BEST POLYNOMIAL APPROXIMATION IN SOBOLEV-LAGUERRE SPACE

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ABSTRACT. We investigate the limiting behavior as γ tends to ∞ of the best polynomial approximations in the Sobolev-Laguerre space $W^{N,2}([0,\infty);e^{-x})$ with respect to the Sobolev-Laguerre inner product

$$\phi(f,g) := \int_0^\infty f(x)g(x)e^{-x}dx + \gamma\int_0^\infty f^{(N)}(x)g^{(N)}(x)e^{-x}dx$$

where $\gamma > 0$ and $N \geq 1$ is an integer.

We also give conjectures for the same problem concerning to Sobolev-Laguerre and Sobolev-Legendre inner products :

$$\phi_1(f,g) := \sum_{k=0}^{N-1}\int_0^\infty f^{(k)}(x)g^{(k)}(x)e^{-x}dx + \gamma\int_0^\infty f^{(N)}(x)g^{(N)}(x)e^{-x}dx$$

and

$$\phi_2(f,g) := \sum_{k=0}^{N-1}\int_{-1}^1 f^{(k)}(x)g^{(k)}(x)dx + \gamma\int_{-1}^1 f^{(N)}(x)g^{(N)}(x)dx.$$

1991 AMS SUBJECT CLASSIFICATION : 41A10, 33C45
KEY WORDS : BEST POLYNOMIAL APPROXIMATION, ORTHOGONAL POLYNOMIALS.

1. Introduction

Polynomial approximation of functions in various weighted Sobolev spaces has been studied by many authors from different points of view (see [1] $\sim$ [9]). In particular, Cohen [4] studied the behavior of the best polynomial approximations for functions in the Sobolev-Legendre space $W^{1,2}[-1,1]$ with the Sobolev inner product

$$(f,g)_s = \int_{-1}^1 f(x)g(x)dx + \gamma\int_{-1}^1 f'(x)g'(x)dx \tag{1.1}$$

as γ tends to ∞. Sobolev orthogonal polynomials with respect to $(\cdot,\cdot)_s$ were studied in detail by Althammer [1] and Gröbner [7]. Motivated by the work of Cohen [4], we consider the best polynomial approximations in the Sobolev-Laguerre space $W^{N,2}[0,\infty;e^{-x}]$ with the Sobolev inner products

$$\phi(f,g) = \int_0^\infty f(x)g(x)e^{-x}dx + \gamma\int_0^\infty f^{(N)}(x)g^{(N)}(x)e^{-x}dx, \tag{1.2}$$

and

$$\phi_1(f,g) := \sum_{k=0}^{N-1}\int_0^\infty f^{(k)}(x)g^{(k)}(x)e^{-x}dx + \gamma\int_0^\infty f^{(N)}(x)g^{(N)}(x)e^{-x}dx, \tag{1.3}$$

where $\gamma > 0$ and $N \geq 1$ is a positive integer. Sobolev-Laguerre orthogonal polynomials with respect to $\phi(\cdot,\cdot)$ for $N = 1$ were studied by Brenner [2] and Marcellán et al [10]. See also [11] for algebraic and differential properties of general Sobolev orthogonal polynomials including Sobolev-Jacobi and Sobolev-Laguerre orthogonal polynomials.

Concerning to the limiting behavior of the best polynomial approximations in $W^{N,2}([0,\infty); e^{-x})$, we also need to consider the following so-called discrete-continuous Sobolev inner product

$$\psi(f,g) = \sum_{k=0}^{N-1} f^{(k)}(0)g^{(k)}(0) + \int_0^\infty f^{(N)}(x)g^{(N)}(x)e^{-x}dx. \tag{1.4}$$

Let $\{S_n^{(\gamma)}(x)\}_{n=0}^\infty$ and $\{Q_n(x)\}_{n=0}^\infty$ be the monic Sobolev orthogonal polynomials with respect to $\phi(\cdot,\cdot)$ and $\psi(\cdot,\cdot)$ respectively. In Section 2, we investigate algebraic properties of $\{S_n^{(\gamma)}(x)\}_{n=0}^\infty$, $\{Q_n(x)\}_{n=0}^\infty$, and $\{S_n^{(\infty)}(x) \equiv \lim_{\gamma\to\infty} S_n^{(\gamma)}(x)\}_{n=0}^\infty$. In Section 3, we study the limiting behavior as γ tends to ∞ of the best polynomial approximations to $f \in W^{N,2}([0,\infty); e^{-x})$ with the Sobolev-Laguerre inner product (1.2). Finally in Section 4, we give three conjectures relating to the limiting behavior of the best polynomial approximations in the Sobolev-Laguerre and Sobolev-Legendre inner products

$$\sum_{k=0}^{N-1} \int_0^\infty f^{(k)}(x)g^{(k)}(x)e^{-x}dx + \gamma \int_0^\infty f^{(N)}(x)g^{(N)}(x)e^{-x}dx$$

and

$$\sum_{k=0}^{N-1} \int_{-1}^1 f^{(k)}(x)g^{(k)}(x)dx + \gamma \int_{-1}^1 f^{(N)}(x)g^{(N)}(x)dx.$$

2. Algebraic Properties

Let

$$L_n^{(\alpha)}(x) = (-1)^n n! \sum_{k=0}^n \frac{(-1)^k(\alpha+k+1)_{n-k}}{k!(n-k)!}x^k \ (n \geq 0,\ \alpha \in \mathbb{R}) \tag{2.1}$$

be the monic Laguerre polynomial system satisfying

$$xy''(x) + (\alpha + 1 - x)y' + ny = 0,\ n \geq 0, \tag{2.2}$$

where

$$(a)_j = \begin{cases} 1 & \text{if } \ j = 0 \\ a(a+1)\cdots(a+j-1) & \text{if } \ j = 1, 2, \cdots \end{cases}$$

is the Pochhammer symbol. Then, the following are well known (see [13]) :

$$L_n^{(\alpha)}(x)' = nL_{n-1}^{(\alpha+1)}(x),\ n \geq 0 \ ; \tag{2.3}$$

$$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) + nL_{n-1}^{(\alpha+1)}(x),\ n \geq 0 \ ; \tag{2.4}$$

$$L_{n+1}^{(\alpha)}(x) = [x - (2n + \alpha + 1)]L_n^{(\alpha)}(x) - n(n + \alpha)L_{n-1}^{(\alpha)}(x),\ n \geq 1. \tag{2.5}$$

By the Favard-Shohat theorem, if $\alpha \neq -1, -2, \cdots$, then $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ is the monic orthogonal polynomial system.

If $\alpha = -N$ is a negative integer, then we have from (2.1) and (2.4), for $n \geq N$,

$$L_n^{(-N)}(x) = x^N L_{n-N}^{(N)}(x) = \sum_{l=0}^{N} \tbinom{N}{l} \left[{n \atop l}\right] L_{n-l}^{(0)}(x), \tag{2.6}$$

where

$$\left[{N \atop l}\right] = \begin{cases} N(N-1)\cdots(N-l+1) = (N-l+1)_l & \text{if } l \leq N, \\ 0 & \text{if } l > N. \end{cases}$$

In particular, we can see from (2.6) that $L_n^{(-N)}(x)$ for $n \geq N$ has $x = 0$ as a zero of order N and has $n - N$ positive zeros. Moreover, positive zeros of $L_n^{(-N)}(x)$ and $L_{n+1}^{(-N)}(x)$ for $n \geq N$ interlace each other.

We now set

$$\phi(S_n^{(\gamma)}(x), S_n^{(\gamma)}(x)) = s_n(\gamma),\ n \geq 0, \tag{2.7}$$

$$\psi\,(Q_n(x), Q_n(x)) = q_n,\ n \geq 0. \tag{2.8}$$

and for i, $j \geq 0$,

$$\begin{aligned}\phi_{ij} := \phi(x^i, x^j) &= \textstyle\int_0^\infty x^{i+j} e^{-x} dx + \gamma \int_0^\infty (x^i)^{(N)} (x^j)^{(N)} e^{-x} dx \\ &= \sigma_{i+j} + \gamma \left[{i \atop N}\right]\left[{j \atop N}\right] \sigma_{i+j-2N}\end{aligned}$$

where $\sigma_i = \int_0^\infty x^i e^{-x} dx = i!$. Let

$$\Delta_n(\phi) := \begin{vmatrix} \phi_{00} & \phi_{01} & \cdots & \phi_{0n} \\ \phi_{10} & \phi_{11} & \cdots & \phi_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n0} & \phi_{n1} & \cdots & \phi_{nn} \end{vmatrix},\ n \geq 0$$

be the Hankel determinant of $\phi(\cdot, \cdot)$.

Then

$$S_0^{(\gamma)}(x) = 1 \text{ and } S_n^{(\gamma)}(x) = \frac{1}{\triangle_{n-1}(\phi)} \begin{vmatrix} \phi_{00} & \phi_{01} & \cdots & \phi_{0n} \\ \phi_{10} & \phi_{11} & \cdots & \phi_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n-1,0} & \phi_{n-1,1} & \cdots & \phi_{n-1,n} \\ 1 & x & \cdots & x^n \end{vmatrix},\ n \geq 1.$$

Since $\triangle_n(\phi)$ is a polynomial in γ of degree $\begin{cases} 0 & \text{if } 0 \leq n < N \\ n-N+1 & \text{if } n \geq N \end{cases}$, $\lim_{\gamma \to \infty} S_n^{(\gamma)}(x)$ $:= S_n^{(\infty)}(x)$, $n \geq 0$, exists.

Proposition 2.1. *We have*

$$L_n^{(-N)}(x) = S_n^{(\gamma)}(x) + \sum_{i=n-N}^{n-1} d_i^{(n)}(\gamma) S_i^{(\gamma)}(x), \ n \ge N \tag{2.9}$$

where $d_{n-N}^{(n)}(\gamma) \neq 0$ *and*

$$d_i^{(n)}(\gamma) = s_i^{-1}(\gamma) \sum_{l=n-i}^{N} \binom{N}{l} \left[{n \atop l}\right] \int_0^\infty L_{n-l}^{(0)}(x) S_i^{(\gamma)}(x) e^{-x} dx, \ n-N \le i \le n-1. \tag{2.10}$$

Proof. Expand $L_n^{(-N)}(x)$ as

$$L_n^{(-N)}(x) = S_n^{(\gamma)}(x) + \sum_{i=0}^{n-1} d_i^{(n)}(\gamma) S_i^{(\gamma)}(x)$$

where $d_i^{(n)}(\gamma) = s_i^{-1}(\gamma)\phi(L_n^{(-N)}, S_i^{(\gamma)})$. From (2.3) and (2.6), if $0 \le i \le n-N-1$, then

$$\phi(L_n^{(-N)}, S_i^{(\gamma)}) = \sum_{l=0}^{N} \binom{N}{l} \left[{n \atop l}\right] \int_0^\infty L_{n-l}^{(0)}(x) S_i^{(\gamma)} e^{-x} dx + \gamma \left[{n \atop N}\right] \int_0^\infty L_{n-N}^{(0)}(x) (S_i^{(\gamma)}(x))^{(N)} e^{-x} dx = 0.$$

Hence (2.9) and (2.10) hold. In particular,

$$\begin{aligned} d_{n-N}^{(n)}(\gamma) &= s_{n-N}^{-1}(\gamma) \left[{n \atop N}\right] \int_0^\infty L_{n-N}^{(0)}(x) S_{n-N}^{(\gamma)}(x) e^{-x} dx \\ &= s_{n-N}^{-1}(\gamma) \left[{n \atop N}\right] \int_0^\infty (L_{n-N}^{(0)}(x))^2 e^{-x} dx \neq 0. \quad \square \end{aligned}$$

Proposition 2.2. *MSOPS* $\{Q_n(x)\}_0^\infty$ *relative to* $\psi(\cdot,\cdot)$ *is*

$$Q_n(x) = \begin{cases} x^n, & 0 \le n \le N, \\ L_n^{(-N)}(x), & n \ge N. \end{cases} \tag{2.11}$$

Furthermore,

$$Q_n^{(N)}(x) = \left[{n \atop N}\right] L_{n-N}^{(0)}(x), \ n \ge N \tag{2.12}$$

so that $Q_n^{(N)}(x)$, for $n \ge N$, has $n-N$ positive zeros.

Proof. For $0 \le m \le N-1$ and $0 \le m \le n-1$,

$$\psi(Q_m, Q_n) = \sum_{k=0}^{N-1} Q_m^{(k)}(0) Q_n^{(k)}(0) = 0.$$

By induction on m, we obtain that $Q_n^{(m)}(0) = 0$ for $0 \le m \le N-1$ and $0 \le m \le n-1$ so that

$$Q_n(x) = \begin{cases} x^n & 0 \le n \le N, \\ x^N \pi_{n-N}(x) & n \ge N. \end{cases}$$

Let m and $n \ge N$. Then $\psi(Q_m, Q_n) = \int_0^\infty Q_m^{(N)} Q_n^{(N)} e^{-x} dx = q_n \delta_{mn}$ so that $\{Q_n^{(N)}(x)\}_{n=N}^\infty$ is an orthogonal polynomial system relative to e^{-x} on $[0,\infty)$ so that (2.12) holds.

On the other hand, from (2.2) and (2.12),

$$x Q_n^{(N+2)}(x) + (1-x) Q_n^{(N+1)}(x) + n Q_n^{(N)}(x) = 0, \ n \ge N.$$

Hence

$$xQ_{n+N}^{(N+2)}(x) + (1-x)Q_{n+N}^{(N+1)}(x) + nQ_{n+N}^{(N)}(x) = 0,\ n \geq 0. \tag{2.13}$$

By induction on $k = N, N-1, \cdots, 0$, we can see from (2.13) that

$$xQ_{n+N}^{(k+2)}(x) + [k-N+1-x]Q_{n+N}^{(k+1)}(x) + [n+N-k]Q_{n+N}^{(k)}(x) = 0,\ n \geq 0,\ 0 \leq k \leq N.$$

In particular, for $k = 0$,

$$xQ''_{n+N}(x) + (-N+1-x)\,Q'_{n+N}(x) + (n+N)Q_{n+N}(x) = 0,\ n \geq 0$$

so that $Q_n(x) = L_n^{(-N)}(x)$, $n \geq N$. $\square$

By Propositions 2.1 and 2.2, we obtain the following relation :

$$Q_n(x) = S_n^{(\gamma)}(x) + \sum_{i=n-N}^{n-1} d_i^{(n)}(\gamma) S_i^{(\gamma)}(x),\ n \geq N.$$

Proposition 2.3. *We have*

$$\int_0^\infty S_n^{(\infty)}(x)x^m e^{-x}dx = 0,\ n \geq m+1 \text{ and } 0 \leq m \leq N-1, \tag{2.14}$$

$$\int_0^\infty [S_n^{(\infty)}(x)]^{(N)} x^m e^{-x}dx = 0,\ n \geq m+N+1 \tag{2.15}$$

so that

$$S_n^{(\infty)}(x) = L_n^{(0)}(x),\ 0 \leq n \leq N,$$

$$S_n^{(\infty)}(x) = L_n^{(-N)}(x) + \pi_{N-1}(x) = \sum_{l=0}^{N} \binom{N}{l} {\textstyle\left[{N \atop l}\right]} L_{n-l}^{(0)}(x) + \pi_{N-1}(x),\ n \geq N \tag{2.16}$$

where $\pi_{N-1}(x) \in \mathbf{P_{n-1}}$, the space of polynomials of degree $\leq n-1$.

Proof. If $0 \leq m \leq N-1$ and $n \geq m+1$, then

$$\begin{aligned}\int_0^\infty S_n^{(\infty)}(x)x^m e^{-x}dx &= \lim_{\gamma\to\infty} \int_0^\infty S_n^{(\gamma)}(x)x^m e^{-x}dx \\ &= \lim_{\gamma\to\infty} \left[\phi(S_n^{(\gamma)}, x^m) - \gamma \int_0^\infty (S_n^{(\gamma)}(x))^{(N)}(x^m)^N e^{-x}dx\right] = 0.\end{aligned}$$

Hence, (2.14) holds. In particular, from (2.14), we have $S_n^{(\infty)}(x) = L_n^{(0)}(x)$, $0 \leq n \leq N$. If $n \geq m+N+1$, then

$$\begin{aligned}\int_0^\infty (S_n^{(\infty)}(x))^{(N)} x^m e^{-x}dx &= \tfrac{1}{(m+1)_N} \int_0^\infty (S_n^{(\infty)}(x))^{(N)}(x^{m+N})^{(N)} e^{-x}dx \\ &= \lim_{\gamma\to\infty} \tfrac{1}{(m+1)_N}\tfrac{1}{\gamma}\left[\phi(S_n^{(\gamma)}, x^{m+N}) - \int_0^\infty S_n^{(\gamma)}(x)x^{m+N}e^{-x}dx\right] \\ &= -\lim_{\gamma\to\infty} \tfrac{1}{(m+1)_N}\tfrac{1}{\gamma} \int_0^\infty S_n^{(\gamma)}(x)x^{m+N}e^{-x}dx = 0\end{aligned}$$

so that

$$\left(S_n^{(\infty)}(x)\right)^{(N)} = {\textstyle\left[{n \atop N}\right]} L_{n-N}^{(0)}(x),\ n \geq N. \tag{2.17}$$

Integrating (2.17) N-times and using (2.3), we obtain (2.16). $\square$

Theorem 2.4. *The following relation holds*

$$S_n^{(\infty)}(x) = S_n^{(\gamma)}(x) + \sum_{i=N}^{n-1} \tilde{d}_i^{(n)}(\gamma) S_i^{(\gamma)}(x),\ \ n \geq N \tag{2.18}$$

where

$$\tilde{d}_i^{(n)}(\gamma) = s_i^{-1}(\gamma) \int_0^\infty S_n^{(\infty)}(x) S_n^{(\gamma)}(x) e^{-x} dx,\ \ N \leq i \leq n-1. \tag{2.19}$$

Proof. Expand $S_n^{(\infty)}(x)$ as

$$S_n^{(\infty)}(x) = S_n^{(\gamma)}(x) + \sum_{i=0}^{n-1} \tilde{d}_i^{(n)}(\gamma) S_i^{(\gamma)}(x)$$

where

$$\tilde{d}_i^{(n)}(\gamma) = s_i^{-1}(\gamma) \phi(S_n^{(\infty)}, S_i^{(\gamma)}),\ \ 0 \leq i \leq n-1.$$

¿From (2.14) and (2.17), if $0 \leq i \leq N-1 < n$, then

$$\phi(S_n^{(\infty)}, S_i^{(\gamma)}) = \int_0^\infty S_n^{(\infty)}(x) S_i^{(\gamma)}(x) e^{-x} dx + \gamma \left[{n \atop N}\right] \int_0^\infty L_{n-N}^{(0)}(x) (S_i^{(\gamma)}(x))^{(N)} e^{-x} dx = 0.$$

Hence, $\tilde{d}_i^{(n)}(\gamma) = 0$ for $0 \leq i \leq N-1$ and $n \geq N$ so that (2.18) and (2.19) holds. □

Lemma 2.5. *We have for any polynomials $f(x)$ and $g(x)$*

$$\begin{aligned}\int_0^\infty f^{(N)}(x) g(x) e^{-x} dx = &\sum_{t=0}^{N-1} \sum_{l=0}^{t} \binom{t}{l} (-1)^l \left[f^{(N-t-1)}(x) g^{(l)}(x) e^{-x} \right]_0^\infty \\ &+ \sum_{l=0}^{N} (-1)^{l+1} \binom{N}{l} \int_0^\infty f(x) g^{(N-l)}(x) e^{-x} dx.\end{aligned} \tag{2.20}$$

Proof. We shall prove (2.20) by induction on $N \geq 1$. If $N = 1$, then

$$\int_0^\infty f'(x) g(x) e^{-x} dx = [f(x) g(x) e^{-x}]_0^\infty - \int_0^\infty f(x) (g(x) e^{-x})' dx$$

so that (2.20) holds for $N = 1$. Assume that (2.20) is true up to N. Then

$$\begin{aligned}&\int_0^\infty f^{(N+1)}(x) g(x) e^{-x} dx = \left[f^{(N)}(x) g(x) e^{-x} \right]_0^\infty - \int_0^\infty f^{(N)}(x) (g'(x) - g(x)) e^{-x} dx \\ &= \left[f^{(N)}(x) g(x) e^{-x} \right]_0^\infty + \sum_{t=0}^{N-1} \sum_{l=0}^{t+1} (-1)^l \left\{ \binom{t}{l} + \binom{t}{l-1} \right\} \left[f^{(N-t-1)}(x) g^{(l)}(x) e^{-x} \right]_0^\infty \\ &\quad + \sum_{l=-1}^{N} (-1)^{l+1} \left\{ \binom{N}{l} + \binom{N}{l+1} \right\} \int_0^\infty f(x) g^{(N-l)}(x) e^{-x} dx \\ &= \sum_{t=0}^{N-1} \sum_{l=0}^{t} \binom{t}{l} (-1)^l \left[f^{(N-t-1)}(x) g^{(l)}(x) e^{-x} \right]_0^\infty + \sum_{l=0}^{N} (-1)^{l+1} \binom{N}{l} \int_0^\infty f(x) g^{(N-l)}(x) e^{-x} dx\end{aligned}$$

so that (2.20) is also true for $N + 1$. □

Theorem 2.6. *(Rodrigues type formula) For $0 \leq n \leq N-1$,*

$$S_n^{(\gamma)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \partial_x^k x^n,\ \ (\partial_x = \tfrac{d}{dx}) \tag{2.21}$$

and for $n \geq N$,

$$S_n^{(\gamma)}(x) = (-1)^{n+1}\left[\sum_{k=1}^{m}\gamma^k \sum_{l_1=0}^{N}\cdots\sum_{l_k=0}^{N}(-1)^{l_1+\cdots+l_k}\binom{N}{l_1}\cdots\binom{N}{l_k} \{e^x\lambda^{(n)}(x)\}^{(2kN-l_1-\cdots-l_k)} + e^x\lambda^{(n)}(x)\right] \tag{2.22}$$

where $m = [\frac{n}{N}] + 1$ *and* $-e^x\lambda(x)$ *is a monic polynomial of degree* n.

Proof. By the Sobolev orthogonality, we have

$$S_n^{(\gamma)}(x) = min\left\{\phi(y,y) = \int_0^\infty \left\{y(x)^2 + \gamma(y^{(N)}(x))^2\right\} e^{-x}dx \mid y^{(n)}(x) = n!\right\}.$$

Hence, $S_n^{(\gamma)}(x)$ must be a stationary point of the functional

$$\mathbf{I}[y] = \int_0^\infty \left\{y(x)^2 + \gamma(y^{(N)}(x))^2\right\} e^{-x}dx + 2\int_0^\infty \lambda(x)(y^{(n)}(x) - n!)dx$$

where $\lambda(x)$ is the Lagrangian multiplier so that $\frac{1}{2}f'(0) = 0$ where $f(\epsilon) = \mathbf{I}[y(x)+\epsilon\eta(x)]$ and $\eta(x)$ is an arbitrary function in $C^N[0,\infty)$.

Hence, by Lemma 2.5,

$$\begin{aligned}\tfrac{1}{2}f'(0) &= \int_0^\infty (y(x)\eta(x) + \gamma y^{(N)}(x)\eta^{(N)}(x))e^{-x}dx + \int_0^\infty \lambda(x)\eta^{(n)}(x)dx \\ &= \int_0^\infty \eta(x)[y(x)e^{-x} + \gamma\sum_{l=0}^{N}(-1)^{l+1}\binom{N}{l}y^{(2N-l)}(x)e^{-x} + (-1)^n\lambda^{(n)}(x)]dx \\ &\quad -\gamma\sum_{i=0}^{N-1}\left[\sum_{l=0}^{N-1-i}\binom{N-1-i}{l}(-1)^l y^{(N+l)}(0)\right]\eta^{(i)}(0) \\ &\quad -\sum_{i=0}^{n-1}(-1)^{n-1-i}\lambda^{(n-1-i)}(0)\eta^{(i)}(0) + \sum_{i=0}^{n-1}(-1)^{n-1-i}\lambda^{(n-1-i)}(\infty)\eta^{(i)}(\infty) = 0.\end{aligned}$$

Hence, for $0 \leq n \leq N-1$:

$$y(x)e^{-x} + (-1)^n\lambda^{(n)}(x) = 0, \tag{2.23}$$

$$\lambda^{(k)}(\infty) = 0,\ 0 \leq k \leq n-1, \tag{2.24}$$

$$\lambda^{(k)}(0) = 0,\ 0 \leq k \leq n-1. \tag{2.25}$$

¿From (2.23), (2.24), and (2.25), and since $y(x)$ is a monic polynomial of degree n,

$$\lambda(x) = -e^{-x}x^n$$

so that $y(x)e^{-x} = (-1)^n\sum_{k=0}^{n}\binom{n}{k}\partial_x^{n-k}e^{-x}\partial_x^k x^n$. Hence, (2.21) holds.

For $n \geq N$:

$$e^{-x}\left[\gamma\sum_{l=0}^{N}(-1)^l\binom{N}{l}y^{(2N-l)}(x) - y(x)\right] = (-1)^n\lambda^{(n)}(x), \tag{2.26}$$

$$\lambda^{(k)}(\infty) = 0,\ 0 \leq k \leq n-1, \tag{2.27}$$

$$\lambda^{(k)}(0) = 0,\ 0 \leq k \leq n-1-N, \tag{2.28}$$

$$(-1)^{n-1-i}\lambda^{(n-1-i)}(0) + \gamma\sum_{l=0}^{N-1-i}(-1)^l\binom{N-1-i}{l}y^{(N+l)}(0) = 0,\ 0 \leq i \leq N-1. \tag{2.29}$$

¿From (2.26), (2.27), and (2.28),

$$\lambda(x) = -e^{-x}x^{n-N}\left(x^N + \sum_{k=0}^{N-1}\lambda_k x^k\right).$$

Then from (2.26),

$$\begin{aligned} y(x) = S_n^{(\gamma)}(x) &= \gamma\sum_{l=0}^{N}(-1)^l\tbinom{N}{l}y^{(2N-l)}(x) - h_n(x) \\ &= \gamma\sum_{l=0}^{N}(-1)^l\tbinom{N}{l}\left[\gamma\sum_{l_1=0}^{N}(-1)^{l_1}\tbinom{N}{l_1}y^{(2N-l_1)}(x) - h_n(x)\right]^{(2N-l)} - h_n(x) \\ &= -\sum_{k=1}^{m}\gamma^k\sum_{l_1=0}^{N}\cdots\sum_{l_k=0}^{N}(-1)^{l_1+\cdots+l_k}\tbinom{N}{l_1}\cdots\tbinom{N}{l_k}h_n{}^{(2kN-l_1-\cdots-l_k)}(x) - h_n(x) \end{aligned}$$

where $m = [\frac{n}{N}]+1$ and $h_n(x) = (-1)^n e^x\lambda^{(n)}(x)$. Hence (2.22) holds. □

Remark. Rodrigues type formula for Sobolev-Legendre and Sobolev-Laguerre orthogonal polynomials for $N = 1$ were obtained by W. Gröbner [7] and J. Brenner [2].

3. Best Polynomial Approximations

We now set

$$\begin{aligned} \mathbf{E} := \{f : \mathbb{R}_+ \to \mathbb{R} \mid f(x) \in C^{N-1}(\mathbb{R}_+),\ f^{(N-1)}(x) \in AC(\mathbb{R}_+), \\ f(x) \text{ and } f^{(N)}(x) \in L^2(\mathbb{R}_+ : e^{-x}dx)\} \end{aligned}$$

where $\mathbb{R}_+ := [0,\infty)$ and for any $f \in \mathbf{E}$, let

$$B_n^{(\gamma)}(x) = \sum_{k=0}^{n} s_k^{-1}(\gamma)\phi(f, S_k^{(\gamma)})S_k^{(\gamma)}(x)$$

and

$$B_n(x) = \sum_{k=0}^{n} q_k^{-1}\psi\left(Q_k, f\right)Q_k(x)$$

be the best polynomial approximations to $f(x)$ in $\mathbf{P_n}$ with respect to $\phi(\cdot,\cdot)$ and $\psi(\cdot,\cdot)$ respectively. Set

$$R_n^{(\gamma)}(x) = f(x) - B_n^{(\gamma)}(x),\ R_n(x) = f(x) - B_n(x)$$

to be the deviations. Then

$$\phi(R_n^{(\gamma)}, x^k) = 0,\ 0 \le k \le n, \tag{3.1}$$

$$\psi(R_n, x^k) = 0,\ 0 \le k \le n. \tag{3.2}$$

Theorem 3.1. $B_n^{(\infty)}(x) := \lim_{\gamma\to\infty} B_n^{(\gamma)}(x)$, $n \ge 0$ *exists and*

$$B_n^{(\infty)}(x) = B_n(x),\ n \ge 2N-1. \tag{3.3}$$

Proof. Set $B_n^{(\gamma)}(x)=\sum\limits_{k=0}^{n}b_k(\gamma)x^k$. Then from (3.1),

$$\phi(B_n^{(\gamma)},x^j)=\sum_{k=0}^{n}b_k(\gamma)\phi(x^k,x^j)=\phi(f,x^j),\ 0\le j\le n.$$

Hence,

$$\left[\phi(x^k,x^j)\right]_{j,k=0}^{n}\left[b_k(\gamma)\right]_{k=0}^{n}=\left[\phi(f,x^j)\right]_{j=0}^{n}\ \text{so that}\ b_k(\gamma)=\frac{\triangle_n^{(k)}(\phi)}{\triangle_n(\phi)},\ 0\le k\le n$$

where $\triangle_n^{(k)}(\phi)$ is the determinant $\triangle_n(\phi)$ where the k-th column of $\triangle_n(\phi)$ is replaced by $\left[\phi(f,x^j)\right]_{j=0}^{n}$. As polynomials in γ, $\triangle_n^{(k)}(\phi)$ is of degree $\le\max(0,n-N+1)$ and $\triangle_n(\phi)$ is of degree $\max(0,n-N+1)$.

Hence, $\lim\limits_{\gamma\to\infty}B_n^{(\gamma)}(x)=B_n^{(\infty)}(x)$ exists.

Then for any $\pi\in\mathbf{P_n}$

$$\frac{1}{\gamma}\int_0^\infty(f-B_n^{(\gamma)})^2e^{-x}dx+\int_0^\infty[\partial_x^N(f-B_n^{(\gamma)})]^2e^{-x}dx$$

$$\le\frac{1}{\gamma}\int_0^\infty(f-\pi)^2e^{-x}dx+\int_0^\infty\left[\partial_x^N(f-\pi)\right]^2e^{-x}dx.$$

Let γ tends to ∞. Then

$$\int_0^\infty[\partial_x^N(f-B_n^{(\infty)})]^2e^{-x}dx\le\int_0^\infty\left[f^{(N)}-\pi^{(N)}\right]^2e^{-x}dx,\ \pi\in\mathbf{P_n}. \tag{3.4}$$

That is, $\partial_x^NB_n^{(\infty)}(x)$ is the best polynomial approximation of degree $\le n-N$ to $f^{(N)}(x)$ in $L^2(R_+:e^{-x}dx)$. Hence,

$$\int_0^\infty x^ke^{-x}\partial_x^NR_n^{(\infty)}(x)dx=0,\ 0\le k\le n-N, \tag{3.5}$$

where $R_n^{(\infty)}(x)=f(x)-B_n^{(\infty)}(x)$. On the other hand, from (3.1),

$$\phi(R_n^{(\gamma)},x^k)=\int_0^\infty R_n^{(\gamma)}(x)x^ke^{-x}dx=0,\ 0\le k\le N-1\ \text{and}\ n\ge k.$$

Let γ tend to ∞. Then

$$\int_0^\infty R_n^{(\infty)}(x)x^ke^{-x}dx=0,\ 0\le k\le N-1\ \text{and}\ n\ge k. \tag{3.6}$$

We set $a_{jk}^{(n)}:=\int_0^\infty x^ke^{-x}\partial_x^jR_n^{(\infty)}(x)dx$, $0\le j\le N$ and $k\ge0$. Then from (3.5) and (3.6), we have

$$a_{0k}^{(n)}=0\ \text{and}\ a_{Nk}^{(n)}=0\ \text{for}\ 0\le k\le N-1\ \text{and}\ n\ge 2N-1. \tag{3.7}$$

By induction on j, we obtain that, for $1\le j\le N$ and $0\le k\le N-1$,

$$a_{jk}^{(n)}=-\sum_{l=0}^{j-1}\left[\sum_{m=0}^{j-l-1}(-1)^{m+1}\binom{j-l-1}{m}\left[{k\atop m}\right]\delta_{0,k-m}\right]\left(\partial_x^lR_n^{(\infty)}\right)(0),\ n\ge k. \tag{3.8}$$

Hence, from (3.7) and (3.8), we have for $0\le k\le N-1$ and $n\ge 2N-1$

$$\begin{aligned}a_{Nk}^{(n)}&=-\sum_{l=0}^{N-1}\left[\sum_{m=0}^{N-l-1}(-1)^m\binom{N-l-1}{m}\left[{k\atop m}\right]\delta_{0,k-m}\right]\left(\partial_x^lR_n^{(\infty)}\right)(0)\\&=-\sum_{l=0}^{N-1-k}(-1)^k\binom{N-l-1}{k}k!(\partial_x^lR_n^{(\infty)})(0)=0.\end{aligned}$$

Hence,

$$\partial_x^k R_n^{(\infty)}(0) = 0 \text{ for } 0 \le k \le N-1 \text{ and } n \ge 2N-1. \tag{3.9}$$

¿From (3.4) and (3.9), if $n \ge 2N-1$, then

$$\psi\left(f - B_n^{(\infty)}, f - B_n^{(\infty)}\right) = \int_0^\infty \left[\partial_x^N (f - B_n^{(\infty)})\right]^2 e^{-x} dx$$

$$\le \int_0^\infty \left[\partial_x^N (f - \pi)\right]^2 e^{-x} dx = \psi(f-\pi, f-\pi)$$

for any $\pi \in \mathbf{P_n}$. Hence $B_n^{(\infty)}(x) = B_n(x)$ if $n \ge 2N-1$. □

Theorem 3.2. *Let $n \ge N$. Then*

(1) *For $0 \le k \le N-1$, $R_n^{(k)}(x)$ has at least $n-N+1$ nodal zeros(i.e. zeros of odd multiplicities) in $(0,\infty)$ so that $R_n^{(k)}(x)$ has at least $n-N+2$ zeros in $[0,\infty)$ including 0 ;*

(2) *If $f \in C^N[0,\infty)$, then $R_n^{(N)}(x)$ has at least $n-N+1$ nodal zeros in $(0,\infty)$.*

Proof. From (3.2) and (3.9),

$$\begin{aligned}\psi(R_n, x^k) &= \sum_{j=0}^{N-1} R_n^{(j)}(0)(x^k)^{(j)}(0) + \int_0^\infty R_n^{(N)}(x)(x^k)^{(N)} e^{-x} dx \\ &= \int_0^\infty R_n^{(N)}(x)[{}^k_N]x^{k-N} e^{-x} dx = 0, \; 0 \le k \le n.\end{aligned}$$

Hence,

$$\int_0^\infty R_n^{(N)}(x) x^m e^{-x} dx = 0, \; 0 \le m \le n-N. \tag{3.10}$$

Now (2) follows from (3.10). By induction on $k = N, N-1, \cdots, 2, 1, 0$, we obtain

$$\int_0^\infty R_n^{(k)}(x) x^m e^{-x} dx = 0, \; 0 \le m \le n-N \text{ and } 0 \le k \le N. \tag{3.11}$$

Hence (1) holds since $R_n^{(k)}(0) = 0$, $0 \le k \le N-1$. □

Theorem 3.3. *We have*

$$B_n^{(\gamma)}(x) - B_n(x) = \sum_{k=n-N+1}^{n} \beta_k^{(n)} S_k^{(\gamma)}(x), \; n \ge N \tag{3.12}$$

where

$$\beta_{n-N+1}^{(n)} = s_{n-N+1}^{-1}(\gamma) \int_0^\infty (f - B_n) x^{n-N+1} e^{-x} dx. \tag{3.13}$$

Proof. Let

$$B_n^{(\gamma)}(x) - B_n(x) = \sum_{k=0}^{n} \beta_k^{(n)} S_k^{(\gamma)}(x).$$

Then

$$\beta_k^{(n)} = s_k^{-1}(\gamma)\phi(B_n^{(\gamma)} - B_n, S_k^{(\gamma)}), \; 0 \le k \le n.$$

¿From (3.1) and (3.11), for $0 \leq k \leq n-N$,

$$\phi(B_n^{(\gamma)} - B_n, x^k) = \phi(f - B_n, x^k)$$

$$= \int_0^\infty (f - B_n) x^k e^{-x} dx + \gamma \int_0^\infty (f^{(N)} - B_n^{(N)})(x^k)^{(N)} e^{-x} dx = 0.$$

Hence, (3.12) holds.

Since

$$\phi(B_n^{(\gamma)} - B_n, S_{n-N+1}^{(\gamma)}) = \phi(B_n^{(\gamma)} - B_n, x^{n-N+1}) = \int_0^\infty (f - B_n) x^{n-N+1} e^{-x} dx,$$

(3.13) holds. □

Theorem 3.3 means that in the expansion by $\{S_n^{(\gamma)}(x)\}_{n=0}^\infty$, $B_n^{(\gamma)}(x)$ and $B_n(x)$ differ only in the last N coefficients.

4. Conjectures

Instead of $\phi(\cdot,\cdot)$, we now consider a Sobolev-Laguerre inner product

$$\phi_1(f,g) = \sum_{k=0}^{N-1} \int_0^\infty f^{(k)}(x) g^{(k)}(x) e^{-x} dx + \gamma \int_0^\infty f^{(N)}(x) g^{(N)}(x) e^{-x} dx$$

where $\gamma > 0$, $N \geq 1$ is an integer and let $\tilde{B}_n^{(\gamma)}(x)$ be the best polynomial approximation to $f \in W^{N,2}([0,\infty); e^{-x})$ with respect to $\phi_1(\cdot,\cdot)$.

Conjecture 1. $\lim_{\gamma \to \infty} \tilde{B}_n^{(\gamma)}(x) = B_n(x)$, $n \geq 2N-1$.

For $N = 1$, Conjecture 1 is true by Theorem 3.1 so that we assume $N \geq 2$. As in the proof of Theorem 3.1, proving the Conjecture 1 is equivalent to showing

(i) $\partial_x^N \tilde{B}_n^{(\infty)}(x)$ is the best polynomial approximation to $f^{(N)}(x)$ in $\mathbf{P_{n-N}}$ with respect to $L^2(\mathbb{R}_+; e^{-x}dx)$

and

(ii) $[\partial_x^l \tilde{R}_n^{(\infty)}](0) = 0,\ 0 \leq k \leq N-1$ and $n \geq 2N-1$,

where $\tilde{B}_n^{(\infty)}(x) := \lim_{\gamma\to\infty} \tilde{B}_n^{(\gamma)}(x)$ and $R_n^{(\infty)}(x) := f(x) - \tilde{B}_n^{(\infty)}(x)$. It is easy to prove (i). In order to prove (ii), let us proceed as in the proof of Theorem 3.1. First, (i) implies that

$$\int_0^\infty x^k e^{-x} \partial_x^N R_n^{(\infty)}(x) dx = 0,\ 0 \leq k \leq n-N. \tag{4.1}$$

On the other hand, we can obtain from $\phi_1(f(x) - B_n^{(\gamma)}(x), x^k) = 0$, $0 \leq k \leq n$

$$\sum_{j=0}^{k} \int_0^\infty [{k \atop j}] x^{k-j} e^{-x} \partial_x^j \tilde{R}_n^{(\infty)}(x) dx = 0,\ 0 \leq k \leq N-1 \text{ and } n \geq k. \tag{4.2}$$

We now set

$$a_{jk}^{(n)} := \int_0^\infty x^k e^{-x} \partial_x^j [\tilde{R}_n^{(\infty)}(x)] dx,\ 0 \leq j \leq N \text{ and } k \geq 0.$$

Then by (4.1) and (4.2), we have

$$a_{0k}^{(n)} = \begin{cases} 0 & \text{if} \quad k=0 \text{ and } n \geq 0 \\ k![\partial_x^{k-1}\tilde{R}_n^{(\infty)}](0) & \text{if} \quad 1 \leq k \leq N-1 \text{ and } n \geq k \end{cases} \tag{4.3}$$

and

$$a_{jk}^{(n)} = \sum_{l=0}^{j}(-1)^l\tbinom{j}{l}\left[{k \atop l}\right]a_{0,k-l} - \sum_{l=0}^{j-1}\left[\sum_{m=0}^{j-l-1}(-1)^m\tbinom{j-l-1}{m}\left[{k \atop m}\right]\delta_{0,k-m}\right]\left[\partial_x^l\tilde{R}_n^{(\infty)}\right](0),$$
$$1 \leq j \leq N \text{ and } k \geq 0. \tag{4.4}$$

Hence, we have by (4.1), (4.3), and (4.4),

$$a_{Nk}^{(n)} = k!\left\{\sum_{l=0}^{k-1}(-1)^{k+l+1}\tbinom{N}{k-1-l}\left[\partial_x^l\tilde{R}_n^{(\infty)}\right](0) - \sum_{l=0}^{N-1-k}(-1)^k\tbinom{N-l-1}{k}\left[\partial_x^l\tilde{R}_n^{(\infty)}\right](0)\right\} = 0,$$
$$0 \leq k \leq N-1. \tag{4.5}$$

For $k=0$, (4.5) becomes

$$\sum_{l=0}^{N-1}\left[\partial_x^l\tilde{R}_n^{(\infty)}\right](0) = 0$$

so that (ii) holds if we can show that the following homogeneous system of equations has trivial solution :

$$\sum_{l=0}^{k-1}(-1)^{k+l}\tbinom{N}{k-1-l}b_l + \sum_{l=0}^{N-1-k}(-1)^k\tbinom{N-l-1}{k}b_l = 0, \; 1 \leq k \leq N-1. \tag{4.6}$$

In other words, we only need to show that $|A_{N-1}| \neq 0$, where A_{N-1} is the coefficients matrix of the system (4.6). Either by a direct computation for N small or by a numeric computation for $2 \leq N \leq 100$, we can see that $|A_{N-1}| \neq 0$.

Furthermore, we conjecture :

Conjecture 2. $$|A_{N-1}| = \prod_{i=0}^{N-1}\frac{(3i+1)!}{(N+i)!},$$

which is the number of alternating sign matrix of order N (see [12], [14]). We can check Conjecture 2 numerically up to $N=20$.

Finally, let us consider Sobolev-Legendre inner products

$$\phi_2(f,g) = \sum_{k=0}^{N-1}\int_{-1}^{1}f^{(k)}(x)g^{(k)}(x)dx + \gamma\int_{-1}^{1}f^{(N)}(x)g^{(N)}(x)dx$$

and

$$\psi_2(f,g) = \sum_{k=0}^{N-1}f^{(k)}(-1)g^{(k)}(-1) + \int_{-1}^{1}f^{(N)}(x)g^{(N)}(x)dx.$$

For any f in $W^{N,2}[-1,1] := \{f : [-1,1] \to \mathbb{R} \mid f(x) \in C^{N-1}[-1,1],\ f^{(N-1)}(x) \in AC[-1,1], \text{ and } f^{(k)}(x) \in L^2[-1,1] \text{ for } 0 \leq k \leq N\}$, let $\hat{B}_n^{(\gamma)}(x)$ and $\hat{B}_n(x)$ be the best polynomial approximations to f with respect to $\phi_2(\cdot,\cdot)$ and $\psi_2(\cdot,\cdot)$ respectively. Then, we conjecture :

Conjecture 3. $$\lim_{\gamma\to\infty}\hat{B}_n^{(\gamma)}(x) = \hat{B}_n(x),\ n \geq 3N-1.$$

For $N = 1$, Conjecture 3 was proved by E. A. Cohen [4], which motivates this work. As in the case of Sobolev-Laguerre inner products, proving Conjecture 3 can be reduced to showing

$$\partial_x^k(f(x) - \hat{B}_n^{(\infty)}(x))|_{x=-1} = 0,\ 0 \le k \le N-1 \text{ and } n \ge 3N-1, \tag{4.7}$$

where $\hat{B}_n^{(\infty)}(x) := \lim_{\gamma\to\infty} \hat{B}_n^{(\gamma)}(x)$. In fact, we can show (4.7) for small N by direct or numerical computation.

Acknowledgements. The work of D. H. Kim and S. H. Kim was partially supported by BK-21 project and the work of K. H. Kwon was partially supported by KOSEF (98-0701-03-01-5). Authors thanks Prof. D. S. Kim for suggesting Conjecture 2.

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