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The degree complexity of smooth surfaces of codimension 2

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ABSTRACT

For a given term order, the degree complexity of a projective scheme is defined by the maximal degree of the reduced Gröbner basis of its defining saturated ideal in generic coordinates (Bayer and Mumford, 1993). It is well known that the degree complexity with respect to the graded reverse lexicographic order is equal to the Castelnuovo–Mumford regularity (Bayer and Stillman, 1987). However, much less is known if one uses the graded lexicographic order (Ahn, 2008; Conca and Sidman, 2005).

In this paper, we study the degree complexity of a smooth irreducible surface in $\mathbb{P}^4$ with respect to the graded lexicographic order and its geometric meaning. As in the case of a smooth curve (Ahn, 2008), we expect that this complexity is closely related to the invariants of the double curve of a surface under a generic projection. As results, we prove that except in a few cases, the degree complexity of a smooth surface S of degree d with $h^0(\mathcal{I}_S(2)) \neq 0$ in $\mathbb{P}^4$ is given by $2 + \binom{\deg Y_1(S)-1}{2} - g(Y_1(S))$, where $Y_1(S)$ is a double curve of degree $\binom{d-1}{2} - g(S \cap H)$ under a generic projection of S. In particular, this complexity is actually obtained at the monomial

$$x_0 x_1 x_3^{\binom{\deg Y_1(S)-1}{2} - g(Y_1(S))}$$

where $k[x_0, x_1, x_2, x_3, x_4]$ is a polynomial ring defining $\mathbb{P}^4$. Exceptional cases are a rational normal scroll, a complete intersection surface of (2, 2)-type, or a Castelnuovo surface of degree 5 in $\mathbb{P}^4$

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whose degree complexities are in fact equal to their degrees. This complexity can also be expressed in terms of degrees of defining equations of I_S in the same manner as the result of A. Conca and J. Sidman (Conca and Sidman, 2005). We also provide some illuminating examples of our results via calculations done with *Macaulay 2* (Grayson and Stillman, 1997).

1. Introduction

In Bayer and Mumford (1993), D. Bayer and D. Mumford introduced the degree complexity of a homogeneous ideal I with respect to a given term order τ as the maximal degree of the reduced Gröbner basis of I, and this is exactly the highest degree of minimal generators of the initial ideal of I. Even though the degree complexity depends on the choice of coordinates, it is constant in generic coordinates since the initial ideal of I is invariant under a generic change of coordinates, which is the so-called the generic initial ideal of I (Eisenbud, 1995).

For the graded lexicographic order (resp. the graded reverse lexicographic order), we denote by $M(I)$ (resp. $m(I)$) the degree complexity of I *in generic coordinates*. For a projective scheme X, the degree complexity of X can also be defined as $M(I_X)$ (resp. $m(I_X)$) for the graded lexicographic order (resp. the graded reverse lexicographic order) where I_X is the defining saturated ideal of X.

D. Bayer and M. Stillman have shown in Bayer and Stillman (1987) that $m(I)$ is exactly equal to the Castelnuovo–Mumford regularity $\operatorname{reg}(I)$. Then what can we say about $M(I)$? A. Conca and J. Sidman proved in Conca and Sidman (2005) that if I_C is the defining ideal of a smooth irreducible complete intersection curve C of type (a, b) in $\mathbb{P}^3$ then $M(I_C)$ is $1 + \frac{ab(a-1)(b-1)}{2}$ with the exception of the case $a = b = 2$, where $M(I_C)$ is 4. Recently, J. Ahn has shown in Ahn (2008) that if I_C is the defining ideal of a non-degenerate smooth integral curve of degree d and genus $g(C)$ in $\mathbb{P}^r$ (for $r \geq 3$), then $M(I_C) = 1 + \binom{d-1}{2} - g(C)$ with two exceptional cases.

In this paper, we would like to compute the degree complexity of a smooth surface S in $\mathbb{P}^4$ with respect to the graded lexicographic order. Our main results are, with the exception of three cases, if $S \subset \mathbb{P}^4$ is a smooth irreducible surface of degree d with $h^0(\mathcal{I}_S(2)) \neq 0$, then the degree complexity $M(I_S)$ of S is given by $2 + \binom{\deg Y_1(S)-1}{2} - g(Y_1(S))$, where $Y_1(S)$ is a smooth double curve of S in $\mathbb{P}^3$ under a generic projection and $\deg Y_1(S) = \binom{d-1}{2} - g(S \cap H)$. Moreover, this complexity is actually obtained at the monomial

$$x_0 x_1 x_3^{\binom{\deg Y_1(S)-1}{2} - g(Y_1(S))}$$

where $k[x_0, x_1, x_2, x_3, x_4]$ is a polynomial ring defining $\mathbb{P}^4$.

On the other hand, $M(I_S)$ can also be expressed in terms of degrees of defining equations of I_S in the same manner as the result of Conca and Sidman (Conca and Sidman, 2005) (see Theorem 4.9). Note that if S is a locally Cohen–Macaulay surface with $h^0(\mathcal{I}_S(2)) \neq 0$ then there are two types of surfaces S. One is a complete intersection of $(2, \alpha)$-type and the other is arithmetically Cohen–Macaulay of degree $2\alpha - 1$. For those cases, $\deg Y_1(S)$, $g(Y_1(S))$ and $g(S \cap H)$ can be obtained in terms of α.

Consequently, if S is a complete intersection of $(2, \alpha)$-type for some $\alpha \geq 3$ then $M(I_S) = \frac{1}{2}(\alpha^4 - 4\alpha^3 + 5\alpha^2 - 2\alpha + 4)$. If S is arithmetically Cohen–Macaulay of degree $2\alpha - 1$, $\alpha \geq 4$, then $M(I_S) = \frac{1}{2}(\alpha^4 - 6\alpha^3 + 13\alpha^2 - 12\alpha + 8)$ (see Theorem 4.9). Exceptional cases are a rational normal scroll, a complete intersection surface of $(2, 2)$-type, or a Castelnuovo surface of degree 5 in $\mathbb{P}^4$. In these cases, $M(I_S) = \deg(S)$ (see Proposition 4.5).

The main ideas are divided into two parts: one is to show that the degree complexity $M(I_S)$ is given by the maximum of $\operatorname{reg}(\operatorname{Gin}_{\mathrm{GLex}}(K_i(I_S))) + i$ for $i = 0, 1$ and the other part is to compare the schemes of multiple loci defined by partial elimination ideals and their classical scheme structures defined by the Fitting ideals of an $\mathcal{O}_{\mathbb{P}^3}$-module $\pi_* \mathcal{O}_S$ where π is a generic projection of S to $\mathbb{P}^3$.

2. Notations and basic facts

- We work over an algebraically closed field k of characteristic zero.
- Let $R = k[x_0, \dots, x_r]$ be a polynomial ring over k. For a closed subscheme X in $\mathbb{P}^r$, we denote the defining saturated ideal of X by

$$I_X = \bigoplus_{m=0}^{\infty} H^0(\mathcal{I}_X(m)).$$

- For a homogeneous ideal I, the Hilbert function of R/I is defined by $H(R/I, m) := \dim_k(R/I)_m$ for any non-negative integer m. We denote its corresponding Hilbert polynomial by $P_{R/I}(z) \in \mathbb{Q}[z]$. If $I = I_X$ then we simply write $P_X(z)$ instead of $P_{R/I_X}(z)$.
- We write $\rho_a(X) = (-1)^{\dim(X)}(P_X(0) - 1)$ for the arithmetic genus of X.
- For a homogeneous ideal $I \subset R$, consider a minimal free resolution

$$\cdots \to \bigoplus_j R(-i-j)^{\beta_{i,j}(I)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}(I)} \to I \to 0$$

  of I as a graded R-modules. We say that I is m-regular if $\beta_{i,j}(I) = 0$ for all $i \geq 0$ and $j > m$. The Castelnuovo–Mumford regularity of I is defined by

$$\operatorname{reg}(I) := \min\{\, m \mid I \text{ is } m\text{-regular}\}.$$

- Given a term order τ, we define the initial term $\mathrm{in}_\tau(f)$ of a homogeneous polynomial $f \in R$ to be the greatest monomial of f with respect to τ. If $I \subset R$ is a homogeneous ideal, we also define the initial ideal $\mathrm{in}_\tau(I)$ to be the ideal generated by $\{\mathrm{in}_\tau(f) \mid f \in I\}$. A set $G = \{g_1, \dots, g_n\} \subset I$ is said to be a Gröbner basis if

$$(\mathrm{in}_\tau(g_1), \dots, \mathrm{in}_\tau(g_n)) = \mathrm{in}_\tau(I).$$

- For an element $\alpha = (\alpha_0, \dots, \alpha_r) \in \mathbb{N}^r$ we define the notation $x^\alpha = x_0^{\alpha_0} \cdots x_r^{\alpha_r}$ for monomials. Its degree is $\mid \alpha \mid = \sum_{i=0}^{r} \alpha_i$.

  For two monomial terms x^α and x^β, the *graded lexicographic order* is defined by $x^\alpha \geq_{\mathrm{GLex}} x^\beta$ if and only if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and if the left most nonzero entry of $\alpha - \beta$ is positive. The *graded reverse lexicographic order* is defined by $x^\alpha \geq_{\mathrm{GRLex}} x^\beta$ if and only if we have $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and if the right most nonzero entry of $\alpha - \beta$ is negative.
- In characteristic 0, we say that a monomial ideal I has the Borel-fixed property if, for any monomial m such that $x_i m \in I$, then $x_j m \in I$ for all $j \leq i$.
- Given a homogeneous ideal $I \subset R$ and a term order τ, there is a Zariski open subset $U \subset GL_{r+1}(k)$ such that $\mathrm{in}_\tau(g(I))$ is constant. We will call $\mathrm{in}_\tau(g(I))$ for $g \in U$ the generic initial ideal of I and denote it by $\mathrm{Gin}_\tau(I)$. Generic initial ideals have the Borel-fixed property (see (Eisenbud, 1995; Green, 1998)).
- For a homogeneous ideal $I \subset R$, let $m(I)$ and $M(I)$ denote the maximum of the degrees of minimal generators of $\mathrm{Gin}_{\mathrm{GRLex}}(I)$ and $\mathrm{Gin}_{\mathrm{GLex}}(I)$ respectively.
- If I is a Borel fixed monomial ideal then $\operatorname{reg}(I)$ is exactly the maximal degree of minimal generators of I (see (Bayer and Stillman, 1987; Green, 1998)). This implies that

$$m(I) = \operatorname{reg}(\mathrm{Gin}_{\mathrm{GRLex}}(I)) \quad \text{and} \quad M(I) = \operatorname{reg}(\mathrm{Gin}_{\mathrm{GLex}}(I)).$$

3. Gröbner bases of partial elimination ideals

Definition 3.1. Let I be a homogeneous ideal in R. If $f \in I_d$ has leading term $\mathrm{in}(f) = x_0^{d_0} \cdots x_r^{d_r}$, we will set $d_0(f) = d_0$, the leading power of x_0 in f. We let

$$\widetilde{K}_i(I) = \bigoplus_{d \geq 0} \{f \in I_d \mid d_0(f) \leq i\}.$$

If $f \in \widetilde{K}_i(I)$, we may write uniquely

$$f = x_0^i \bar{f} + g,$$

where $d_0(g) < i$. Now we define $K_i(I)$ as the image of $\widetilde{K}_i(I)$ in $\bar{R} = k[x_1 \dots x_r]$ under the map $f \to \bar{f}$ and we call $K_i(I)$ the i-th partial elimination ideal of I. □

Remark 3.1. We have an inclusion of the partial elimination ideals of I:

$$I \cap \bar{R} = K_0(I) \subset K_1(I) \subset \cdots \subset K_i(I) \subset K_{i+1}(I) \subset \cdots \subset \bar{R}.$$

Note that if I is in generic coordinates and $i_0 = \min\{i \mid I_i \neq 0\}$ then $K_i(I) = \bar{R}$ for all $i \geq i_0$. □

The following result gives the precise relationship between partial elimination ideals and the geometry of the projection map from $\mathbb{P}^r$ to $\mathbb{P}^{r-1}$. For a proof of this proposition, see (Green, 1998, Propostion 6.2).

Proposition 3.2. *Let $X \subset \mathbb{P}^r$ be a reduced closed subscheme and let I_X be the defining ideal of X. Suppose $p = [1, 0, \dots, 0] \in \mathbb{P}^r \setminus X$ and that $\pi : X \to \mathbb{P}^{r-1}$ is the projection from the point $p \in \mathbb{P}^r$ to the hyperplane where $x_0 = 0$. Then, set-theoretically, $K_i(I_X)$ is the ideal of $\{q \in \pi(X) \mid \mathrm{mult}_q(\pi(X)) > i\}$.* □

For each $i \geq 0$, note that we can give a scheme structure on the set

$$Y_i(X) := \{q \in \pi(X) \mid \mathrm{mult}_q(\pi(X)) > i\}$$

from the i-th partial elimination ideal $K_i(I)$. Let

$$Z_i(X) := \mathrm{Proj}(\bar{R}/K_i(I_X)),$$

where $\bar{R} = k[x_1 \dots x_r]$. Then it follows from Proposition 3.2 that

$$Z_i(X)_{\mathrm{red}} = Y_i(X).$$

Remark 3.3. Let $X \subset \mathbb{P}^r$ be a smooth variety of codimension two and let $\pi : X \to \mathbb{P}^{r-1}$ be a generic projection of X. A classical scheme structure on the set $Y_i(X)$ is given by i-th Fitting ideal of the $\mathcal{O}_{\mathbb{P}^{r-1}}$-module $\pi_*\mathcal{O}_X$ (see (Kleiman et al., 1996; Mezzetti and Portelli, 1997)). Throughout this paper, we use the notation $Y_i(X)$ in the sense that it is a closed subscheme defined by the Fitting ideal of $\pi_*\mathcal{O}_X$, as distinguished from the notation $Z_i(X)$. We show that if $S \subset \mathbb{P}^4$ is a smooth surface lying in a quadric hypersurface then $Y_1(S)$ and $Z_1(S)$ have the same reduced scheme structure (see Theorem 4.2), which will be used in the proof of Proposition 4.5. □

It is natural to ask: what is a Gröbner basis of $K_i(I)$? Recall that any non-zero polynomial f in R can be uniquely written as $f = x_0^t \bar{f} + g$ where $d_0(g) < t$. Conca and Sidman (Conca and Sidman, 2005) show that if G is a Gröbner basis for an ideal I then the set

$$G_i = \{\bar{f} \mid f \in G \text{ with } d_0(f) \leq i\}$$

is a Gröbner basis for $K_i(I)$. However if I is in generic coordinates then there is a more refined Gröbner basis for $K_i(I)$, which plays an important role in this paper.

Proposition 3.4. *Let I be a homogeneous ideal in generic coordinates and G be a Gröbner basis for I with respect to the graded lexicographic order. Then, for each $i \geq 0$,*

(a) *the i-th partial elimination ideal $K_i(I)$ is in generic coordinates;*
(b) *$G_i = \{\bar{f} \mid f \in G \text{ with } d_0(f) = i\}$ is a Gröbner basis for $K_i(I)$.* □

Proof. (a) is in fact proved in Proposition 3.3 in Conca and Sidman (2005). For a proof of (b), it suffices to show that $\langle \mathrm{in}(G_i)\rangle = \mathrm{in}(K_i(I))$ by the definition of Gröbner bases. Since $G_i \subset K_i(I)$, we only need to show that $\langle \mathrm{in}(G_i)\rangle \supset \mathrm{in}(K_i(I))$. Now, we denote $\mathcal{G}(I)$ by the set of minimal generators of I. Let $m \in \mathrm{in}(K_i(I))$ be a monomial. Then there is a monomial generator $M \in \mathcal{G}(\mathrm{in}(K_i(I)))$ such that M divides m.

We claim that $x_0^i M \in \mathcal{G}(\mathrm{in}(I))$ if and only if $M \in \mathcal{G}(\mathrm{in}(K_i(I)))$.

If the claim is proved then we will be done. Indeed, for $M \in \mathcal{G}(\text{in}(K_i(I)))$, we see that $x_0^i M \in \mathcal{G}(\text{in}(I))$. This implies that there exists a polynomial $f = x_0^i \bar{f} + g \in G$ with $d_0(g) < i$ such that

$$\text{in}(f) = x_0^i \text{in}(\bar{f}) = x_0^i M.$$

This means that $M = \text{in}(\bar{f}) \in \langle \text{in}(G_i) \rangle$. Thus we have $m \in \langle \text{in}(G_i) \rangle$.
Here is a proof of the claim: suppose that $x_0^i M \in \mathcal{G}(\text{in}(I))$ then we can say that $x_0^i M \in \text{in}(I)$. Thus there is a polynomial $f = x_0^i \bar{f} + g \in I$ such that $d_0(g) < i$ and $\text{in}(f) = x_0^i \text{in}(\bar{f}) = x_0^i M$. By the definition of partial elimination ideals, we have that $\bar{f} \in K_i(I)$, which means $M \in \text{in}(K_i(I))$. Assume that $M \notin \mathcal{G}(\text{in}(K_i(I)))$. Then for some monomial $N \in \mathcal{G}(\text{in}(K_i(I)))$ such that N divides M. This implies that

$$x_0^i N \in \text{in}(I) \quad \text{and} \quad x_0^i N \mid x_0^i M,$$

which contradicts the fact that $x_0^i M$ is a minimal generator of $\text{in}(I)$. Thus M is contained in $\mathcal{G}(\text{in}(K_i(I)))$.

Conversely, suppose that there is $M \in \mathcal{G}(\text{in}(K_i(I)))$ such that $x_0^i M \notin \mathcal{G}(\text{in}(I))$. Then we may choose a monomial $x_0^j N \in \mathcal{G}(\text{in}(I))$ satisfying

$$x_0 \nmid N \quad \text{and} \quad x_0^j N \mid x_0^i M. \tag{1}$$

Note that (1) implies that $i \geq j \geq 0$. Since $N \in \text{in}(K_j(I))$ and $K_0(I) \subset K_1(I) \subset \cdots$, it is obvious that $N \in \text{in}(K_i(I))$ and N divides M. Now, we claim that N can be chosen to be different from M. If $N = M$ then j must be less than i. Denote N by $x_1^{j_1} \cdots x_r^{j_r}$ and choose a nonzero $j_t \in \{j_1, \dots, j_r\}$. By (a), note that $K_i(I)$ is in generic coordinates and so we may assume that $\text{in}(K_i(I))$ has the Borel-fixed property. Therefore, if we set $N' = N/x_{j_t}$ then $x_0^{j+1} N' \in \text{in}(I)$. Replace $x_0^j N$ by $N'' = x_0^{j+1} N'$. Then $N' \in \text{in}(K_{j+1}(I))$. Since $j+1 \leq i$, we can say that $N' \in \text{in}(K_i(I))$ and N' divides M with $N' \neq M$. This contradicts the assumption that $M \in \mathcal{G}(\text{in}(K_i(I)))$. □

Remark 3.5. The condition "in generic coordinates" is crucial in Proposition 3.4 (b) as the following example shows. Let $I = (x_0^2, x_0x_1, x_0x_2, x_3)$ be a monomial ideal. Then $G = \{x_0^2, x_0x_1, x_0x_2, x_3\}$ is a Gröbner basis for I. Then we can easily check that

$$G_1 = \{\bar{f} \mid f \in G \text{ with } d_0(f) \leq 1\} = (x_1, x_2, x_3),$$
$$G_1' = \{\bar{f} \mid f \in G \text{ with } d_0(f) = 1\} = (x_1, x_2).$$

This shows that G_1' is not a Gröbner basis for $K_1(I)$. □

We have the following corollary from Proposition 3.4.

Corollary 3.6. *For a homogeneous ideal $I \subset R = k[x_0, \dots, x_r]$ in generic coordinates, we have*

$$M(I) = \max\{M(K_i(I)) + i \mid 0 \leq i \leq \beta\},$$

where $\beta = \min\{j \mid I_j \neq 0\}$. □

Proof. Note that $K_\beta(I) = \bar{R}$ for $\beta = \min\{j \mid I_j \neq 0\}$ by definition. We know that $M(I)$ can be obtained from the maximal degree of generators in $\text{Gin}(I)$. Remember that $\mathcal{G}(I)$ is the set of minimal generators of I. Then by Proposition 3.4, every generator of $\text{Gin}(I)$ is of the form $x_0^i M$ where $M \in \mathcal{G}(\text{Gin}(K_i(I)))$ for some i. This means that $M(I) \leq M(\text{Gin}(K_i(I))) + i$ for some i. On the other hand, if for each i, we choose $M \in \mathcal{G}(K_i(I))$, then by Proposition 3.4, $x_0^i M$ is contained in $\mathcal{G}(\text{Gin}(I))$. Hence we conclude that

$$M(I) = \max\{M(K_i(I)) + i \mid 0 \leq i \leq \beta\}. \quad \square$$

Corollary 3.6 together with the following theorem can be used to obtain the degree-complexities of the smooth surface lying in a quadric hypersurface in $\mathbb{P}^4$. For a proof of this theorem, see (Ahn, 2008, Theorem 4.4).

Theorem 3.7. *Let C be a non-degenerate smooth curve of degree d and genus $g(C)$ in $\mathbb{P}^r$ for some $r \geq 3$. Then,*

$$M(I_C) = \max\left\{d, 1 + \binom{d-1}{2} - g(C)\right\}. \quad \square$$

4. Degree complexity of smooth irreducible surfaces in $\mathbb{P}^4$

Let S be a non-degenerate smooth irreducible surface of degree d and arithmetic genus $\rho_a(S)$ in $\mathbb{P}^4$ and let I_S be the defining ideal of S in $R = k[x_0, \ldots, x_4]$. In this section, we study the scheme structure of

$$Z_i(S) := \text{Proj}(\bar{R}/K_i(I_S)), \quad \text{where } \bar{R} = k[x_1, x_2, x_3, x_4].$$

arising from a generic projection in order to get a geometric interpretation of the degree-complexity $M(I_S)$ of S in $\mathbb{P}^4$ with respect to the degree lexicographic order.

We recall without proof the standard facts concerning generic projections of surfaces in $\mathbb{P}^4$ to $\mathbb{P}^3$.

Let $S \subset \mathbb{P}^4$ be a non-degenerate smooth irreducible surface of degree d and arithmetic genus $\rho_a(S)$ and $\pi : S \to \pi(S) \subset \mathbb{P}^3$ be a generic projection.

(a) The singular locus of $\pi(S)$ is a curve $Y_1(S)$ with only singularities a number t of ordinary triple points with transverse tangent directions. The inverse image $\pi^{-1}(Y_1(S))$ is a curve with only singularities $3t$ nodes, 3 nodes above each triple point of $Y_1(S)$ (see (Pinkham, 1986)). This implies (using Proposition 3.2) that the ideals $K_j(I_S)$ have finite colength if $j > 2$. This fact is used in the proofs of Proposition 4.6 and Theorem 4.3.

(b) If a smooth surface $S \subset \mathbb{P}^4$ is contained in a quadric hypersurface then there are no ordinary triple points in $Y_1(S)$. This implies that the double curve $Y_1(S)$ is smooth by (a).

(c) The double curve $Y_1(S)$ is irreducible unless S is a projected Veronese surface in $\mathbb{P}^4$ (see (Mezzetti and Portelli, 1997)).

(d) The reduced induced scheme structure on $Y_1(S)$ is defined by the first Fitting ideal of the $\mathcal{O}_{\mathbb{P}^3}$-module $\pi_*\mathcal{O}_S$ (see (Mezzetti and Portelli, 1997)).

(e) The degree of $Y_1(S)$ is $\binom{d-1}{2} - g(S \cap H)$ where $S \cap H$ is a general hyperplane section and the number of apparent triple points t is given in Le Barz (1981) by

$$t = \binom{d-1}{3} - g(S \cap H)(d-3) + 2\chi(\mathcal{O}_S) - 2.$$

The following lemma shows that the Hilbert function of I_S can be obtained from those of partial elimination ideals $K_i(I_S)$.

Lemma 4.1. *Let $S \subset \mathbb{P}^4$ be a smooth surface with defining ideal I_S in $R = k[x_0, x_1, \ldots, x_4]$. Consider a projection $\pi_q : S \longrightarrow \mathbb{P}^3$ from a general point $q = [1, 0, 0, 0, 0] \notin S$. Then,*

$$H(R/I_S, m) = \sum_{i \geq 0} H(\bar{R}/K_i(I_S), m - i).$$

In particular,

$$P_S(z) = P_{Z_0(S)}(z) + P_{Z_1(S)}(z-1) + P_{Z_2(S)}(z-2). \quad \square$$

Proof. The equality on Hilbert functions basically comes from the following combinatorial identity

$$\binom{m+d}{d} = \sum_{i=0}^{m} \binom{m-i+d-1}{d-1}.$$

For a smooth surface $S \subset \mathbb{P}^4$, $Z_i(S) = \emptyset$ for $i \geq 3$ by the (dimension +2)-secant lemma (see (Ran, 1991)) and so $\bar{R}/K_i(I_S)$ is Artinian. Thus $P_{Z_i(S)}(z) = 0$ for $i \geq 3$ (see (Ahn, 2008, Lemma 3.4) for details). $\square$

The following theorem says that the first partial elimination ideal $K_1(I_S)$ gives the reduced induced scheme structure on the double curve $Y_1(S)$ in $\mathbb{P}^3$ (i.e., $I_{Z_1(S)} = I_{Y_1(S)}$).

Theorem 4.2. *Suppose that S is a reduced irreducible surface in $\mathbb{P}^4$. Then,*

(a) *the first partial elimination ideal $K_1(I_S)$ is a saturated ideal, so we have $K_1(I_S) = I_{Z_1(S)}$;*
(b) *if S is a smooth surface contained in a quadric hypersurface, then $K_1(I_S) = I_{Y_1(S)}$, which implies that $K_1(I_S)$ is a radical ideal.* $\square$

Proof. (a) Assume that S is a reduced irreducible surface in $\mathbb{P}^4$ of degree d. Take a general point $q \in \mathbb{P}^4$; we may assume $q = [1, 0, \ldots, 0]$. Then the generic projection of S into $\mathbb{P}^3$ from the point q is defined by a single polynomial $F \in k[x_1, x_2, x_3, x_4]$ of degree d and $K_0(I_S) = (F)$, which is a radical ideal.

Let $\bar{\mathcal{M}} = (x_1, x_2, x_3, x_4)$ be the irrelevant maximal ideal of $\bar{R} = k[x_1, x_2, x_3, x_4]$ and let $\bar{V} = \langle x_1, x_2, x_3, x_4\rangle$ be the vector space over k. By the definition of a saturated ideal, $K_1(I_S)$ is saturated if and only if

$$(K_1(I_S) : \bar{\mathcal{M}}) = K_1(I_S).$$

Hence it is enough to show that

$$(K_1(I_S) : \bar{\mathcal{M}})/K_1(I_S) = 0.$$

For the proof, consider the Koszul complex

$$\cdots \to \mathcal{K}_m^{-p-1} \to \mathcal{K}_m^{-p} \to \mathcal{K}_m^{-p+1} \to \cdots,$$

where $\mathcal{K}_m^{-p} = \wedge^p \bar{V} \otimes K_0(I_S)_{m-p}$. From Corollary 6.7 in Green (1998), the $\bar{R}$-module $(K_1(I_S) : \bar{\mathcal{M}})_d/K_1(I_S)_d$ injects into $H^{-1}(\mathcal{K}_{d+3}^{\bullet})$ for each d. Note that

$$H^{-1}(\mathcal{K}_{d+3}^{\bullet}) = H(\wedge^1 \bar{V} \otimes K_0(I_S)_{d+2}) = \mathrm{Tor}_1^{\bar{R}}(\bar{R}/\bar{\mathcal{M}}, K_0(I_S))_{d+3}.$$

Since the ideal $K_0(I_S)$ is generated by a single polynomial F, we have that

$$\mathrm{Tor}_1^{\bar{R}}(\bar{R}/\bar{\mathcal{M}}, K_0(I_S)) = 0.$$

This proves that $(K_1(I_S) : \bar{\mathcal{M}})/K_1(I_S) = 0$, as we wished.

(b) Consider the graded $\bar{R}$-module homomorphism

$$\varphi : \bar{R}(-1) \oplus \bar{R} \to R/I_S \text{ defined by } \varphi(f, g) = [x_0 f + g]$$

where $[x_0 f + g]$ is the quotient image of the polynomial $x_0 f + g$ in R/I_S. Since the surface S is contained in a quadric hypersurface Q, we assume $q = [1, 0, 0, 0, 0] \notin Q$ and the defining equation of Q is of the form :

$$F = x_0^2 - x_0 L(x_1, x_2, x_3, x_4) - F'(x_1, x_2, x_3, x_4) \in I_S.$$

Now we claim that φ is surjective. Note that

$$x_0^2 \equiv x_0 L(x_1, x_2, x_3, x_4) + F'(x_1, x_2, x_3, x_4) \mod I_S.$$

Hence, this equation can be used to show that, for every homogeneous polynomial $G \in R$, there are polynomials f and g in $\bar{R}$ such that

$$G \equiv x_0 f + g \mod I_S.$$

This implies that the $\bar{R}$-module homomorphism φ is surjective and we have the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_0(I_S) & \longrightarrow & \bar{R} & \longrightarrow & \bar{R}/K_0(I_S) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{K}_1(I_S) & \longrightarrow & \bar{R} \oplus \bar{R}(-1) & \stackrel{\varphi}{\longrightarrow} & R/I_S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1(I_S)(-1) & \longrightarrow & \bar{R}(-1) & \longrightarrow & \bar{R}/K_1(I_S)(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\widetilde{K}_1(I_S) = \{f \in I_S \mid d_0(f) \le 1\}$ is an $\bar{R}$-module. Let $\mathcal{O}_{Z_1(S)}$ be the sheafification of $\bar{R}/K_1(I_S)$. By sheafifying the rightmost vertical sequence, we have

$$0 \longrightarrow \mathcal{O}_{\pi(S)} \longrightarrow \pi_* \mathcal{O}_S \longrightarrow \mathcal{O}_{Z_1(S)}(-1) \longrightarrow 0. \tag{2}$$

Let $\mathcal{I}_{Z_1(S)} = \mathcal{K}_1(I_S)$ be the sheafification of the ideal $K_1(I_S)$. In Kleiman et al. (1996, (3.4.1), p. 302), S. Kleiman, J. Lipman and B. Ulrich proved that

$$\mathcal{I}_{Y_1(S)} = \mathrm{Fitt}_1^{\mathbb{P}^3}(\pi_*\mathcal{O}_S) = \mathrm{Fitt}_0^{\mathbb{P}^3}(\pi_*\mathcal{O}_S/\mathcal{O}_{\pi(S)}) = \mathrm{Ann}_{\mathbb{P}^3}(\mathcal{O}_{Z_1(S)}(-1)),$$

and this defines *the reduced scheme structure* on $Y_1(S)$ (see (Mezzetti and Portelli, 1997, p. 3)).

On the other hand, from the sequence (2), we have

$$\mathcal{I}_{Y_1(S)} = \mathrm{Ann}_{\mathbb{P}^3}(\mathcal{O}_{Z_1(S)}(-1)) = \mathcal{K}_1(I_S) = \mathcal{I}_{Z_1(S)}.$$

Then it follows from (a) that

$$I_{Z_1(S)} = K_1(I_S)^{\mathrm{sat}} = K_1(I_S) = I_{Y_1(S)}.$$

Since $I_{Y_1(S)}$ is a radical ideal, we conclude that $I_{Z_1(S)} = K_1(I_S)$ is also a radical ideal. □

If $S \subset \mathbb{P}^4$ is contained in a quadric hypersurface, then by Theorem 4.2, $K_1(I_S)$ is saturated and radical. So, it defines the reduced scheme structure on $Y_1(S)$. Note also that the double curve $Y_1(S)$ is smooth (see the standard fact (b) in the beginning of this section). We use this fact to prove the following theorem.

Theorem 4.3. *Let S be a smooth irreducible surface of degree d lying on a quadric hypersurface in $\mathbb{P}^4$. Let $Y_1(S)$ be the double curve of genus $g(Y_1(S))$ defined by a generic projection π of S to $\mathbb{P}^3$. Then, we have the following;*

(a) $M(I_S) = \max\{d, 1 + \deg Y_1(S), 2 + \binom{\deg Y_1(S)-1}{2} - g(Y_1(S))\}$;
(b) *$M(I_S)$ can be obtained at one of monomials*

$$x_1^d,\ x_0x_2^{\deg Y_1(S)},\ x_0x_1x_3^{\binom{\deg Y_1(S)-1}{2}-g(Y_1(S))}. \quad \square$$

Proof. Note that by Corollary 3.6,

$$M(I_S) = \max_{0\le i\le\beta}\{\mathrm{reg}(\mathrm{Gin}(K_i(I_S))) + i\},$$

where $\beta = \min\{j \mid K_j(I_S) = \bar{R}\}$. Since S is contained in a quadric hypersurface, $\mathrm{Gin}(I_S)$ contains the monomial x_0^2. This means that $\mathrm{Gin}(K_2(I_S)) = \bar{R}$. On the other hand, $\mathrm{Gin}(K_0(I_S)) = (x_1^d)$ by the Borel fixed property because $\pi(S)$ is a hypersurface of degree d in $\mathbb{P}^3$ and $I_{\pi(S)} = K_0(I_S)$. Thus $\mathrm{Gin}(I_S)$ is of the form

$$(x_0^2, x_0g_1, x_0g_2, \ldots, x_0g_m, x_1^d).$$

Note that $g_1, \ldots g_m$ are monomial generators of $\mathrm{Gin}(K_1(I_S)) = \mathrm{Gin}(I_{Y_1(S)})$ by Proposition 3.4.

Therefore, by Theorem 3.7,

$$\mathrm{reg}(\mathrm{Gin}(K_1(I_S))) = \max\left\{\deg Y_1(S), 1 + \binom{\deg Y_1(S)-1}{2} - g(Y_1(S))\right\}$$

and consequently,

$$M(I_S) = \max\left\{d, 1 + \deg Y_1(S), 2 + \binom{\deg Y_1(S)-1}{2} - g(Y_1(S))\right\}.$$

For a proof of (b), consider $\mathrm{Gin}(K_1(I_S)) = \langle g_1, g_2, \ldots, g_m\rangle$ in (a). Note that the double curve $Y_1(S)$ is smooth in $\mathbb{P}^3$. By a similar argument used in (a), $\mathrm{Gin}(K_1(I_S))$ contains $x_2^{\deg(Y_1(S))}$ because the image of $Y_1(S)$ under a generic projection to $\mathbb{P}^2$ is a plane curve of degree $\deg(Y_1(S))$. Finally, consider all monomial generators of the form $x_1 \cdot h_j(x_2, x_3, x_4)$ in $\{g_1, g_2, \ldots, g_m\}$. Then, $\{h_j(x_2, x_3, x_4) \mid 1 \le j \le m\}$ is a minimal generating set of $\mathrm{Gin}(K_1(I_{Y_1(S)}))$ by Proposition 3.4. Recall that $K_1(I_{Y_1(S)})$ defines $\binom{\deg Y_1(S)-1}{2} - g(Y_1(S))$ distinct nodes in $\mathbb{P}^2$. So, $\mathrm{Gin}(K_1(I_{Y_1(S)}))$ should contain the monomial $x_3^{\binom{\deg Y_1(S)-1}{2}-g(Y_1(S))}$ (see also (Conca and Sidman, 2005, Corollary 5.3)). Therefore, $\mathrm{Gin}(I_S)$ contains monomials $x_1^d, x_0x_2^{\deg(Y_1(S))}$ and $x_0x_1x_3^{\binom{\deg Y_1(S)-1}{2}-g(Y_1(S))}$. □

Remark 4.4. In the proof of Theorem 4.3, we showed that if a smooth irreducible surface S is contained in a quadric hypersurface then $M(I_S)$ is determined by two partial elimination ideals $K_0(I_S)$ and $K_1(I_S)$ since $K_i(I_S) = \bar{R}$ for all $i \geq 2$. □

The following theorem shows that if $d \geq 6$ then $M(I_S)$ is determined by the degree complexity of the first partial elimination ideal $K_1(I_S)$.

Proposition 4.5. *Let S be a smooth irreducible surface of degree d in $\mathbb{P}^4$. Suppose that S is contained in a quadric hypersurface. Then*

$$M(I_S) = \begin{cases} 3 & \text{if } S \text{ is a rational normal scroll with } d = 3 \\ 4 & \text{if } S \text{ is a complete intersection of (2,2)-type} \\ 5 & \text{if } S \text{ is a Castelnuovo surface with } d = 5 \\ 2 + \binom{\deg Y_1(S)-1}{2} - g(Y_1(S)) & \text{for } d \geq 6 \end{cases}$$

where $Y_1(S) \subset \mathbb{P}^3$ is a double curve of degree $\binom{d-1}{2} - g(S \cap H)$ under a generic projection of S to $\mathbb{P}^3$. □

Proof. Since $K_2(I_S) = \bar{R}$, Theorem 4.3 implies that

$$M(I_S) = \max\left\{d,\ 1 + \deg Y_1(S),\ 2 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S))\right\}.$$

If $\deg Y_1(S) \geq 5$ then by the genus bound,

$$1 + \deg Y_1(S) \leq 2 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S)).$$

We claim that if $d \geq 6$, then $d \leq 1 + \deg Y_1(S)$. Notice that from our claim, we have the degree complexity of a surface lying on a quadric hypersurface in $\mathbb{P}^4$ for $d \geq 6$ as follows;

$$M(I_S) = 2 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S)).$$

In the Montreal lecture notes (Eisenbud and Harris, 1982), Eisenbud and Harris gave the genus bound for non-degenerate integral curves of degree d and arithmetic genus ρ_a in $\mathbb{P}^r$. Indeed, if we set

$$\pi(d, r) = \binom{m}{2}(r - 1) + m\epsilon,$$

where $m = [\frac{d-1}{r-1}]$ and $\epsilon = d - m(r - 1) - 1$, then we have the following genus bound:

$$\rho_a \leq \pi(d, r) = \binom{m}{2}(r - 1) + m\epsilon.$$

From the genus bound, we know that

$$g(S \cap H) \leq \pi(d, 3) = \begin{cases} (\frac{d}{2} - 1)^2 & \text{if } d \text{ is even;} \\ (\frac{d-1}{2})(\frac{d-3}{2}) & \text{if } d \text{ is odd.} \end{cases}$$

Then we can show that $\pi(d, 3) \leq \binom{d-1}{2} - d + 1$ if $d = \deg(S \cap H) \geq 6$. Thus, if $d \geq 6$ then

$$d \leq 1 + \binom{d-1}{2} - g(S \cap H) = 1 + \deg Y_1(S).$$

So, our claim is proved and only three cases of $d = 3, 4, 5$ remain.

Case 1: If $\deg S = 3$ then S is a rational normal scroll with $g(S \cap H) = 0$ and the double curve $Y_1(S)$ is a line. So, by a simple computation, $M(I_S) = 3$.

Case 2: If $\deg S = 4$ then S is a complete intersection of (2,2)-type with $g(S \cap H) = 1$ and the double curve $Y_1(S)$ is a plane conic of $\deg Y_1(S) = 2$. So, by a simple computation, $M(I_S) = 4$.

Case 3: If $\deg S = 5$ then S is a Castelnuovo surface with $g(S \cap H) = 2$ and the double curve $Y_1(S) \subset \mathbb{P}^3$ is a smooth elliptic curve of degree 4. In this case, we can also compute

$$M(I_S) = 5 = \deg S > 2 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S)) = 4. \quad \square$$

Proposition 4.6. *Let S be a smooth irreducible surface of degree d and arithmetic genus $\rho_a(S)$ in $\mathbb{P}^4$. Let $Y_i(S)$ be the multiple locus defined by a generic projection of S to $\mathbb{P}^3$ for $i \geq 0$. Assume that S is contained in a quadric hypersurface. Then, the following identity holds;*

$$g(Y_1(S)) = \binom{d-1}{3} - \binom{d-1}{2} + g(S \cap H) - \rho_a(S) + 1. \quad \square$$

Proof. Let $P_S(z)$ be the Hilbert polynomial of a smooth irreducible surface of degree d and arithmetic genus $\rho_a(S)$. Since $Y_2(S) = \emptyset$, $P_{Y_2(S)}(z) = 0$ and, by Lemma 4.1,

$$P_S(z) = P_{Y_0(S)}(z) + P_{Y_1(S)}(z-1). \tag{3}$$

Note that $Y_0(S)$ is the image of a generic projection, which is a hypersurface of degree d in $\mathbb{P}^3$. Plugging $z = 0$ in the Eq. (3), we see from Algebraic geometry (1977, p. 54) that

$$P_S(0) = \rho_a(S) + 1 \quad \text{and} \quad P_{Y_0(S)}(0) = \binom{d-1}{3} + 1,$$

and thus

$$P_{Y_1(S)}(-1) = -\deg Y_1(S) + 1 - g(Y_1(S)) = -\binom{d-1}{2} + g(S \cap H) + 1 - g(Y_1(S)).$$

Therefore, we have the following identity:

$$g(Y_1(S)) = \binom{d-1}{3} - \binom{d-1}{2} + g(S \cap H) - \rho_a(S) + 1. \quad \square$$

Remark 4.7. By Proposition 4.6, when $d \geq 6$, $M(I_S)$ can be expressed with only three invariants of S: its degree, sectional genus, and arithmetic genus, as follows:

$$M(I_S) = \binom{\binom{d-1}{2} - g(S \cap H) - 1}{2} - \binom{d-1}{3} + \binom{d-1}{2} - g(S \cap H) + \rho_a(S) + 1. \quad \square$$

In order to compute $M(I_S)$ in terms of degrees of defining equations as Conca and Sidman did in Conca and Sidman (2005), we need the following remark. This shows that a smooth surface in $\mathbb{P}^4$ has a nice algebraic structure when it is contained in a quadric hypersurface.

Remark 4.8. Let S be a locally Cohen–Macaulay surface lying on a quadric hypersurface Q in $\mathbb{P}^4$. Then S satisfies one of following conditions (see (Kwak, 1999, Theorem 2.1));

(a) S is a complete intersection of $(2, \alpha)$-type.
  (i) $I_S = (Q, F)$, where F is a polynomial of degree α.
  (ii) $\mathrm{reg}(S) = \alpha + 1$.
(b) S is arithmetically Cohen–Macaualy of degree $2\alpha - 1$.
  (i) $I_S = (Q, F_1, F_2)$, where F_1 and F_2 are polynomials of degree α.
  (ii) $\mathrm{reg}(S) = \alpha$. $\square$

From the above Remark 4.8, we can compute $g(S \cap H)$ and $\rho_a(S)$ in terms of the degree of defining equations of S by finding the Hilbert polynomial of S in two ways. Therefore, we have the following Theorem.

Theorem 4.9. *Let $S \subset \mathbb{P}^4$ be a smooth irreducible surface of degree d and arithmetic genus $\rho_a(S)$, which is contained in a quadric hypersurface.*

(a) *Suppose S is of degree 2α, $\alpha \geq 3$. Then,*

$$M(I_S) = \frac{1}{2}(\alpha^4 - 4\alpha^3 + 5\alpha^2 - 2\alpha + 4).$$

(b) *Suppose S is of degree $2\alpha - 1$, $\alpha \geq 4$. Then*

$$M(I_S) = \frac{1}{2}(\alpha^4 - 6\alpha^3 + 13\alpha^2 - 12\alpha + 8). \quad \square$$

Proof. For a proof of (a), by the Koszul complex we have the minimal free resolution of the defining ideal I_S as follows:

$$0 \longrightarrow R(-\alpha - 2) \longrightarrow R(-2) \oplus R(-\alpha) \longrightarrow I_S \longrightarrow 0.$$

Hence the Hilbert function of R/I_S is given by

$$\begin{aligned} H(R/I_S, m) =& \alpha m^2 + (-\alpha^2 + 3\alpha)m + \frac{1}{6}\alpha(2\alpha^2 - 9\alpha + 13) \\ =& \frac{2\alpha}{2}m^2 + (\alpha + 1 - g(S \cap H))\, m + \rho_a(S) + 1. \end{aligned}$$

Hence $g(S \cap H) = (\alpha - 1)^2$ and $\rho_a(S) = \frac{1}{6}\alpha(2\alpha^2 - 9\alpha + 13) - 1$.
If $Y_1(S)$ is the double curve of S then

$$\deg Y_1(S) = \binom{2\alpha - 1}{2} - g(S \cap H) = \alpha(\alpha - 1).$$

By Remark 4.7,

$$g(Y_1(S)) = \binom{2\alpha - 1}{3} - \binom{2\alpha - 1}{2} + g(S \cap H) - \rho_a(S) + 1.$$

Thus we conclude that

$$\begin{aligned} M(I_S) &= 2 + \binom{\alpha(\alpha - 1) - 1}{2} - g(Y_1(S)) \\ &= \binom{\alpha(\alpha - 1) - 1}{2} - \binom{2\alpha - 1}{3} + \binom{2\alpha - 1}{2} - (\alpha - 1)^2 + \rho_a(S) + 1 \\ &= \frac{1}{2}(\alpha^4 - 4\alpha^3 + 5\alpha^2 - 2\alpha + 4). \end{aligned}$$

For a proof of (b), let S be a smooth surface of degree $2\alpha - 1$ lying on a quadric hypersurface in $\mathbb{P}^4$. Note that S is arithmetically Cohen–Macaulay of codimension 2. By the Hilbert–Burch Theorem (Eisenbud, 2005) we have the minimal free resolution of the defining ideal I_S as follows:

$$0 \longrightarrow R(-\alpha - 1)^2 \xrightarrow{\begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \\ F_5 & F_6 \end{pmatrix}} R(-2) \oplus R(-\alpha)^2 \longrightarrow I_S \longrightarrow 0,$$

where L_1, L_2, L_3, L_4 are linear forms and F_5, F_6 are forms of degree $\alpha - 1$. Hence the Hilbert function of R/I_S is given by

$$\begin{aligned} H(R/I_S, m) =& \frac{1}{2}(2\alpha - 1)m^2 + \left(4\alpha - \alpha^2 - \frac{3}{2}\right)m + \frac{1}{3}\alpha^3 - 2\alpha + \frac{11}{3}\alpha - 1 \\ =& \frac{(2\alpha - 1)}{2}m^2 + \left(\frac{2\alpha - 1}{2} + 1 - g(S \cap H)\right)m + \rho_a(S) + 1. \end{aligned}$$

Hence we have that $g(S \cap H) = 2\binom{\alpha-1}{2}$ and $\rho_a(S) = 2\binom{\alpha-1}{3}$.

If $Y_1(S)$ be the double curve of S then

$$\deg Y_1(S) = \binom{2\alpha-2}{2} - g(S \cap H) = \binom{2\alpha-2}{2} - 2\binom{\alpha-1}{2}.$$

On the other hand, we have

$$\begin{aligned} g(Y_1(S)) &= \binom{2\alpha-2}{3} - \binom{2\alpha-2}{2} + g(S \cap H) - \rho_a(S) + 1 \\ &= (\alpha-2)(\alpha^2-3\alpha+1) \end{aligned}$$

and thus we conclude that

$$\begin{aligned} M(I_S) &= 2 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S)) \\ &= \frac{1}{2}(\alpha^4 - 6\alpha^3 + 13\alpha^2 - 12\alpha + 8). \quad \square \end{aligned}$$

Example 4.10 (*Macaulay 2*)**.** We give some examples of $\mathrm{Gin}(I_S)$ and $M(I_S)$ computed by using *Macaulay 2* (Grayson and Stillman, 1997).

(a) Let S be a rational normal scroll in $\mathbb{P}^4$ whose defining ideal is

$$I_S = (x_0x_3 - x_1x_2, x_0x_1 - x_3x_4, x_0^2 - x_2x_4).$$

Using Macaulay 2, we can compute the generic initial ideal of I_S with respect to GLex:

$$\mathrm{Gin}(I_S) = (x_0^2, x_0x_1, x_0x_2, \mathbf{x_1^3}).$$

Thus $\mathrm{reg}(\mathrm{Gin}_{\mathrm{GLex}}(K_0)) = 3$ and $\mathrm{reg}(\mathrm{Gin}_{\mathrm{GLex}}(K_1)) = 1$. Therefore,

$$M(I_S) = \deg S = 3.$$

(b) Let S be a complete intersection of $(2, 2)$-type in $\mathbb{P}^4$. Then,

$$\mathrm{Gin}(I_S) = (x_0^2, x_0x_1, \mathbf{x_1^4}, x_0x_2^2).$$

Hence, we see $M(I_S) = \deg S = 4$.

(c) Let S be a Castelnuovo surface of degree 5 in $\mathbb{P}^4$. Then, we can compute

$$\mathrm{Gin}(I_S) = (x_0^2, x_0x_1^2, \mathbf{x_1^5}, x_0x_1x_2, x_0x_2^4, x_0x_1x_3^2).$$

Hence, we see $M(I_S) = \deg S = 5$.

(d) Let S be a complete intersection of $(2, 3)$-type in $\mathbb{P}^4$. Then, we see that $M(I_S) = 8$ from Theorem 4.9. On the other hand, we can compute the generic initial ideal:

$$\mathrm{Gin}(I_S) = (x_0^2, x_0x_1^2, x_1^6, x_0x_1x_2^2, x_0x_2^6, x_0x_1x_2x_3^2, \mathbf{x_0x_1x_3^6}, x_0x_1x_2x_3x_4^2, x_0x_1x_2x_4^4).$$

This also shows $M(I_S) = 8$.

(e) Let S be a smooth surface of degree 7 lying on a quadric which is not a complete intersection in $\mathbb{P}^4$. Then, the minimal resolution of I_S is given by Hilbert–Burch Theorem and thus we have

$$I_S = (L_1L_4 - L_2L_3, L_1F_5 - L_2F_6, L_3F_5 - L_4F_6),$$

where L_i is a linear form and F_5, F_6 are forms of degree 3. This is the case of $\alpha = 4$ in Theorem 4.9 and we see $M(I_S) = 20$. This can also be obtained by the computation of generic initial ideal of I_S

using *Macaulay 2*:

$$\begin{aligned}\mathrm{Gin}(I_S) = (&x_0^2,\ x_0x_1^3,\ x_1^7,\ x_0x_1^2x_2,\ x_0x_1x_2^4,\ x_0x_2^9,\ x_0x_1^2x_3^2,\ x_0x_1x_2^3x_3^2,\ x_0x_1x_2^2x_3^5,\\ &x_0x_1x_2x_3^8,\ \mathbf{x_0x_1x_3^{18}},\ x_0x_1x_2^2x_3^4x_4,\ x_0x_1^2x_3x_4^2,\ x_0x_1x_2^3x_3x_4^2,\ x_0x_1x_2^2x_3^3x_4^2,\\ &x_0x_1x_2x_3^7x_4^2,\ x_0x_1x_2^3x_4^3,\ x_0x_1^2x_4^4,\ x_0x_1x_2^2x_3^2x_4^4,\ x_0x_1x_2x_3^6x_4^4,\ x_0x_1x_2^2x_3x_4^5,\\ &x_0x_1x_2x_3^5x_4^6,\ x_0x_1x_2^2x_4^7,\ x_0x_1x_2x_3^4x_4^8,\ x_0x_1x_2x_3^3x_4^{10},\ x_0x_1x_2x_3^2x_4^{12},\\ &x_0x_1x_2x_3x_4^{14},\ x_0x_1x_2x_4^{16}).\end{aligned}$$

(f) Let S be a complete intersection of $(2,4)$-type in $\mathbb{P}^4$. Then, we see that $M(I_S) = 38$ from Theorem 4.9. This can be given by the computation of generic initial ideal of I_S:

$$\begin{aligned}\mathrm{Gin}(I_S) = (&x_0^2,\ x_0x_1^3,\ x_1^8,\ x_0x_1^2x_2^2,\ x_0x_1x_2^6,\ x_0x_2^{12},\ x_0x_1^2x_2x_3^2,\ x_0x_1x_2^5x_3^2,\\ &x_0x_1^2x_3^5,\ x_0x_1x_2^4x_3^5,\ x_0x_1x_2^3x_3^7,\ x_0x_1x_2^2x_3^{11},\ x_0x_1x_2x_3^{17},\ \mathbf{x_0x_1x_3^{36}},\\ &x_0x_1^2x_3^4x_4,\ x_0x_1x_2^4x_3^4x_4,\ x_0x_1x_2^3x_3^6x_4,\ x_0x_1x_2^2x_3^{10}x_4,\ x_0x_1^2x_2x_3x_4^2,\\ &x_0x_1x_2^5x_3x_4^2,\ x_0x_1^2x_3^3x_4^2,\ x_0x_1x_2^4x_3^3x_4^2,\ x_0x_1x_2^2x_3^9x_4^2,\ x_0x_1x_2x_3^{16}x_4^2,\\ &x_0x_1^2x_2x_4^3,\ x_0x_1x_2^5x_4^3,\ x_0x_1x_2^4x_3^2x_4^3,\ x_0x_1x_2^3x_3^5x_4^3,\ x_0x_1^2x_3^2x_4^4,\\ &x_0x_1x_2^3x_3^4x_4^4,\ x_0x_1x_2^2x_3^8x_4^4,\ x_0x_1x_2x_3^{15}x_4^4,\ x_0x_1^2x_3x_4^5,\ x_0x_1x_2^4x_3x_4^5,\\ &x_0x_1x_2^3x_3^3x_4^5,\ x_0x_1x_2^2x_3^7x_4^5,\ x_0x_1x_2^4x_4^6,\ x_0x_1x_2x_3^{14}x_4^6,\ x_0x_1^2x_4^7,\\ &x_0x_1x_2^3x_3^2x_4^7,\ x_0x_1x_2^2x_3^6x_4^7,\ x_0x_1x_2^3x_3x_4^8,\ x_0x_1x_2^2x_3^5x_4^8,\ x_0x_1x_2x_3^{13}x_4^8,\\ &x_0x_1x_2^3x_4^9,\ x_0x_1x_2^2x_3^4x_4^{10},\ x_0x_1x_2x_3^{12}x_4^{10},\ x_0x_1x_2^2x_3^3x_4^{11},\ x_0x_1x_2x_3^{11}x_4^{12},\\ &x_0x_1x_2^2x_3^2x_4^{13},\ x_0x_1x_2^2x_3x_4^{14},\ x_0x_1x_2x_3^{10}x_4^{14},\ x_0x_1x_2^2x_4^{16},\ x_0x_1x_2x_3^9x_4^{16},\\ &x_0x_1x_2x_3^8x_4^{18},\ x_0x_1x_2x_3^7x_4^{20},\ x_0x_1x_2x_3^6x_4^{22},\ x_0x_1x_2x_3^5x_4^{24},\ x_0x_1x_2x_3^4x_4^{26},\\ &x_0x_1x_2x_3^3x_4^{28},\ x_0x_1x_2x_3^2x_4^{30},\ x_0x_1x_2x_3x_4^{32},\ x_0x_1x_2x_4^{34}).\quad \square\end{aligned}$$

Even though we cannot compute the generic initial ideals for the cases $\alpha \geq 5$ by using computer algebra systems, we know the degree-complexity of smooth surfaces lying on a quadric by theoretical computations. We give the following data in Tables 1 and 2:

Table 1
The complete intersection S of $(2,\alpha)$-type in $\mathbb{P}^4$.

α	5	6	7	8	9	10	20	50	100
$M(I_S)$	122	302	632	1178	2018	3242	64982	2881202	48024902
$m(I_S)$	6	7	8	9	10	11	21	51	101

Table 2
The smooth surface $S \subset \mathbb{P}^4$ of degree $(2\alpha - 1)$ lying on a quadric.

α	5	6	7	8	9	10	20	50	100
$M(I_S)$	74	202	452	884	1570	2594	58484	2765954	47064404
$m(I_S)$	5	6	7	8	9	10	20	50	100

Remark and Question 4.11. Let S be a non-degenerate smooth surface of degree d and arithmetic genus $\rho_a(S)$, not necessarily contained in a quadric hypersurface in $\mathbb{P}^4$. Our question is: What can be the degree complexity $M(I_S)$ of S? It is expected that $K_1(I_S)$ and $K_2(I_S)$ are radical ideals and the degree complexity $M(I_S)$ is given by

$$M(I_S) = \max \begin{cases} \deg(S) \\ \mathrm{reg}(\mathrm{Gin}_{\mathrm{GLex}}(K_1(I_S))) + 1 \\ \mathrm{reg}(\mathrm{Gin}_{\mathrm{GLex}}(K_2(I_S))) + 2 \end{cases} = \max \begin{cases} d \\ M(I_{Y_1(S)}) + 1 \\ t + 2. \end{cases}$$

Note that t is the number of apparent triple points of $S \subset \mathbb{P}^4$ and $Y_1(S)$ is the double curve (possibly singular with ordinary double points) under a generic projection. □

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References

Ahn, Jeaman, 2008. The degree-complexity of the defining ideal of a smooth integral curve. J. Symbolic Comput. 43 (6–7), 422–441.
Bayer, Dave, Mumford, David, 1993. What can be computed in algebraic geometry?. In: Computational algebraic geometry and commutative algebra (Cortona, 1991). In: Sympos. Math., vol. XXXIV. Cambridge Univ. Press, Cambridge, pp. 1–48.
Bayer, David, Stillman, Michael, 1987. A criterion for detecting m-regularity. Invent. Math. 87 (1), 1–11.
Conca, Aldo, Sidman, Jessica, 2005. Generic initial ideals of points and curves. J. Symbolic Comput. 40 (3), 1023–1038.
Eisenbud, David, 2005. The geometry of syzygies. In: A second course in commutative algebra and algebraic geometry. In: Graduate Texts in Mathematics, vol. 229. Springer-Verlag, New York.
Eisenbud, David, 1995. Commutative Algebra with a View toward Algebraic Geometry. In: Graduate Texts in Mathematics, vol. 150. Springer-Verlag, New York.
Eisenbud, David, Harris, Joe, Curves in projective space. With the collaboration of David Eisenbud. Seminaire de Mathematiques Superieures [Seminar on Higher Mathematics], 85. Presses de l'Universite de Montreal, Montreal, Que., 1982. 138 pp. ISBN: 2-7606-0603-1.
Green, Mark, 1998. Generic Initial Ideals, in Six lectures on Commutative Algebra. In: Elias, J., Giral, J.M., Miró-Roig, R.M., Zarzuela, S. (Eds.), Progress in Mathematics, vol. 166. Birkhäuser, pp. 119–186.
Grayson, Daniel R., Stillman, Michael E., 1997. Macaulay 2: a software system for algebraic geometry and commutative algebra, available over the web at http://www.math.uiuc.edu/Macaulay2.
1977. Algebraic geometry. In: Graduate Texts in Mathematics, vol. 52. Springer-Verlag, New York-Heidelberg.
Kwak, Sijong, 1999. Castelnuovo-Mumford regularity bound for smooth threefolds in $\mathbb{P}^5$ and extremal examples. J. Reine Angew. Math. 509, 21–34.
Kleiman, Steven, Lipman, Joseph, Ulrich, Bernd, 1996. The multiple-point schemes of a finite curvilinear map of codimension one. Ark. Mat. 34 (2), 285–326.
Le Barz, Patrick, 1981. Formules pour les multisecantes des surfaces. (French) [Formulas for the multisecants of surfaces]. C. R. Acad. Sci. Paris Ser. I Math. 292 (17), 797–800.
Mezzetti, Emilia, Portelli, Dario, 1997. A tour through some classical theorems on algebraic surfaces. An. Stiint. Univ. Ovidius Constanta Ser. Mat. 5 (2), 51–78.
Pinkham, Henry C., 1986. A Castelnuovo bound for smooth surfaces. Invent. Math. 83 (2), 321–332.
Ran, Ziv., 1991. The (dimension +2)-secant lemma. Invent. Math. 106 (1), 65–71.