ the "Chebychev-Stirling numbers of the first kind of the first kind". Notation, sometimes, can be problem! We instead refer to the numbers $\{P^{(\alpha,\beta)}s_n^{(j)}\}$ as the "**associated Jacobi-Stirling numbers**". From the definition of these associated Jacobi-Stirling numbers, we immediately obtain the following bi-orthogonality relationships between $\{P^{(\alpha,\beta)}S_n^{(j)}\}$ and $\{P^{(\alpha,\beta)}s_n^{(j)}\}$:

$$\sum_{j=0}^{\max\{n,m\}+1} P^{(\alpha,\beta)}s_n^{(j)} \cdot P^{(\alpha,\beta)}S_j^{(m)} = \delta_{n,m} \quad (n,m \in \mathbb{N}_0),$$

$$\sum_{j=0}^{\max\{n,m\}+1} P^{(\alpha,\beta)}S_n^{(j)} \cdot P^{(\alpha,\beta)}s_j^{(m)} = \delta_{n,m} \quad (n,m \in \mathbb{N}_0).$$

The following table lists a few of these associated Jacobi-Stirling numbers $\{P^{(\alpha,\beta)}s_n^{(j)}\}$:

j/n	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$j=0$	1	0	0	0	0	0
$j=1$	0	1	$-2-\alpha-\beta$	$12+2\alpha^2+4\alpha\beta+10\alpha+2\beta^2+10\beta$	$-144-6\alpha^3-18\alpha^2\beta-54\alpha^2-18\alpha\beta^2-108\alpha\beta-156\alpha-6\beta^3-54\beta^2-156\beta$	$2880+3696\alpha+3696\beta+3408\alpha\beta+1008\alpha^2\beta+1008\alpha\beta^2+1704\alpha^2+1704\beta^2+336\alpha^3+336\beta^3+24\alpha^4+24\beta^4+96\alpha^3\beta+144\alpha^2\beta^2+96\alpha\beta^3$
$j=2$	0	0	1	$-8-3\alpha-3\beta$	$108+11\alpha^2+32\alpha+11\beta^2+22\alpha\beta+108\beta$	$-2304-1988\alpha-554\alpha^2-50\alpha^3-1988\beta-1108\alpha\beta-150\alpha^2\beta-554\beta^2-150\alpha\beta^2-50\beta^3$
$j=3$	0	0	0	1	$-20-6\alpha-6\beta$	$508+35\alpha^2+35\beta^2+270\alpha+270\beta+70\alpha\beta$
$j=4$	0	0	0	0	1	$-40-10\alpha-10\beta$
$j=5$	0	0	0	0	0	1

Table 2: A List of Associated Jacobi-Stirling Numbers (for example, $P^{(\alpha,\beta)}s_4^{(2)} = 108 + 11\alpha^2 + 32\alpha + 11\beta^2 + 22\alpha\beta + 108\beta$)

We now prove the following key result which will allow us, in the next section, to obtain the left-definite spaces associated with the pair $(L^2_{\alpha,\beta}(-1,1), A_k^{(\alpha,\beta)})$. Recall the definition of $\mathcal{P}$ (see the various notation at the end of Section 1), as the space of all polynomials $p : \mathbb{R} \to \mathbb{C}$.

Theorem 4.2. *Let $k \geq 0$. For each $n \in \mathbb{N}$, the n^{th} composite power of the classical Jacobi differential expression $\ell_{\alpha,\beta,k}[\cdot]$, defined in* (1.1), *is Lagrangian symmetrizable, with symmetry factor*

$w_{\alpha,\beta}(t) = (1-t)^{\alpha}(1+t)^{\beta}$, *and is given explicitly by*

$$w_{\alpha,\beta}(t)\ell^{n}_{\alpha,\beta,k}[y](t) = \sum_{j=0}^{n}(-1)^{j}\left(c_{j}^{(\alpha,\beta)}(n,k)(1-t)^{\alpha+j}(1+t)^{\beta+j}y^{(j)}(t)\right)^{(j)}, \tag{4.17}$$

where $c_{j}^{(\alpha,\beta)}(n,k)$ *is defined in* (4.2) *and* (4.3). *Moreover, for* $p,q \in \mathcal{P}$, *the following identity is valid:*

$$\begin{aligned}(\ell^{n}_{\alpha,\beta,k}[p],q)_{\alpha,\beta} &= \int_{-1}^{1}\ell^{n}_{\alpha,\beta,k}[p](t)\overline{q}(t)w_{\alpha,\beta}(t)dt \\ &= \sum_{j=0}^{n}c_{j}^{(\alpha,\beta)}(n,k)\int_{-1}^{1}p^{(j)}(t)\overline{q}^{(j)}(t)(1-t)^{\alpha+j}(1+t)^{\beta+j}dt.\end{aligned} \tag{4.18}$$

Proof. We first establish the identity in (4.18). Since the Jacobi polynomials $\{P_{m}^{(\alpha,\beta)}\}_{m=0}^{\infty}$ form a basis for $\mathcal{P}$, it suffices to show (4.18) is valid for $p = P_{m}^{(\alpha,\beta)}$ and $q = P_{r}^{(\alpha,\beta)}$, for arbitrary $m,r \in \mathbb{N}_0$. From the identity (which follows immediately by induction)

$$\ell^{n}_{\alpha,\beta,k}[P_{m}^{(\alpha,\beta)}](t) = (m(m+\alpha+\beta+1)+k)^{n}P_{m}^{(\alpha,\beta)}(t) \quad (m \in \mathbb{N}_0), \tag{4.19}$$

it follows from (3.7), with this particular choice of p and q, that the left-hand side of (4.18) reduces to

$$(m(m+\alpha+\beta+1)+k)^{n}\delta_{m,r}. \tag{4.20}$$

On the other hand, from (3.9), we see that the right-hand side of (4.18) yields

$$\begin{aligned}&\sum_{j=0}^{n}c_{j}^{(\alpha,\beta)}(n,k)\int_{-1}^{1}\frac{d^{j}(P_{m}^{(\alpha,\beta)}(t))}{dt^{j}}\frac{d^{j}(P_{r}^{(\alpha,\beta)}(t))}{dt^{j}}(1-t)^{\alpha+j}(1+t)^{\beta+j}dt \\ &= \sum_{j=0}^{n}c_{j}^{(\alpha,\beta)}(n,k)\frac{m!\Gamma(\alpha+\beta+m+1+j)}{(m-j)!\Gamma(\alpha+\beta+m+1)}\delta_{m,r}.\end{aligned} \tag{4.21}$$

Comparing (4.20) with (4.21), we see from (4.1) and Theorem 4.1 that the identity in (4.18) is valid.

To prove (4.17), first define the differential expression

$$w_{\alpha,\beta}(t)m_{J}^{(\alpha,\beta)}[y](t) := \sum_{j=0}^{n}(-1)^{j}\left(c_{j}^{(\alpha,\beta)}(n,k)(1-t)^{\alpha+j}(1+t)^{\beta+j}y^{(j)}(t)\right)^{(j)} \quad (-1<t<1). \tag{4.22}$$

For $p,q \in \mathcal{P}$, integration by parts yields

$$\begin{aligned}&\int_{-1}^{1}m_{J}^{(\alpha,\beta)}[p](t)\overline{q}(t)w_{\alpha,\beta}(t)dt \\ &= \left[\sum_{j=1}^{n}(-1)^{j}c_{j}^{(\alpha,\beta)}(n,k)\sum_{r=1}^{j}(-1)^{r+1}\left(p^{(j)}(t)(1-t)^{\alpha+j}(1+t)^{\beta+j}\right)^{(j-r)}\overline{q}^{(r-1)}(t)\right]_{-1}^{+1} \\ &+\sum_{j=0}^{n}c_{j}^{(\alpha,\beta)}(n,k)\int_{-1}^{1}p^{(j)}(t)\overline{q}^{(j)}(t)(1-t)^{\alpha+j}(1+t)^{\beta+j}dt.\end{aligned}$$

Now, for any $p \in \mathcal{P}$ and integer r with $1 \leq r \leq j$, $\left(p^{(j)}(t)(1-t)^{\alpha+j}(1+t)^{\beta+j}\right)^{(j-r)} = p_{j,r}(t)(1-t^2)$ for some $p_{j,r} \in \mathcal{P}$; in particular,

$$\lim_{t\to\pm1}\left(p^{(j)}(t)(1-t^2)^{j}\right)^{(j-r)}\overline{q}^{(r-1)}(t) = 0 \quad (p,q \in \mathcal{P};\ r,j \in \mathbb{N},\ r \leq j).$$

Consequently, we see that

$$\begin{aligned}(4.23)\qquad &\int_{-1}^{1} m_J^{(\alpha,\beta)}[p](t)\overline{q}(t)w_{\alpha,\beta}(t)dt \\ &= \sum_{j=0}^{n} c_j^{(\alpha,\beta)}(n,k)\int_{-1}^{1} p^{(j)}(t)\overline{q}^{(j)}(t)(1-t)^{\alpha+j}(1+t)^{\beta+j}dt \quad (p,q\in\mathcal{P}).\end{aligned}$$

Hence, from (4.18) and (4.23), we see that for all polynomials p and q, we have

$$(\ell_{\alpha,\beta,k}^{n}[p] - m_J^{(\alpha,\beta)}[p], q)_{\alpha,\beta} = 0.$$

¿From the density of $\mathcal{P}$ in $L^2_{\alpha,\beta}(-1,1)$, it follows that

$$(4.24)\qquad \ell_{\alpha,\beta,k}^{n}[p](t) = m_J^{(\alpha,\beta)}[p](t) \quad (t\in(-1,1);\ p\in\mathcal{P}).$$

This latter identity implies that the expression $\ell_{\alpha,\beta,k}^{n}[\cdot]$ has the form given in (4.17). □

For general results on symmetry factors as well as necessary and sufficient conditions on the Lagrangian symmetrizability of ordinary differential expressions with smooth coefficients, we refer the reader to [14] and [15]. We also refer to [12] and [29] where more general results are obtained for composite powers of ordinary quasi-differential expressions defined via Shin-Zettl matrices.

The following corollary lists some additional, and important, properties of the Jacobi differential expressions $\ell_{\alpha,\beta,k}[\cdot]$ and $\ell_{\alpha,\beta,0}[\cdot]$, as well as new orthogonality properties of the classical Jacobi polynomials $\{P_m^{(\alpha,\beta)}\}_{m=0}^{\infty}$; as the reader will see, these properties are important in our left-definite analysis of $\ell_{\alpha,\beta,k}[\cdot]$ that we develop in the next section.

Corollary 4.1. *Let $n\in\mathbb{N}$. Then*

(a) *the n^{th} (composite) power of the classical Jacobi differential expression*

$$\begin{aligned}\ell_{\alpha,\beta,0}[y](t) &:= -(1-t^2)y''(t) + (\alpha-\beta+(\alpha+\beta+2)t)y'(t)\\ &= \frac{1}{w_{\alpha,\beta}(t)}\left(-(1-t)^{\alpha+1}(1+t)^{\beta+1}y'(t)\right)'\end{aligned}$$

is Lagrangian symmetrizable with symmetry factor $w_{\alpha,\beta}(t)=(1-t)^{\alpha}(1+t)^{\beta}$ and has the Lagrangian symmetrizable form

$$\ell_{\alpha,\beta,0}^{n}[y](t) := \frac{1}{w_{\alpha,\beta}(t)}\sum_{j=1}^{n}(-1)^j\left(P^{(\alpha,\beta)}S_n^{(j)}(1-t)^{\alpha+j}(1+t)^{\beta+j}y^{(j)}(t)\right)^{(j)},$$

where $P^{(\alpha,\beta)}S_n^{(j)}$ is defined in (4.4)*;*

(b) *the bilinear form $(\cdot,\cdot)_{n,k}^{(\alpha,\beta)}$, defined on $\mathcal{P}\times\mathcal{P}$ by*

$$(4.25)\qquad (p,q)_{n,k}^{(\alpha,\beta)} := \sum_{j=0}^{n} c_j^{(\alpha,\beta)}(n,k)\int_{-1}^{1} p^{(j)}(t)\overline{q}^{(j)}(t)(1-t)^{\alpha+j}(1+t)^{\beta+j}dt \quad (p,q\in\mathcal{P}),$$

is an inner product when $k>0$ (a pseudo inner product when $k=0$) and, for each $k\geq 0$,

$$(4.26)\qquad (\ell_{\alpha,\beta,k}^{n}[p], q)_{\alpha,\beta} = (p,q)_{n,k}^{(\alpha,\beta)} \quad (p,q\in\mathcal{P});$$

(c) *for each* $k \geq 0$, *the Jacobi polynomials* $\{P_m^{(\alpha,\beta)}\}_{m=0}^{\infty}$ *are orthogonal with respect to* $(\cdot,\cdot)_{n,k}^{(\alpha,\beta)}$; *in fact,*

$$(P_m^{(\alpha,\beta)}, P_r^{(\alpha,\beta)})_{n,k}^{(\alpha,\beta)} = \sum_{j=0}^{n} c_j^{(\alpha,\beta)}(n,k) \int_{-1}^{1} \frac{d^j(P_m^{(\alpha,\beta)}(t))}{dt^j} \frac{d^j(P_r^{(\alpha,\beta)}(t))}{dt^j} (1-t)^{\alpha+j}(1+t)^{\beta+j} dt$$
$$= (m(m+\alpha+\beta+1)+k)^n \delta_{m,r}. \tag{4.27}$$

Proof. The proof of (a) follows immediately from Theorem 4.2 and $k=0$. The proof of (b) is clear since all the numbers $\{c_j(n,k)\}_{j=0}^{n}$ are positive when $k>0$. The identity in (4.26) is a restatement of (4.18). Lastly, (4.27) follows from (4.19), (4.26), and the orthonormality of the Jacobi polynomials in $L^2_{\alpha,\beta}(-1,1)$. □

To illustrate Theorem 4.2 and Corollary 4.1, we list the following examples of powers of the Jacobi differential expression:

$$\begin{aligned} \ell^2_{\alpha,\beta,k}[y](t) = \frac{1}{w_{\alpha,\beta}(t)} &[((1-t)^{\alpha+2}(1+t)^{\beta+2}y'')'' \\ &- ((2k+2+\alpha+\beta)(1-t)^{\alpha+1}(1+t)^{\beta+1}y')' \\ &+ k^2(1-t)^{\alpha}(1+t)^{\beta}y], \end{aligned} \tag{4.28}$$

$$\begin{aligned} \ell^3_{\alpha,\beta,k}[y](t) = &\frac{1}{w_{\alpha,\beta}(t)}[-((1-t)^{\alpha+3}(1+t)^{\beta+3})y''')''' \\ &+ ((3k+8+3\alpha+3\beta)(1-t)^{\alpha+2}(1+t)^{\beta+2}y'')'' \\ &- ((3k^2+6k+4+\alpha^2+3k\alpha+4\alpha+2\alpha\beta+\beta^2+3k\beta+4\beta)(1-t)^{\alpha+1}(1+t)^{\beta+1})y')' \\ &+ k^3(1-t)^{\alpha}(1+t)^{\beta}y], \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} \ell^4_{\alpha,\beta,0}[y](t) = \frac{1}{w_{\alpha,\beta}(t)} &[((1-t)^{\alpha+4}(1+t)^{\beta+4}y^{(4)})^{(4)} \\ &- ((6\alpha+6\beta+20)(1-t)^{\alpha+3}(1+t)^{\beta+3}y''')''' \\ &+ ((7\alpha^2+38\alpha+7\beta^2+38\beta+14\alpha\beta+52)(1-t)^{\alpha+2}(1+t)^{\beta+2}y'')'' \\ &- ((\alpha+\beta+2)^3(1-t)^{\alpha+1}(1+t)^{\beta+1}y')']. \end{aligned} \tag{4.30}$$

Remark 4.2. We note a remarkable, and somewhat mysterious, point concerning the Stirling numbers of the second kind, the Legendre-Stirling numbers and, now more generally, the Jacobi-Stirling numbers. As mentioned in Remark 4.1, the Stirling numbers of the second kind appear in the integral composite powers of the classical Laguerre differential expression $\ell_{Lag,k}[\cdot]$. The Stirling numbers of the second kind $\{S_n^{(j)}\}$ may be defined (see [1, pp. 824-825]) as the coefficient of t^{n-j} in the Taylor series expansion of

$$g_j(t) := \prod_{r=1}^{j} \frac{1}{1-rt} \quad \left(|t| < \frac{1}{j}\right). \tag{4.31}$$

Furthermore, the r^{th} Laguerre polynomial $y = L_r^{\alpha}(t)$ $(r \in \mathbb{N}_0)$ is a solution of

$$\ell_{Lag}[y](t) = ry(t),$$

where the Laguerre differential expression $\ell_{Lag}[\cdot]$ is defined in (4.12); observe that this eigenvalue $\lambda_r = r$ also appears in the denominator of the rational generating function $g_j(t)$ defined in (4.31).

This is not a coincidence. Indeed, the same phenomenon occurs with the Hermite and the general Jacobi equations. More specifically, powers of the Hermite equation, defined by

$$\begin{aligned}\ell_H[y](t) &:= \exp(t^2)\left(-(\exp(-t^2)y'(t)\right)' \quad (t \in (-\infty, \infty)) \\ &= -y''(t) + 2ty'(t)\end{aligned}$$

are given by

$$\ell_H^n[y](t) = \exp(t^2)\sum_{j=0}^{n}(-1)^j 2^{n-j} S_n^{(j)}\left(\exp(-t^2)y^{(j)}(t)\right)^{(j)} \quad (n \in \mathbb{N});$$

the reader can check that $2^{n-j}S_n^{(j)}$ is also the coefficient of the Taylor series expansion of

$$h_j(t) := \prod_{r=1}^{j}\frac{1}{1-2rt} \quad \left(|t| < \frac{1}{2j}\right). \tag{4.32}$$

Moreover, $y = H_r(t)$, the r^{th} Hermite polynomial is a solution of $\ell_H[y](t) = 2ry(t)$; again notice that the eigenvalue $\lambda_r = 2r$ and the denominator term in (4.32) agree. In the general Jacobi case, which includes the Legendre case studied in [10], we see, from Theorem 4.1, that the Jacobi-Stirling number $P^{(\alpha,\beta)}S_n^{(j)}$ is the coefficient of t^{n-j} in the expansion of

$$f_j^{(\alpha,\beta)}(t) := \prod_{r=1}^{j}\frac{1}{1-r(r+\alpha+\beta+1)t} \quad \left(|t| < \frac{1}{j(j+\alpha+\beta+1)}\right).$$

Moreover, the r^{th} Jacobi polynomial $y = P_r^{(\alpha,\beta)}(t)$ is a solution of

$$\ell_{\alpha,\beta,0}[y](t) = r(r+\alpha+\beta+1)y(t);$$

again, notice the agreement between the eigenvalue $\lambda_r = r(r+\alpha+\beta+1)$ and the denominator term in $f_j^{(\alpha,\beta)}(t)$ above. These are intriguing results between the eigenvalues associated with the classical Jacobi, Laguerre, and Hermite expressions and the corresponding generating functions (4.5), (4.31), and (4.32) for the powers of these differential expressions. There is also some mystery concerning this connection. Indeed, the problem of determining integral composite powers of these expressions is completely an algebraic problem, independent of any functional or operator analysis. Why then do the generating functions (4.5), (4.31), and (4.32) for these powers involve the eigenvalues of those self-adjoint operators which have the corresponding Jacobi, Laguerre, and Hermite polynomials as eigenfunctions instead of the eigenvalues for some other self-adjoint operator generated from these differential expression? The answer could be that this is a new, and remarkable, property of these classical orthogonal polynomials and the second-order differential equations that they satisfy.

5. The left-definite theory for the Jacobi equation

For the results that follow in this section, we assume $k > 0$, where k is the parameter in the Jacobi expression (1.1). We remind the reader of the definition of the space $AC_{\text{loc}}^{(n)}(I)$ for $n \in \mathbb{N}$; see the notation at the end of Section 1. Notice that if $f \in AC_{\text{loc}}^{(n)}(-1,1)$, then $f^{(n+1)}(t)$ exists for almost all $t \in (-1,1)$. We also recall the Hilbert spaces $L^2_{\alpha+j,\beta+j}(-1,1)$ for each $j \in \mathbb{N}_0$ (see (3.1), (3.2), and (3.3)). Lastly, we remind the reader of the Jacobi self-adjoint operator $A_k^{(\alpha,\beta)}$, defined in (3.12) and (3.13), and its properties that are given in Section 3; in particular, from (3.14), this operator is bounded below in $L^2_{\alpha,\beta}(-1,1)$ by kI.

Definition 5.1. *Let $k > 0$. For each $n \in \mathbb{N}$, define*

$$V_n^{(\alpha,\beta)} := \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{loc}^{(n-1)}(-1,1); f^{(j)} \in L^2_{\alpha+j,\beta+j}(-1,1) \ (j = 0, 1, \ldots n)\} \tag{5.1}$$

and let $(\cdot,\cdot)_{n,k}^{(\alpha,\beta)}$ *and* $\|\cdot\|_{n,k}^{(\alpha,\beta)}$ *denote, respectively, the inner product*

$$(f,g)_{n,k}^{(\alpha,\beta)} := \sum_{j=0}^{n} c_j^{(\alpha,\beta)}(n,k) \int_{-1}^{1} f^{(j)}(t)\overline{g}^{(j)}(t)(1-t)^{\alpha+j}(1+t)^{\beta+j}dt \quad (f,g \in V_n^{(\alpha,\beta)}), \tag{5.2}$$

(*see* (4.25) *and* (4.26)) *and the norm* $\|f\|_{n,k}^{(\alpha,\beta)} := \left((f,f)_{n,k}^{(\alpha,\beta)}\right)^{1/2}$, *where the numbers* $c_j^{(\alpha,\beta)}(n,k)$ *are defined in* (4.2) *and* (4.3)*. Finally, let* $W_{n,k}^{(\alpha,\beta)}(-1,1) := (V_n^{(\alpha,\beta)}, (\cdot,\cdot)_{n,k}^{(\alpha,\beta)})$.

The inner product $(\cdot,\cdot)_{n,k}^{(\alpha,\beta)}$, defined in (5.2), is a Sobolev inner product and, in the context of the theory of differential operators, is more commonly called the *Dirichlet inner product* associated with the symmetrizable differential expression $\ell_{\alpha,\beta,k}^{n}[\cdot]$ given in (4.17).

Notice, from the definition in (5.2) and the non-negativity of each of the numbers $c_j^{(\alpha,\beta)}(n,k)$ $(j = 0,1,\ldots,n)$, that

$$\begin{aligned}\left(\|f\|_{n,k}^{(\alpha,\beta)}\right)^2 &= \sum_{j=0}^{n} c_j^{(\alpha,\beta)}(n,k) \left\|f^{(j)}\right\|_{\alpha+j,\beta+j}^2 \\ &\geq c_j^{(\alpha,\beta)}(n,k) \left\|f^{(j)}\right\|_{\alpha+j,\beta+j}^2 \quad (j=0,1,\ldots,n;\ f \in V_n^{(\alpha,\beta)}),\end{aligned} \tag{5.3}$$

where $\|\cdot\|_{\alpha+j,\beta+j}$ is defined in (3.7). In particular, from (4.2) and (5.3), we see that

$$\left(\|f\|_{n,k}^{(\alpha,\beta)}\right)^2 \geq k^n \|f\|_{\alpha,\beta}^2 \quad (f \in W_{n,k}^{(\alpha,\beta)}(-1,1)); \tag{5.4}$$

see item (iv) in Definition 2.1.

We remark that, for each $r > 0$, the r^{th} left-definite inner product $(\cdot,\cdot)_{r,k}^{(\alpha,\beta)}$ is given abstractly, through the Hilbert space spectral theorem (see [22]), by

$$(f,g)_{r,k}^{(\alpha,\beta)} := \int_{\mathbb{R}} \lambda^r dE_{f,g}^{(\alpha,\beta)}(k) \quad (f,g \in V_{r,k}^{(\alpha,\beta)} := \mathcal{D}((A_k^{(\alpha,\beta)})^{r/2})),$$

where $E^{(\alpha,\beta)}(k)$ is the spectral resolution of the identity for the self-adjoint operator $A_k^{(\alpha,\beta)}$; see [16] for further details connecting the left-definite theory with the spectral theorem. However, we are able to determine this inner product *in terms of the differential expression* $\ell_{\alpha,\beta,k}^{r}[\cdot]$ only when $r \in \mathbb{N}$; see also Remark 2.1. Clearly, further work along this line needs to be addressed; perhaps the theory of fractional differential expressions can be utilized to extend the results of this paper as well those results in [9], [10], and [16].

One of our aims in this section is to show that $W_{n,k}^{(\alpha,\beta)}(-1,1)$ is the n^{th} left-definite space associated with the pair $(L_{\alpha,\beta}^2(-1,1), A_k^{(\alpha,\beta)})$, where $A_k^{(\alpha,\beta)}$ is the self-adjoint Jacobi operator defined in (3.12) and (3.13). From this, we are able to obtain, for each $n \in \mathbb{N}$, an explicit representation of the n^{th} left-definite operator $B_{n,k}^{(\alpha,\beta)}$ as well as an explicit representation of each integral (composite) power $\left(A_k^{(\alpha,\beta)}\right)^n$ of $A_k^{(\alpha,\beta)}$.

We begin by showing that $W_{n,k}^{(\alpha,\beta)}(-1,1)$ is a complete inner product space.

Theorem 5.1. *Let* $k > 0$*. For each* $n \in \mathbb{N}$, $W_{n,k}^{(\alpha,\beta)}(-1,1)$ *is a Hilbert space.*

Proof. Let $n \in \mathbb{N}$. Suppose $\{f_m\}_{m=1}^{\infty}$ is Cauchy in $W_{n,k}^{(\alpha,\beta)}(-1,1)$. Since $c_n^{(\alpha,\beta)}(n,k)$ is positive, we see from (5.3) that $\{f_m^{(n)}\}_{m=1}^{\infty}$ is Cauchy in $L_{\alpha+n,\beta+n}^2(-1,1)$ and hence there exists $g_{n+1} \in L_{\alpha+n,\beta+n}^2(-1,1)$ such that

$$f_m^{(n)} \to g_{n+1} \text{ in } L_{\alpha+n,\beta+n}^2(-1,1),$$

so $g_{n+1} \in L^1_{\rm loc}(-1,1)$. Fix $t,\ t_0 \in (-1,1)$ (t_0 will be chosen shortly) and assume $t_0 \le t$. From Hölder's inequality, we see that as $m \to \infty$,

$$\begin{aligned}
&\int_{t_0}^{t} \left| f_m^{(n)}(t) - g_{n+1}(t) \right| dt \\
=&\int_{t_0}^{t} \left| f_m^{(n)}(t) - g_{n+1}(t) \right| (1-t)^{(\alpha+n)/2}(1+t)^{(\beta+n)/2}(1-t)^{-(\alpha+n)/2}(1+t)^{-(\beta+n)/2} dt \\
\le& \left(\int_{t_0}^{t} \left| f_m^{(n)}(t) - g_{n+1}(t) \right|^2 (1-t)^{\alpha+n}(1+t)^{\beta+n} dt \right)^{1/2} \cdot \left(\int_{t_0}^{t} (1-t)^{-(\alpha+n)}(1+t)^{-(\beta+n)} dt \right)^{1/2} \\
=&M(t_0,t) \left(\int_{t_0}^{t} \left| f_m^{(n)}(t) - g_{n+1}(t) \right|^2 (1-t)^{\alpha+n}(1+t)^{\beta+n} dt \right)^{1/2} \to 0.
\end{aligned}$$

It now follows, since $f_m^{(n-1)} \in AC_{\rm loc}(-1,1)$, that

$$f_m^{(n-1)}(t) - f_m^{(n-1)}(t_0) = \int_{t_0}^{t} f_m^{(n)}(t) dt \to \int_{t_0}^{t} g_{n+1}(t) dt. \tag{5.5}$$

Furthermore, from the definition of $(\cdot,\cdot)_{n,k}^{(\alpha,\beta)}$, we see that $\{f_m^{(n-1)}\}_{m=0}^{\infty}$ is Cauchy in $L^2_{\alpha+n-1,\beta+n-1}(-1,1)$; hence, there exists $g_n \in L^2_{\alpha+n-1,\beta+n-1}(-1,1)$ such that

$$f_m^{(n-1)} \to g_n \text{ in } L^2_{\alpha+n-1,\beta+n-1}(-1,1).$$

Repeating the above argument, we see that $g_n \in L^1_{\rm loc}(-1,1)$ and, for any $t, t_1 \in (-1,1)$,

$$f_m^{(n-2)}(t) - f_m^{(n-2)}(t_1) = \int_{t_1}^{t} f_m^{(n-1)}(t) dt \to \int_{t_1}^{t} g_n(t) dt. \tag{5.6}$$

Moreover, from [22, Theorem 3.12], there exists a subsequence $\{f_{m_{k,n-1}}^{(n-1)}\}$ of $\{f_m^{(n-1)}\}_{m=1}^{\infty}$ such that

$$f_{m_{k,n-1}}^{(n-1)}(t) \to g_n(t) \quad (\text{a.e. } t \in (-1,1)).$$

Choose $t_0 \in \mathbb{R}$ in (5.5) such that $f_{m_{k,n-1}}^{(n-1)}(t_0) \to g_n(t_0)$ and then pass through this subsequence in (5.5) to obtain

$$g_n(t) - g_n(t_0) = \int_{t_0}^{t} g_{n+1}(t) dt \quad (\text{a.e. } t \in (-1,1)).$$

That is to say,

$$g_n \in AC_{\rm loc}(-1,1) \text{ and } g_n'(t) = g_{n+1}(t) \quad (\text{a.e. } t \in (-1,1)). \tag{5.7}$$

Again, from the definition of $(\cdot,\cdot)_{n,k}^{(\alpha,\beta)}$, we see that $\{f_m^{(n-2)}\}_{m=1}^{\infty}$ is Cauchy in $L^2_{\alpha+n-2,\beta+n-2}(-1,1)$ so there exists $g_{n-1} \in L^2_{\alpha+n-2,\beta+n-2}(-1,1)$ such that

$$f_m^{(n-2)} \to g_{n-1} \text{ in } L^2_{\alpha+n-2,\beta+n-2}(-1,1).$$

As above, we find that $g_{n-1} \in L^1_{\rm loc}(-1,1)$; moreover, for any $t, t_2 \in (-1,1)$

$$f_m^{(n-3)}(t) - f_m^{(n-3)}(t_2) = \int_{t_2}^{t} f_m^{(n-2)}(t) dt \to \int_{t_2}^{t} g_{n-1}(t) dt,$$

and there exists a subsequence $\{f_{m_{k,n-2}}^{(n-2)}\}$ of $\{f_m^{(n-2)}\}$ such that

$$f_{m_{k,n-2}}^{(n-2)}(t) \to g_{n-1}(t) \quad (\text{a.e. } t \in (-1,1)).$$

In (5.6), choose $t_1 \in (-1,1)$ such that $f^{(n-2)}_{m_{k,n-2}}(t_1) \to g_{n-1}(t_1)$ and pass through the subsequence $\{f^{(n-2)}_{m_{k,n-2}}\}$ in (5.6) to obtain

$$g_{n-1}(t) - g_{n-1}(t_1) = \int_{t_1}^{t} g_n(t)dt \quad (\text{a.e. } t \in (-1,1)).$$

Consequently, $g_{n-1} \in AC^{(1)}_{\text{loc}}(-1,1)$ and $g''_{n-1}(t) = g'_n(t) = g_{n+1}(t)$ a.e. $t \in (-1,1)$. Continuing in this fashion, we obtain $n+1$ functions $g_{n-j+1} \in L^2_{\alpha+n-j,\beta+n-j}(-1,1)$ $(j = 0,1,\ldots,n)$ such that
(i) $f^{(n-j)}_m \to g_{n-j+1}$ in $L^2_{\alpha+n-j,\beta+n-j}(-1,1)$ $(j = 0,1,\ldots,n)$,
(ii) $g_1 \in AC^{(n-1)}_{\text{loc}}(-1,1)$, $g_2 \in AC^{(n-2)}_{\text{loc}}(-1,1), \ldots, g_n \in AC_{\text{loc}}(-1,1)$,
(iii) $g'_{n-j}(t) = g_{n-j+1}(t)$ a.e. $t \in (-1,1)$ $(j = 0,1,\ldots,n-1)$,
(iv) $g^{(j)}_1 = g_{j+1}$ $(j = 0,1,\ldots,n)$.
In particular, we see that $f^{(j)}_m \to g^{(j)}_1$ in $L^2_{\alpha+j,\beta+j}(-1,1)$ for $j = 0,1,\ldots,n$ and $g_1 \in V^{(\alpha,\beta)}_n$. Hence, we see that

$$\begin{aligned}\left(\|f_m - g_1\|^{(\alpha,\beta)}_{n,k}\right)^2 &= \sum_{j=0}^{n} c^{(\alpha,\beta)}_j(n,k) \int_{-1}^{1} \left|f^{(j)}_m(t) - g^{(j)}_1(t)\right|^2 (1-t)^{\alpha+j}(1+t)^{\beta+j}dt \\ &= \sum_{j=0}^{n} c^{(\alpha,\beta)}_j(n,k) \left\|f^{(j)}_m - g^{(j)}_1\right\|^2_{\alpha+j,\beta+j} \\ &\to 0 \text{ as } m \to \infty.\end{aligned}$$

Thus $W^{(\alpha,\beta)}_{n,k}(-1,1)$ is complete and, consequently, so is the proof of this theorem. □

We next establish the completeness of the Jacobi polynomials $\{P^{(\alpha,\beta)}_m\}_{m=0}^{\infty}$ in each $W^{(\alpha,\beta)}_{n,k}(-1,1)$.

Theorem 5.2. *Let $k > 0$. The Jacobi polynomials $\{P^{(\alpha,\beta)}_m\}_{m=0}^{\infty}$ form a complete orthogonal set in the space $W^{(\alpha,\beta)}_{n,k}(-1,1)$. Equivalently, the space $\mathcal{P}$ of polynomials is dense in $W^{(\alpha,\beta)}_{n,k}(-1,1)$.*

Proof. Let $f \in W^{(\alpha,\beta)}_{n,k}(-1,1)$; in particular, $f^{(n)} \in L^2_{\alpha+n,\beta+n}(-1,1)$. Consequently, from the completeness and orthonormality of the Jacobi polynomials $\{P^{(\alpha+n,\beta+n)}_m\}_{m=0}^{\infty}$ in $L^2_{\alpha+n,\beta+n}(-1,1)$, it follows that

$$\sum_{m=0}^{r} c^{(\alpha,\beta)}_{m,n} P^{(\alpha+n,\beta+n)}_m \to f^{(n)} \text{ as } r \to \infty \text{ in } L^2_{\alpha+n,\beta+n}(-1,1), \tag{5.8}$$

where the numbers $\{c^{(\alpha,\beta)}_{m,n}\}_{m=0}^{\infty} \subset \ell^2$ are the Fourier coefficients of $f^{(n)}$, relative to the orthonormal basis $\{P^{(\alpha+n,\beta+n)}_m\}_{m=0}^{\infty}$ of $L^2_{\alpha+n,\beta+n}(-1,1)$, defined by

$$c^{(\alpha,\beta)}_{m,n} = \int_{-1}^{1} f^{(n)}(t) P^{(\alpha+n,\beta+n)}_m(t)(1-t)^{\alpha+n}(1+t)^{\beta+n}dt \quad (m \in \mathbb{N}_0). \tag{5.9}$$

For $r \geq n$, define the polynomials

$$p_r(t) = \sum_{m=n}^{r} \frac{c^{(\alpha,\beta)}_{m-n,n}\left((m-n)!\right)^{1/2}\left(\Gamma(\alpha+\beta+m+1)\right)^{1/2}}{(m!)^{1/2}\left(\Gamma(\alpha+\beta+m+n+1)!\right)^{1/2}} P^{(\alpha,\beta)}_m(t). \tag{5.10}$$

Then, using the derivative formula (3.8) for the Jacobi polynomials, we see that

$$\begin{aligned} &p_r^{(j)}(t) \\ &= \sum_{m=n}^{r} \frac{c_{m-n,n}^{(\alpha,\beta)}\left((m-n)!\right)^{1/2}\left(\Gamma(\alpha+\beta+m+j+1)!\right)^{1/2}}{\left(\Gamma(\alpha+\beta+m+n+1)!\right)^{1/2}\left((m-j)!\right)^{1/2}} P_{m-j}^{(\alpha+j,\beta+j)}(t) \quad (j=0,1,\ldots,n), \end{aligned} \tag{5.11}$$

and, in particular, from (5.8),

$$p_r^{(n)} = \sum_{m=n}^{r} c_{m-n,n}^{(\alpha,\beta)} P_{m-n}^{(\alpha+n,\beta+n)} \to f^{(n)} \text{ in } L^2_{\alpha+n,\beta+n}(-1,1) \quad (r \to \infty).$$

Furthermore, from [22, Theorem 3.12], there exists a subsequence $\{p_{r_j}^{(n)}\}$ of $\{p_r^{(n)}\}$ such that

$$p_{r_j}^{(n)}(t) \to f^{(n)}(t) \quad (\text{a.e. } t \in (-1,1)). \tag{5.12}$$

Returning to (5.11), observe that since $\dfrac{\left((m-n)!\right)^{1/2}\left(\Gamma(\alpha+\beta+m+j+1)!\right)^{1/2}}{\left(\Gamma(\alpha+\beta+m+n+1)!\right)^{1/2}\left((m-j)!\right)^{1/2}} \to 0$ as $m \to \infty$ for $j = 0, 1, \ldots, n-1$, we see that

$$\left\{ \frac{c_{m-n,n}^{(\alpha,\beta)}\left((m-n)!\right)^{1/2}\left(\Gamma(\alpha+\beta+m+j+1)!\right)^{1/2}}{\left(\Gamma(\alpha+\beta+m+n+1)!\right)^{1/2}\left((m-j)!\right)^{1/2}} \right\}_{m=n}^{\infty} \in \ell^2.$$

Hence, from the completeness of the Jacobi polynomials $\{P_m^{(\alpha+j,\beta+j)}\}_{m=0}^{\infty}$ in $L^2_{\alpha+j,\beta+j}(-1,1)$ and the Riesz-Fischer theorem (see [22, Chapter 4, Theorem 4.17]), there exists $g_j \in L^2_{\alpha+j,\beta+j}(-1,1)$ such that

$$p_r^{(j)} \to g_j \text{ in } L^2_{\alpha+j,\beta+j}(-1,1) \text{ as } r \to \infty \ (j = 0, 1, \ldots, n-1). \tag{5.13}$$

Since, for a.e. $a, t \in (-1,1)$,

$$p_{r_j}^{(n-1)}(t) - p_{r_j}^{(n-1)}(a) = \int_a^t p_{r_j}^{(n)}(u)du \to \int_a^t f^{(n)}(u)du = f^{(n-1)}(t) - f^{(n-1)}(a) \quad (j \to \infty),$$

we see that, as $j \to \infty$,

$$p_{r_j}^{(n-1)}(t) \to f^{(n-1)}(t) + c_1 \quad (\text{a.e. } t \in (-1,1)), \tag{5.14}$$

where c_1 is some constant. From (5.13), with $j = n-1$, we deduce that

$$g_{n-1}(t) = f^{(n-1)}(t) + c_1 \quad (\text{a.e. } t \in (-1,1)).$$

Next, from (5.14) and one integration, we obtain

$$p_{r_j}^{(n-2)}(t) \to f^{(n-2)}(t) + c_1 t + c_2 \quad (j \to \infty),$$

for some constant c_2 and hence, from (5.13),

$$g_{n-2}(t) = f^{(n-2)}(t) + c_1 t + c_2 \quad (\text{a.e. } t \in (-1,1)).$$

We continue this process to see that, for $j = 0, 1, \ldots n-1$,

$$g_j(t) = f^{(j)}(t) + q_{n-j-1}(t) \quad (\text{a.e. } t \in (-1,1)),$$

where q_{n-j-1} is a polynomial of degree $\leq n-j-1$ satisfying

$$q'_{n-j-1}(t) = q_{n-j-2}(t).$$

Combined with (5.13), we see that, as $r \to \infty$,

$$p_r^{(j)} \to f^{(j)} + q_{n-j-1} \text{ in } L^2_{\alpha+j,\beta+j}(-1,1) \quad (j = 0, 1, \ldots, n).$$

For each $r \geq n$, define the polynomial

$$\pi_r(t) := p_r(t) - q_{n-1}(t),$$

and observe that, for $j = 0, 1, \ldots, n$,

$$\begin{aligned} \pi_r^{(j)} &= p_r^{(j)} - q_{n-1}^{(j)} \\ &= p_r^{(j)} - q_{n-j-1} \\ &\to f^{(j)} \text{ in } L^2_{\alpha+j,\beta+j}(-1,1). \end{aligned}$$

Hence, as $r \to \infty$,

$$\left(\|f - \pi_r\|_{n,k}^{(\alpha,\beta)}\right)^2 = \sum_{j=0}^{n} c_j^{(\alpha,\beta)}(n,k) \int_{-1}^{1} \left| f^{(j)}(t) - \pi_r^{(j)}(t) \right|^2 (1-t)^{\alpha+j}(1+t)^{\beta+j} dt \to 0.$$

This shows that $\mathcal{P}$ is dense in $W_{n,k}^{(\alpha,\beta)}(-1,1)$ and completes the proof of this theorem. □

The next result, which gives a simpler characterization of the function space $V_n^{(\alpha,\beta)}$, follows from ideas presented in the above proof of Theorem 5.2. Due to the importance of this theorem (which can be further seen in the statement of Corollary 5.1 below), we provide the following proof.

Theorem 5.3. *For each* $n \in \mathbb{N}$,

$$\begin{aligned} V_n^{(\alpha,\beta)} &= \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{\text{loc}}^{(n-1)}(-1,1); f^{(n)} \in L^2_{\alpha+n,\beta+n}(-1,1)\} \\ &= \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{\text{loc}}^{(n-1)}(-1,1); (1-t^2)^{n/2} f^{(n)} \in L^2_{\alpha,\beta}(-1,1)\} \end{aligned} \tag{5.15}$$

Proof. On account of (3.4), it is clear that the two sets on the right-hand side of (5.15) are equal. Let $n \in \mathbb{N}$ and recall the definition of $V_n^{(\alpha,\beta)}$ in (5.1). Define

$$\widetilde{V}_n^{(\alpha,\beta)} = \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{\text{loc}}^{(n-1)}(-1,1); f^{(n)} \in L^2_{\alpha+n,\beta+n}(-1,1)\}.$$

It is clear that $V_n^{(\alpha,\beta)} \subset \widetilde{V}_n^{(\alpha,\beta)}$. Conversely, suppose $f \in \widetilde{V}_n^{(\alpha,\beta)}$ so $f^{(n)} \in L^2_{\alpha+n,\beta+n}(-1,1)$ and $f \in AC_{\text{loc}}^{(n-1)}(-1,1)$. As shown in Theorem 5.2, as $r \to \infty$,

$$\sum_{m=0}^{r} c_{m,n}^{(\alpha,\beta)} P_m^{(n,n)} \to f^{(n)} \quad \text{in } L^2_{\alpha+n,\beta+n}(-1,1),$$

where $\{c_{m,n}^{(\alpha,\beta)}\}$ are the Fourier coefficients of $f^{(n)}$, defined in (5.9), relative to the orthonormal basis $\{P_m^{(\alpha+n,\beta+n)}\}_{m=0}^{\infty}$ of $L^2_{\alpha+n,\beta+n}$.

For $r \geq n$, let $p_r(t)$ be the polynomial that is defined in (5.10). Then, for any $j \in \mathbb{N}_0$, the j^{th} derivative of p_r is given in (5.11) and, as in Theorem 5.2,

$$p_r^{(n)} \to f^{(n)} \text{ as } r \to \infty \text{ in } L^2_{\alpha+n,\beta+n}(-1,1);$$

moreover, as shown in Theorem 5.2, there exists polynomials q_{n-j-1} of degree $\leq n-j-1$, for $j = 0, 1, \ldots, n-1$, satisfying $q'_{n-j-1}(t) = q_{n-j-2}(t)$ with

$$\begin{aligned} p_r^{(j)} &\to f^{(j)} + q_{n-j-1} \text{ as } r \to \infty \text{ in } L^2_{\alpha+j,\beta+n}(-1,1), \\ &= f^{(j)} + q_{n-1}^{(j)}. \end{aligned}$$

Consequently, for each $j = 0, 1, \ldots, n-1$, $\{p_r^{(j)} - q_{n-1}^{(j)}\}_{r=n}^{\infty}$ converges in $L^2_{\alpha+j,\beta+j}(-1,1)$ to $f^{(j)}$. From the completeness of $L^2_{\alpha+j,\beta+j}(-1,1)$, we conclude that $f^{(j)} \in L^2_{\alpha+j,\beta+j}(-1,1)$ for $j = 0, 1, \ldots, n-1$. That is to say, $f \in V_n^{(\alpha,\beta)}$. This completes the proof. □

We are now in position to prove the main result of this section.

Theorem 5.4. *For $k>0$, let $A_k^{(\alpha,\beta)}:\mathcal{D}(A_k^{(\alpha,\beta)})\subset L^2_{\alpha,\beta}(-1,1)\to L^2_{\alpha,\beta}(-1,1)$ be the Jacobi self-adjoint operator, defined in* (3.12) *and* (3.13), *having the Jacobi polynomials $\{P_m^{(\alpha,\beta)}\}_{m=0}^\infty$ as eigenfunctions. For each $n\in\mathbb{N}$, let $V_n^{(\alpha,\beta)}$ be given as in* (5.1) *or* (5.15) *and let $(\cdot,\cdot)_{n,k}^{(\alpha,\beta)}$ denote the inner product defined in* (5.2). *Then $W_{n,k}^{(\alpha,\beta)}(-1,1)=(V_n^{(\alpha,\beta)},(\cdot,\cdot)_{n,k}^{(\alpha,\beta)})$ is the n^{th} left-definite space for the pair $(L^2_{\alpha,\beta}(-1,1),A_k^{(\alpha,\beta)})$. Moreover, the Jacobi polynomials $\{P_m^{(\alpha,\beta)}\}_{m=0}^\infty$ form a complete orthogonal set in $W_{n,k}^{(\alpha,\beta)}(-1,1)$ satisfying the orthogonality relation* (4.27). *Furthermore, define*

$$B_{n,k}^{(\alpha,\beta)}:\mathcal{D}(B_{n,k}^{(\alpha,\beta)})\subset W_{n,k}^{(\alpha,\beta)}(-1,1)\to W_{n,k}^{(\alpha,\beta)}(-1,1)$$

by

$$B_{n,k}^{(\alpha,\beta)}f=\ell_{\alpha,\beta,k}[f]\quad(f\in\mathcal{D}(B_{n,k}^{(\alpha,\beta)}):=V_{n+2}^{(\alpha,\beta)}),$$

where $\ell_{\alpha,\beta,k}[\cdot]$ is the Jacobi differential expression defined in (1.1). *Then $B_{n,k}^{(\alpha,\beta)}$ is the n^{th} left-definite operator associated with the pair $(L^2_{\alpha,\beta}(-1,1),A_k^{(\alpha,\beta)})$. Furthermore, the Jacobi polynomials $\{P_m^{(\alpha,\beta)}\}_{m=0}^\infty$ form a complete set of eigenfunctions of $B_{n,k}^{(\alpha,\beta)}$ and the spectrum of $B_{n,k}^{(\alpha,\beta)}$ is given by*

$$\sigma(B_{n,k}^{(\alpha,\beta)})=\{m(m+\alpha+\beta+1)+k\mid m\in\mathbb{N}_0\}=\sigma(A_k^{(\alpha,\beta)}).$$

Proof. Let $n\in\mathbb{N}$; in order to show that $W_{n,k}^{(\alpha,\beta)}(-1,1)$ is the n^{th} left-definite space for the pair $(L^2_{\alpha,\beta}(-1,1),A_k^{(\alpha,\beta)})$, we must show that the five items listed in Definition 2.1 are each satisfied.

(i) $W_{n,k}^{(\alpha,\beta)}(-1,1)$ is complete:

The proof of the completeness is given in Theorem 5.1; see also Theorem 5.3 where an alternative characterization of the underlying vector space $V_n^{(\alpha,\beta)}$ is given.

(ii) $\mathcal{D}((A_k^{(\alpha,\beta)})^n)\subset W_{n,k}^{(\alpha,\beta)}(-1,1)\subset L^2_{\alpha,\beta}(-1,1)$:

Let $f\in\mathcal{D}((A_k^{(\alpha,\beta)})^n)$. Since the Jacobi polynomials $\{P_m^{(\alpha,\beta)}\}_{m=0}^\infty$ form a complete orthonormal set in $L^2_{\alpha,\beta}(-1,1)$, we see that

$$p_j\to f\text{ in }L^2_{\alpha,\beta}(-1,1)\quad(j\to\infty),\tag{5.16}$$

where

$$p_j(t):=\sum_{m=0}^{j}c_m^{(\alpha,\beta)}P_m^{(\alpha,\beta)}(t)\quad(t\in(-1,1)),$$

and $\{c_m^{(\alpha,\beta)}\}_{m=0}^\infty$ are the Fourier coefficients of f in $L^2_{\alpha,\beta}(-1,1)$ defined by

$$c_m^{(\alpha,\beta)}=(f,P_m^{(\alpha,\beta)})_{\alpha,\beta}=\int_{-1}^{1}f(t)P_m^{(\alpha,\beta)}(t)(1-t)^\alpha(1+t)^\beta dt\quad(m\in\mathbb{N}_0).$$

Since $(A_k^{(\alpha,\beta)})^nf\in L^2_{\alpha,\beta}(-1,1)$, we see that

$$\sum_{m=0}^{j}\widetilde{c}_m^{(\alpha,\beta)}P_m^{(\alpha,\beta)}\to(A_k^{(\alpha,\beta)})^nf\text{ in }L^2_{\alpha,\beta}(-1,1)\quad(j\to\infty),$$

where, from (4.19),

$$\begin{aligned}\widetilde{c}_m^{(\alpha,\beta)}&:=((A_k^{(\alpha,\beta)})^nf,P_m^{(\alpha,\beta)})_{\alpha,\beta}=(f,(A_k^{(\alpha,\beta)})^nP_m^{(\alpha,\beta)})_{\alpha,\beta}\\&=(m(m+\alpha+\beta+1)+k)^n(f,P_m^{(\alpha,\beta)})_{\alpha,\beta}\\&=(m(m+\alpha+\beta+1)+k)^nc_m^{(\alpha,\beta)};\end{aligned}$$

that is to say,

$$(A_k^{(\alpha,\beta)})^n p_j \to (A_k^{(\alpha,\beta)})^n f \text{ in } L^2_{\alpha,\beta}(-1,1) \quad (j \to \infty).$$

Moreover, from (4.26), we see that

$$\begin{aligned}\left(\|p_j - p_r\|_{n,k}^{(\alpha,\beta)}\right)^2 &= ((A_k^{(\alpha,\beta)})^n [p_j - p_r], p_j - p_r)_{\alpha,\beta} \\ &\to 0 \text{ as } j, r \to \infty;\end{aligned}$$

that is to say, $\{p_j\}_{j=0}^\infty$ is Cauchy in $W_{n,k}^{(\alpha,\beta)}(-1,1)$. From Theorem 5.1, we see that there exists $g \in W_{n,k}^{(\alpha,\beta)}(-1,1) \subset L^2_{\alpha,\beta}(-1,1)$ such that

$$p_j \to g \text{ in } W_{n,k}^{(\alpha,\beta)}(-1,1) \quad (j \to \infty).$$

Furthermore, from (5.4), we see that

$$\|p_j - g\|_{\alpha,\beta} \leq k^{-n/2} \|p_j - g\|_{n,k}^{(\alpha,\beta)}$$

and, hence

$$p_j \to g \text{ in } L^2_{\alpha,\beta}(-1,1). \tag{5.17}$$

Comparing (5.16) and (5.17), we see that $f = g \in W_{n,k}^{(\alpha,\beta)}(-1,1)$; this completes the proof of (ii).

<u>(iii) $\mathcal{D}((A_k^{(\alpha,\beta)})^n)$ is dense in $W_{n,k}^{(\alpha,\beta)}(-1,1)$:</u>

Since polynomials are contained in $\mathcal{D}((A_k^{(\alpha,\beta)})^n)$ and are dense in $W_{n,k}^{(\alpha,\beta)}(-1,1)$ (see Theorem 5.2), it is clear that (iii) is valid. Furthermore, from Theorem 5.2, we see that the Jacobi polynomials $\{P_m^{(\alpha,\beta)}\}_{m=0}^\infty$ form a complete orthogonal set in $W_{n,k}^{(\alpha,\beta)}(-1,1)$; see also (4.27).

<u>(iv) $(f,f)_{n,k}^{(\alpha,\beta)} \geq k^n (f,f)_{\alpha,\beta}$ for all $f \in V_n^{(\alpha,\beta)}$:</u>

This was observed in (5.4).

<u>(v) $(f,g)_{n,k}^{(\alpha,\beta)} = ((A_k^{(\alpha,\beta)})^n f, g)_{\alpha,\beta}$ for $f \in \mathcal{D}((A_k^{(\alpha,\beta)})^n)$ and $g \in V_n^{(\alpha,\beta)}$:</u>

Observe that this identity is true for any $f, g \in \mathcal{P}$; indeed, this is seen in (4.26). Let $f \in \mathcal{D}((A_k^{(\alpha,\beta)})^n) \subset W_{n,k}^{(\alpha,\beta)}(-1,1)$ and $g \in W_{n,k}^{(\alpha,\beta)}(-1,1)$; since polynomials are dense in both $W_{n,k}^{(\alpha,\beta)}(-1,1)$ and $L^2_{\alpha,\beta}(-1,1)$ and convergence in $W_{n,k}^{(\alpha,\beta)}(-1,1)$ implies convergence in $L^2_{\alpha,\beta}(-1,1)$ (from (iv) above), there exists sequences of polynomials $\{p_j\}_{j=0}^\infty$ and $\{q_j\}_{j=0}^\infty$ such that, as $j \to \infty$,

$$p_j \to f \text{ in } W_{n,k}^{(\alpha,\beta)}(-1,1),\ (A_k^{(\alpha,\beta)})^n p_j \to (A_k^{(\alpha,\beta)})^n f \text{ in } L^2_{\alpha,\beta}(-1,1)$$

(see the proof of part (ii) of this Theorem), and

$$q_j \to g \text{ in } W_{n,k}^{(\alpha,\beta)}(-1,1) \text{ and } L^2_{\alpha,\beta}(-1,1).$$

Hence, from (4.26),

$$((A_k^{(\alpha,\beta)})^n [f], g)_{\alpha,\beta} = \lim_{j\to\infty} ((A_k^{(\alpha,\beta)})^n [p_j], q_j)_{\alpha,\beta} = \lim_{j\to\infty} (p_j, q_j)_{n,k}^{(\alpha,\beta)} = (f,g)_{n,k}^{(\alpha,\beta)}.$$

This proves (v). The rest of the proof follows immediately from Theorems 2.1, 2.2, and 2.3. □

Remark 5.1. Observe that, for $n = 1$, the identity given in part (v) of the above proof, namely

$$(f,g)_{1,k}^{(\alpha,\beta)} = (A_k^{(\alpha,\beta)} f, g)_{\alpha,\beta} \quad (f \in \mathcal{D}(A_k^{(\alpha,\beta)}),\ g \in V_1^{(\alpha,\beta)}),$$

extends the Dirichlet identity for $A_k^{(\alpha,\beta)}$, given in (3.15).

Remark 5.2. Theorem 5.4 generalizes the left-definite theory of the Legendre differential expression $\ell_{0,0,k}[\cdot]$ developed in [10]; see also [27], where the first left-definite theory is developed in a different manner.

The next corollary follows immediately from Theorems 2.1, 5.3, 5.4 and the observation made in (3.4). Remarkably, it characterizes the domain of each of the integral composite powers of $A_k^{(\alpha,\beta)}$. Furthermore, the characterization given in (5.18) below of the domain $\mathcal{D}(A_k^{(\alpha,\beta)})$ of $A_k^{(\alpha,\beta)}$ is new and generalizes the results in [3] and [8] for the special case of the Legendre differential operator $A_k^{(0,0)}$.

Corollary 5.1. *Let $k > 0$. For each $n \in \mathbb{N}$, the domain $\mathcal{D}((A_k^{(\alpha,\beta)})^n)$ of the n^{th} composite power $(A_k^{(\alpha,\beta)})^n$ of the self-adjoint Jacobi operator $A_k^{(\alpha,\beta)}$, defined in* (3.12) *and* (3.12), *is given by*

$$\begin{aligned}\mathcal{D}((A_k^{(\alpha,\beta)})^n) = V_{2n}^{(\alpha,\beta)} &= \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{\text{loc}}^{(2n-1)}(-1,1);\ f^{(2n)} \in L^2_{\alpha+2n\beta+2n}(-1,1)\}\\ &= \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{\text{loc}}^{(2n-1)}(-1,1);\ (1-t^2)^n f^{(2n)} \in L^2_{\alpha,\beta}(-1,1).\end{aligned}$$

In particular,

$$\mathcal{D}(A_k^{(\alpha,\beta)}) = V_2^{(\alpha,\beta)} = \{f : (-\infty,\infty) \to \mathbb{C} \mid f \in AC_{\text{loc}}^{(1)}(-1,1); (1-t^2)f'' \in L^2_{\alpha,\beta}(-1,1)\}. \tag{5.18}$$

Lastly, we note, from Theorems 2.2 and 5.4, that the domain of the first left-definite operator $B_{1,k}^{(\alpha,\beta)}$ is given explicitly by

$$\mathcal{D}(B_{1,k}^{(\alpha,\beta)}) = V_3^{(\alpha,\beta)} = \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{loc}^{(2)}(-1,1); (1-t^2)^{3/2} f''' \in L^2_{\alpha,\beta}(-1,1)\};$$

this characterization, as well, extends results in [3] and [8] for the Legendre differential operator $A_k^{(0,0)}$.

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Department of Mathematics, University of Birmingham, Birmingham, B15 2TT, England, UK.
E-mail address: w.n.everitt@bham.ac.uk

Division of Applied Mathematics, KAIST, 373-1 Kusong-dong, Yusong-ku, Taejon 305-701, Korea
E-mail address: khkwon@jacobi.kaist.ac.kr

Department of Mathematics and Statistics, Utah State University, Logan, Utah, 84322-3900, USA.
E-mail address: lance@math.usu.edu

Division of Applied Mathematics, KAIST, 373-1 Kusong-dong, Yusong-ku, Taejon 305-701, Korea
E-mail address: ykj@jacobi.kaist.ac.kr