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On syzygies of non-complete embedding of projective varieties

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Abstract Let X be a non-degenerate, not necessarily linearly normal projective variety in $\mathbb{P}^r$. Recently the generalization of property N_p to non-linearly normal projective varieties have been considered and its algebraic and geometric properties are studied extensively. One of the generalizations is the property $N_{d,p}$ for the saturated ideal I_X (Eisenbud et al. in Compos Math 141: 1460–1478, 2005) and the other is the property N_p^S for the graded module of the twisted global sections of $\mathcal{O}_X(1)$ (Kwak and Park in J Reine Angew Math 582: 87–105, 2005). In this paper, we are interested in the algebraic and geometric meaning of properties N_p^S for every $p \geq 0$ and the syzygetic behaviors of isomorphic projections and hyperplane sections of a given variety with property N_p^S.

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1 Introduction

Some classical questions concerning linear systems on algebraic varieties and defining equations of the projective embedding have been studied extensively from the early Twentieth

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century by Italian algebraic geometers [3] and reformulated by Eisenbud and Goto [5], Eisenbud et al. [6], Ein and Lazarsfeld [7], Green [8] and Mumford [11].

Along these lines, we are interested in Castelnuovo–Mumford regularity of projective varieties and the natural generalization of property N_p to non-complete embeddings of projective varieties. Recently, property N_p has been extended to the case of not necessarily linearly normal projective varieties by Birkenhake [2], Alzati and Russo [1] and Eisenbud–Green–Hulek–Popescu [6] and the second and third authors [9], respectively.

In [9], the authors also introduce property N_p^S for not necessarily linearly normal varieties. Assume that X is a projective variety over an algebraically closed field K and $\mathcal{L} \in \mathrm{Pic} X$ is a very ample line bundle on X and $R(\mathcal{L}) = \bigoplus_{\ell \in \mathbb{Z}} H^0(X, \mathcal{L}^{\otimes \ell})$ is the graded ring of twisted global sections of $\mathcal{L}$. For a very ample subsystem

$$W \subset H^0(X, \mathcal{L}),$$

let $S = \oplus_{\ell \geq 0} Sym^{\ell}(W)$ be the homogeneous coordinate ring of $\mathbb{P}(W)$. Then $R(\mathcal{L})$ is a finitely generated graded module over S, so it has a minimal graded free resolution. As a generalization of property N_p defined by Green and Lazarsfeld, we can say that $X \subset \mathbb{P}(W)$ satisfies property N_p^S if $R(\mathcal{L})$ has the minimal free resolution of the form

$$\begin{array}{ccccccccc}
\cdots \to & E_p & \to & E_{p-1} & \to \cdots \to & E_1 & \to & E_0 & \to R(\mathcal{L}) \to 0 \\
 & \| & & \| & & \| & & \| & \\
 & S(-p-1)^{\oplus \beta_{p,1}} & & S(-p)^{\oplus \beta_{p-1,1}} & & S(-2)^{\oplus \beta_{1,1}} & & S \oplus S(-1)^t &
\end{array}$$

as a graded S-module where $t = \mathrm{codim}(W, H^0(X, \mathcal{L}))$. In other words, property N_p^S means that the maps between E_i, $1 \leq i \leq p$, are given by matrices of linear forms. Note that if $t = 0$, i.e., $W = H^0(X, \mathcal{L})$, then property N_p^S coincides with property N_p. In [9], two basic facts for property N_p^S are proved. Briefly speaking, the followings hold:

(a) If $X \subset \mathbb{P}(W)$ satisfies property N_1^S, then it is k-normal for all $k \geq t+1$, cut out by hypersurfaces of degree $\leq t+2$, and satisfies a Castelnuovo–Mumford regularity condition(Theorem 1.1 in [9]).
(b) When $H^1(X, \mathcal{L}^{\ell}) = 0$ for $\ell \geq 2$, the notions properties N_p^S behave well under isomorphic linear projections(Theorem 1.2 in [9]).

The aim of this paper is to continue the study of property N_p^S. We are interested in the algebraic and geometric meanings of properties N_p^S for every $p \geq 0$ and the syzygetic behavior of isomorphic projections and hyperplane sections of a given variety with property N_p^S. Obviously these questions could not be raised up for the complete embedding of projective varieties with property N_p because hyperplane sections and isomorphic projections of a linearly normal variety are not necessarily linearly normal. In general, it is hard to find defining equations or to control the degree bound of defining equations of a non-linearly normal projective variety X in $\mathbb{P}(W)$. More precisely, this paper is intended to investigate the following three problems:

1. Extend Theorem 1.1 in [9] for all $p \geq 0$. That is, for every $p \geq 0$, describe the geometric and algebraic meaning of property N_p^S.
2. Prove Theorem 1.2 in [9] without the assumption "$H^1(X, \mathcal{L}^{\ell}) = 0$ for $\ell \geq 2$".
3. Describe the syzygetic behavior of hyperplane sections for a variety with property N_p^S (a generalization of Theorem 3.b.7 in [8]).

The answers are given in Sects. 3, 4, and 5. Roughly speaking, our results describe basic properties of N_p^S, i.e. its relation to higher order normality, Castelnuovo–Mumford regularity, and property $N_{d,p}$(Theorem 3 and Corollary 2). Also the syzygetic behavior of isomorphic linear projections(Theorem 2) and hyperplane sections(Theorem 4) are provided. It should be mentioned that Theorem 2 and 3 are extensions of Theorems 1.1 and 1.2 in [9]. We actually remove all cohomological assumptions given in the Theorem 1.2 in [9]. Corollary 3 suggest an interesting answer for the following natural question:

($\star$) If a projective variety $X \subset \mathbb{P}^r$ has simple syzygy modules for first few steps, e.g., property N_p holds for some $p \geq 2$, then what can be expected for linear isomorphic projections $X \subset \mathbb{P}^{r-t}$?

The answer should be closely related to higher syzygy modules of the equations defining $X \subset \mathbb{P}^{r-t}$. For precise statements, we recall two definitions about the minimal free resolution of the homogeneous ideal. Let X be a non-degenerate projective variety admitting a minimal free solution;

$$\cdots \to L_i \to L_{i-1} \to \cdots \to L_1 \to S \to S_X \to 0$$

where $L_i = \bigoplus_j S(-i-j)^{\oplus \beta_{i,j}}$. $X \subset \mathbb{P}^r$ is said to be m-regular if one of the following conditions holds:

($R1$) $\beta_{i,j} = 0$ for all $j \geq m$. That is, the ith syzygy module L_i is generated by elements of degree $\leq m + i - 1$ for all $i \geq 1$.
($R2$) $H^i(\mathbb{P}^r, \mathcal{I}_X(m-i)) = 0$ for every $i \geq 1$.
($R3$) The truncation $(I_X)_{\geq m} = \oplus_{\ell \geq m} H^0(\mathbb{P}^r, \mathcal{I}_X(\ell))$ of I_X in degrees $\geq m$ admits a linear resolution: that is, $(I_X)_{\geq m}$ is generated by forms of degree m, and the maps in the minimal free resolution of $(I_X)_{\geq m}$ are given by matrices of linear forms.

For the equivalence of the three statements, see [5]. Note that the Castelnuovo–Mumford regularity is an invariant related to the whole minimal free resolution. On the other hand, the first few modules of syzygies of I_X may be simpler. And property $N_{d,p}$ is a criterion to express this. For $d \geq 2$ and $p \geq 1$, $X \subset \mathbb{P}^r$ is said to satisfy property $N_{d,p}$ if one of the following conditions holds:

($N_{d,p}$ 1) $\beta_{i,j} = 0$ for $1 \leq i \leq p$ and all $j \geq d$.
($N_{d,p}$ 2) The truncation $(I_X)_{\geq d} = \oplus_{\ell \geq d} H^0(\mathbb{P}^r, \mathcal{I}_X(\ell))$ of I_X in degrees $\geq d$ is generated in degree d and the minimal free resolution of $(I_X)_{\geq d}$ is linear until p steps. That is,

$$\cdots \to S^{m_p}(-d-p+1) \to \cdots \to S^{m_2}(-d-1) \to S^{m_1}(-d) \to (I_X)_{\geq d} \to 0.$$

Note that property $N_{2,p}$ is the same as property N_p if it is projectively normal. property $N_{d,1}$ clearly implies that I_X is generated by equations of degree at most d. For details, we refer the reader to see Remark 1.5 in [6]. With these notions in mind, we prove that if $X \subset \mathbb{P}^r$ satisfies property N_p, then every isomorphic projection $X \subset \mathbb{P}^{r-t}$, $1 \leq t \leq p-1$, is k-normal for all $k \geq t+1$ and satisfies property $N_{2+t,p-t}$. (Theorem 2, Corollaries 2 and 3). This suggests an answer for the question ($\star$). Recall that the Castelnuovo-Mumford regularity of $X \subset \mathbb{P}^{r-t}$ satisfies

$$\operatorname{Reg}(X) \geq \min\left\{ j+1 \mid H^i(X, \mathcal{O}_X(j-i)) = 0 \quad \text{for all } i \geq 1 \right\}.$$

For an example, if $K_X = \mathcal{O}_X$, then $\mathrm{Reg}(X) \geq \dim X + 1$. However, Corollary 3 gives that first a few steps of the minimal resolution are linear. Therefore our results say something new and interesting for first few steps of the syzygy modules of the non-linearly normal embedding $X \subset \mathbb{P}^{r-t}$.

For the second problem, we succeed in proving Theorem 1.2 without cohomological assumptions. See Theorem 2. Therefore our result implies Corollary 3.3 in [1]. We provide a couple of examples which can be explained only by Theorem 2.

The third problem is motivated by Green's Theorem (3.b.7) in [8] where he clarified the relation between the graded betti numbers of a linearly normal regular projective variety $X \subset \mathbb{P}^r$ and those of a general hyperplane section $Y \subset \mathbb{P}^{r-1}$. Indeed they have the same graded betti numbers if $H^1(X, \mathcal{O}_X(j)) = 0$ for all $j \geq 0$. In particular, $X \subset \mathbb{P}^r$ satisfies property N_p if and only if $Y \subset \mathbb{P}^{r-1}$ satisfies property N_p. In this paper we study the relation for projective varieties which is not necessarily linearly normal and not necessarily regular. In Theorem 4, we prove that if $X \subset \mathbb{P}^r$ satisfies property N_p^S, then a hyperplane section $Y \subset \mathbb{P}^{r-1}$ also satisfies property N_p^S under the assumption $H^1(X, \mathcal{O}_X(j)) = 0$ for all $j \geq 1$. More interesting result is Corollary 5. We prove that when $H^1(X, \mathcal{O}_X(j)) = 0$ for all $j \geq 1$ (in particular, X is not necessarily regular), if $X \subset \mathbb{P}^r$ satisfies property N_p, then a hyperplane section $Y \subset \mathbb{P}^{r-1}$ is k-normal for all $k \geq 2$ and satisfies property $N_{2,p}$. Briefly speaking, Green's Theorem still holds for irregular varieties except the linear normality.

Remark 1 In the present paper we study syzygies of the homogeneous coordinate ring S_X and the graded module $\bigoplus_{\ell \geq 0} H^0(X, \mathcal{O}_X(\ell))$ of twisted sections of $\mathcal{O}_X(1)$ in the case of non-complete embedding $X \subset \mathbb{P}^{r-t}$, $1 \leq t \leq p-1$ when $X \subset \mathbb{P}^r$ satisfies property N_p. One can easily construct examples such that $X \subset \mathbb{P}^{r-t}$ admits a $(t+2)$-secant line and hence it fails to satisfy property $N_{t+1,1}$. Therefore it does not make sense to improve our result for arbitrary isomorphic projections.

On the other hand, it seems very interesting and important to find out geometric conditions on the center of the projection such that $X \subset \mathbb{P}^{r-t}$ satisfies property $N_{\alpha,q}$, $\alpha \leq t+1$ (which is better than the uniform bound $N_{t+2,q}$ in Corollary 2) for some $q \geq 1$. Classically property $N_{2,q}$ has been paid attention because given varieties are cut out by quadrics and have many geometric properties. So it is quite natural to ask about property $N_{2,q}$ under isomorphic projections.

Recently the third author studied this problem when $X \subset \mathbb{P}^r$ is a smooth rational normal scroll and $t = 1$ [13,14]. For example, let $X \subset \mathbb{P}^r$ be a rational normal curve of degree r and let X^k denote the k^{th} higher secant variety of X. For a point $Q \in X^k \setminus X^{k-1}$, $k \geq 4$, $\pi_Q(X) \subset \mathbb{P}^{r-1}$ satisfies property $N_{2,p}$ if and only if $p \leq k-3$. Note that for a point $Q \in X^3 \setminus Sec(X)$, $\pi_Q(X)$ has a trisecant line in $\mathbb{P}^{r-1}$. In particular, the converse of Theorem 1.1 in [6] holds for $\pi_Q(X)$(see Theorem 1.1 and Corollary 1.2 in [13]). For higher dimensional case, the problem is more complicated. In this case, the converse of Theorem 1.1 in [6] fails to hold and the graded betti numbers of $\pi_Q(X) \subset \mathbb{P}^{r-1}$ depend on a smooth rational normal scroll $Y \subset \mathbb{P}^{r-1}$ containing $\pi_Q(X)$ as a divisor. For details, see Theorem 1.3 and Example 1 in [14].

Notations

Throughout this paper all varieties are defined over an algebraically closed field K. For a finite dimensional K-vector space V, $\mathbb{P}(V)$ is the projective space of one-dimensional quotients of V.

2 Minimal free resolution and Koszul cohomology

2.1 Long exact sequence for Koszul cohomology

For a finite dimensional vector space V of dimension $r+1$ over an algebraically closed field K, we form the symmetric algebra

$$S := \oplus_{\ell \geq 0} Sym^{\ell}(V).$$

For a given graded S-module $B = \oplus_{\ell \geq 0} B_{\ell}$, the Koszul complex

$$\cdots \to \bigwedge^{p+1} V \otimes B_{q-1} \to \bigwedge^{p} V \otimes B_q \to \bigwedge^{p-1} V \otimes B_{q+1} \to \cdots$$

gives the homology group $\mathcal{K}_{p,q}(B, V)$ which is called the *Koszul cohomology*. On the other hand, consider the minimal graded free resolution

$$\cdots \to \oplus_j S(-i-j)^{\beta_{i,j}} \to \cdots \to \oplus_j S(-1-j)^{\beta_{1,j}} \to \oplus_j S(-j)^{\beta_{0,j}} \to B \to 0$$

of B as a graded S-module. By the elementary property of the Tor-functor, we have

$$\dim_K \mathcal{K}_{p,q}(B, V) = \dim_K \mathrm{Tor}_p^{p+q}(B, K) = \beta_{p,q}.$$

Also there is a long exact sequence for Koszul cohomology:

Proposition 1 (Green [8]) *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of graded S-modules with degrees preserving maps. Then there is a long exact sequence*

$$\begin{aligned} \cdots &\to \mathcal{K}_{1,q-1}(A, V) \to \mathcal{K}_{1,q-1}(B, V) \to \mathcal{K}_{1,q-1}(C, V) \\ &\to \mathcal{K}_{0,q}(A, V) \to \mathcal{K}_{0,q}(B, V) \to \mathcal{K}_{0,q}(C, V) \to 0. \end{aligned}$$

Proof Since $\mathcal{K}_{p,q}(-, V) \cong \mathrm{Tor}_p^{p+q}(-, K)$, the assertion comes immediately from the long exact sequence of Tor-functor. □

Now we turn to geometric cases. Let $X \subset \mathbb{P}(V)$ be a projective subvariety. By applying Proposition 1 to the short exact sequence

$$0 \to I_X \to S \to S_X \to 0$$

where I_X is the homogeneous ideal of X and S_X is the homogeneous coordinate ring of X, one can easily check that $X \subset \mathbb{P}(V)$ satisfies property $N_{d,p}$ if and only if

$$\mathcal{K}_{i,j}(S_X, V) = 0 \quad \text{for} \quad 0 \leq i \leq p \quad \text{and all} \quad j \geq d.$$

This criterion will be used in the proof of Theorem 3.

2.2 Koszul cohomology of coherent sheaves

Let $\mathcal{F}$ be a non-zero coherent sheaf on $\mathbb{P}(V)$. Consider the associated graded S-module

$$F = \bigoplus_{\ell \geq 0} H^0\big(\mathbb{P}(V), \mathcal{F}(\ell)\big).$$

There is the following general connection between syzygies of F and some cohomology groups related to $\mathcal{F}$.

Theorem 1 (Theorem 5.8 [4]) *There is an exact sequence*

$$0 \to \mathcal{K}_{i,j}(F, V) \to H^1(\mathbb{P}(V), \bigwedge^{i+1} \mathcal{M} \otimes \mathcal{F}(j-1)) \to \bigwedge^{i+1} V \otimes H^1(\mathbb{P}(V), \mathcal{F}(j-1))$$

where $\mathcal{M} = \Omega_{\mathbb{P}(V)}(1)$.

Corollary 1 *Let* $X \subset \mathbb{P}(V)$ *be a projective variety satisfying* k*-normality for all* $k \geq 2$. *Then* $X \subset \mathbb{P}(V)$ *satisfies property* $N_{2,p}$ *if and only if*

$$H^1(\mathbb{P}(V), \bigwedge^{i} \mathcal{M} \otimes \mathcal{I}_X(j)) = 0 \quad \text{for} \quad 1 \leq i \leq p \quad \text{and} \quad j \geq 2.$$

Proof By definition property $N_{2,p}$ holds if and only if

$$\mathcal{K}_{i,j}(I_X, V) = 0 \quad \text{for} \quad 0 \leq i \leq p-1 \quad \text{and} \quad j \geq 3.$$

Since $X \subset \mathbb{P}(V)$ is k-normal for all $k \geq 2$, the exact sequence in Theorem 1 implies that

$$\mathcal{K}_{i,j}(I_X, V) \cong H^1\big(\mathbb{P}(V), \bigwedge^{i+1} \mathcal{M} \otimes \mathcal{I}_X(j-1)\big)$$

for all $j \geq 3$. This completes the proof. □

3 Isomorphic projections

As one can find in [9], it is very interesting to consider the syzygetic behavior of isomorphic projections with respect to property N_p^S. The aim of this section is to reprove Theorem 1.2 in [9] without any cohomological assumption.

Theorem 2 *Let* $X \subset \mathbb{P}^r$ *be a reduced non-degenerate projective variety which satisfies property* N_p^S. *If* $X \subset \mathbb{P}^{r-t}$ *is an isomorphic linear projection where* $0 \leq t \leq p$, *then* $X \subset \mathbb{P}^{r-t}$ *satisfies property* N_{p-t}^S.

Proof Clearly it suffices to show the statement when $t = 1$. When $X \subset \mathbb{P}^{r-1}$ is an isomorphic linear projection of $X \subset \mathbb{P}^r$, we use the following notations:

- $V = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$
- $W = H^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(1))$
- $0 \to \mathcal{M}_W \to W \otimes \mathcal{O}_X \to \mathcal{O}_X(1) \to 0$: the restriction of the Euler sequence on $\mathbb{P}^{r-1}$ to X

Then we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{M}_W & \to & W \otimes \mathcal{O}_X & \to & \mathcal{O}_X(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{M}_V & \to & V \otimes \mathcal{O}_X & \to & \mathcal{O}_X(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_X & = & \mathcal{O}_X & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & & .
\end{array}
$$

Now consider the following commutative diagram induced from the above one:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bigwedge^{i+1} \mathcal{M}_W & \to & \bigwedge^{i+1} \mathcal{M}_V & \to & \bigwedge^{i} \mathcal{M}_W & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bigwedge^{i+1} W \otimes \mathcal{O}_X & \to & \bigwedge^{i+1} V \otimes \mathcal{O}_X & \to & \bigwedge^{i} W \otimes \mathcal{O}_X & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bigwedge^{i} \mathcal{M}_W \otimes \mathcal{O}_X(1) & \to & \bigwedge^{i} \mathcal{M}_V \otimes \mathcal{O}_X(1) & \to & \bigwedge^{i-1} \mathcal{M}_W \otimes \mathcal{O}_X(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

This gives the following commutative diagram of cohomology groups:

$$
\begin{array}{ccccccc}
\bigwedge^{i+1} V \otimes H^0(\mathcal{O}_X(j-1)) & \xrightarrow{\gamma_{i,j}} & \bigwedge^{i} W \otimes H^0(\mathcal{O}_X(j-1)) & \to & 0 & & \\
{\scriptstyle\alpha_{i,j}}\downarrow & & \downarrow{\scriptstyle\beta_{i,j}} & & & & \\
H^0(\bigwedge^{i} \mathcal{M}_V(j)) & \xrightarrow{\delta_{i,j}} & H^0(\bigwedge^{i-1} \mathcal{M}_W(j)) & \to & H^1(\bigwedge^{i} \mathcal{M}_W(j)) & \xrightarrow{\eta_{i,j}} & H^1(\bigwedge^{i} \mathcal{M}_V(j)) \\
\downarrow & & \downarrow & & & & \\
0 & & H^1(\bigwedge^{i} \mathcal{M}_W(j-1)) & & & & \\
 & & \downarrow{\scriptstyle\xi_{i,j}} & & & & \\
 & & \bigwedge^{i} W \otimes H^1(\mathcal{O}_X(j-1)) & & & &
\end{array}
$$

From this diagram, observe the followings by following:

1. $\alpha_{i,j}$ is surjective for $0 \leq i \leq p$ and $j \geq 2$ since property N_p^S holds.
2. $\gamma_{i,j}$ is always surjective by linear algebra.

3. For $0 \le i \le p$ and $j \ge 2$, the following holds:

$$\begin{aligned} \beta_{i,j} \text{ is surjective.} &\iff \delta_{i,j} \quad \text{is surjective.} \\ &\iff \eta_{i,j} \quad \text{is injective.} \\ &\iff \xi_{i,j} \quad \text{is injective.} \end{aligned}$$

4. $X \subset \mathbb{P}^{r-1}$ satisfies property N^S_{p-1} if and only if

$$\xi_{i,j} \text{ is injective for all } 1 \le i \le p \text{ and } j \ge 2.$$

Also we have the following commutative diagram of cohomology groups:

$$\begin{array}{ccc} H^1\big(X, \bigwedge^i \mathcal{M}_W(j)\big) & \overset{\eta_{i,j}}{\to} & H^1\big(X, \bigwedge^i \mathcal{M}_V(j)\big) \\ {\scriptstyle \xi_{i,j+1}}\big\downarrow & & {\scriptstyle \rho_{i,j+1}}\big\downarrow \\ \bigwedge^i W \otimes H^1\big(X, \mathcal{O}_X(j)\big) & \overset{\mu_{i,j}}{\to} & \bigwedge^i V \otimes H^1\big(X, \mathcal{O}_X(j)\big) \end{array}$$

From this diagram, observe the followings:

1. $\rho_{i,j+1}$ is injective for $1 \le i \le p+1$ and $j \ge 1$ since property N^S_p holds.
2. $\mu_{i,j}$ is always injective by linear algebra.

Thus for $1 \le i \le p+1$ and $j \ge 1$,

$$\eta_{i,j} \text{ is injective if and only if } \xi_{i,j+1} \text{ is injective.}$$

Note that $\xi_{i,j+1}$ is injective if j is sufficiently large and hence the following holds for $1 \le i \le p$:

$$\begin{aligned} \xi_{i,j+1} \text{ is injective.} &\Longrightarrow \eta_{i,j} \quad \text{and} \quad \xi_{i,j} \text{ are injective.} \\ &\Longrightarrow \eta_{i,j-1} \quad \text{and} \quad \xi_{i,j-1} \text{ are injective.} \\ &\cdots \qquad \cdots\cdots\cdots \\ &\Longrightarrow \eta_{i,2} \quad \text{and} \quad \xi_{i,2} \text{ are injective.} \end{aligned}$$

Therefore it is proved that $\xi_{i,j}$ is injective for all $1 \le i \le p$ and $j \ge 2$. □

4 Algebraic and geometric meanings of property N^S_p

From Theorem 1.1 in [9], a reduced non-degenerate projective variety $X \subset \mathbb{P}^r$ satisfying property N^S_1 is k-normal for all $k \ge t+1$ where $t = h^0(X, \mathcal{O}_X(1)) - r - 1$. Also the homogeneous ideal I_X is generated by forms of degree $\le t+2$. The aim of this section is to prove some algebraic and geometric meanings of the properties N^S_p for all $p \ge 1$. Our main result is the following:

Theorem 3 *Let $X \subset \mathbb{P}^r$ be a reduced non-degenerate projective variety. Assume that*

(i) *property N^S_p holds for some $p \ge 1$, and*
(ii) *k-normality holds for all $k \ge k_0$.*

Then $X \subset \mathbb{P}^r$ satisfies property $N_{k_0+1,p}$. In particular, the homogeneous ideal I_X of X is generated by forms of degree $\le k_0 + 1$.

Proof Since X is reduced, we have the short exact sequence

$$(*) \quad 0 \to S_X \to R \to M \to 0$$

of graded S-modules where $S_X = S/I_X$ is the homogeneous coordinate ring of $X \subset \mathbb{P}^r$,

$$R = \bigoplus_{\ell \in \mathbb{Z}} H^0(X, \mathcal{O}_X(\ell))$$

is the graded ring of twisted global sections of $\mathcal{O}_X(1)$, and

$$M = \bigoplus_{\ell \in \mathbb{Z}} H^1(\mathbb{P}^r, \mathcal{I}_X(\ell))$$

is the Hartshorne–Rao module. The basic idea is that if we know some information of two terms in the short exact sequence $(*)$, then we can obtain some property of the rest term. Since k-normality holds for all $k \geq k_0$, we have

$$M = H^1\big(\mathbb{P}^r, \mathcal{I}_X(1)\big) \oplus H^1\big(\mathbb{P}^r, \mathcal{I}_X(2)\big) \oplus \cdots \oplus H^1\big(\mathbb{P}^r, \mathcal{I}_X(k_0 - 1)\big).$$

In particular, $\mathcal{K}_{i,j}(M, V) = 0$ for all $i \geq 0$ and $j \geq k_0$. The property N_p^S of $X \subset \mathbb{P}^r$ guarantees that

$$\mathcal{K}_{i,j}(R, V) = 0 \quad \text{for} \quad 0 \leq i \leq p \quad \text{and all} \quad j \geq 2.$$

Therefore we obtain the desired vanishing of $\mathcal{K}_{i,j}(S_X, V)$ from the long exact sequence for Koszul cohomology group in Proposition 1. □

As we already mentioned in the beginning of this section, property N_1^S gives useful information about higher order normality. Thus Theorem 3 implies the following Corollary 2 which refines Theorem 1.1 in [9].

Corollary 2 *Let $X \subset \mathbb{P}^r$ be a reduced non-degenerate projective variety which satisfies property N_p^S for some $p \geq 1$. Let $t := h^0(X, \mathcal{O}_X(1)) - r - 1$. Then the following holds:*

(i) *X is k-normal for all $k \geq t + 1$.*
(ii) *X is $\max\{t+2, m+1\}$-regular where m is the regularity of $\mathcal{O}_X$ with respect to $\mathcal{O}_X(1)$.*
(iii) *The minimal free resolution of I_X satisfies property $N_{t+2,p}$. In particular, I_X is generated by equations of degree at most $t + 2$.*

Proof By Theorem 1.1 in [9], we already know the followings:

1. $X \subset \mathbb{P}^r$ is k-normal for all $k \geq t + 1$.
2. I_X is generated by forms of degree $\leq t + 2$.
3. $X \subset \mathbb{P}^r$ is $\max\{m + 1, t + 2\}$-regular.

We would like to point out that the proof of Theorem 1.1 in [9] is available for arbitrary reduced projective varieties. Thus it remains to prove that property $N_{t+2,p}$ holds. And this is immediately checked by using Theorem 3 because the property N_p^S holds for X and X is k-normal for $k \geq t + 1$ □

By combining Theorem 2 and Corollary 2, we have the following result for isomorphic linear projections of projectively normal varieties.

Corollary 3 *Let $X \subset \mathbb{P}^r$ be a non-degenerate projective variety satisfying property N_p for some $p \geq 2$. Then for every linear isomorphic projection $X \subset \mathbb{P}^{r-t}$, $1 \leq t \leq p - 1$,*

(*a*) *k-normality holds for all* $k \geq t+1$,
(*b*) $I_{X/\mathbb{P}^{r-t}}$ *is generated by forms of degree* $\leq t+2$, *and*
(*c*) *the minimal free resolution of* $I_{X/\mathbb{P}^{r-t}}$ *satisfies property* $N_{t+2,p-t}$.

Proof By Theorem 2, $X \subset \mathbb{P}^{r-t}$ satisfies property N^S_{p-t}. Thus the assertion comes immediately from Corollary 2. □

As an example, for a variety $X \subset \mathbb{P}^r$ satisfying property N_2, one point isomorphic projection in $\mathbb{P}^{r-1}$ is k-normal for all $k \geq 2$ which is already known by Alzati and Russo (Corollary 3.3 in [1]). However Corollary 3. (*b*) and (*c*) say something new for defining equations and their first few syzygies. Here we give $X \subset \mathbb{P}^r$ which satisfies property N_p for some $p \geq 2$ and $H^1(X, \mathcal{L}^{\otimes 2}) \neq 0$.

Example 1 (1) Let $X = \mathbb{G}(k,n) \subset \mathbb{P}^N$ be the Plüker embedding of the Grassmannian which parameterize $\mathbb{P}^k$ in $\mathbb{P}^n$. Then X satisfies property N_2 ([6], Remark 3.9). Let $C \subset \mathbb{P}^r$ be a linear curve section of X. Then it satisfies property N_2 by Green's Theorem (3.b.7) in [8]. By Corollary 3, every isomorphic projection $C' \subset \mathbb{P}^{r-1}$ is k-normal for $k \geq 2$ and the homogeneous ideal $I_{C'}$ is generated by quadrics and cubics and satisfies property $N_{3,1}$. Note that the regularity of C' is bigger than or equal to 5 if $k(n-k-1) \geq 4$. This result can not be covered by Theorem 1.2 in [9] because $K_{C'} = \mathcal{O}_{C'}(kn - k^2 - k - 2)$.

(2) By the same argument, if one can consider the linear curve section $C \subset \mathbb{P}^r$ of the Segre embedding of $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$, $m, n \geq 4$, then C satisfies property N_3 ([6], Remark 3.7) and $H^1(C, \mathcal{O}_C(2)) \neq 0$. The isomorphic projection $C' \subset \mathbb{P}^{r-2}$ is k-normal for $k \geq 3$ and the homogeneous ideal I_C is generated by forms of degree less than or equal to 4 while the regularity of C' is bigger than or equal to 5.

Corollary 4 *Let X be a projective variety and let* $L \in PicX$ *be very ample such that* $X \subset \mathbb{P}H^0(X,L)$ *satisfies property* N_p, $p \geq 2$. *Suppose that there exists a very ample linear subsystem* $V \subset H^0(X,L)$ *of codimension* $c \leq p-1$ *such that* $X \subset \mathbb{P}(V)$ *is 2-normal. Then for general* $W \subset H^0(X,L)$ *of codimension* $e \leq c$, *the isomorphic projection*

$$X \subset \mathbb{P}(W)$$

is j*-normal for all* $j \geq 2$ *and property* $N_{3,p-e}$ *holds.*

Proof First recall that for a fixed codimension c, the 2-normality is an open condition on the Grassmannian of c-codimensional subspaces of $H^0(X,L)$. Also note that if $X \subset \mathbb{P}(W_0)$ is 2-normal for a subspace $W_0 \subset H^0(X,L)$, then for all $W_0 \subset W \subset H^0(X,L)$, the corresponding embedding $X \subset \mathbb{P}(W)$ is also 2-normal. So we know that for general $W \subset H^0(X,L)$ of codimension $= e \leq c$,

$$X \subset \mathbb{P}(W)$$

is 2-normal. This guarantees that j-normality holds for all $j \geq 2$ since $X \subset \mathbb{P}H^0(X,L)$ is projectively normal (see Lemma 1.1 in [1]). Since property N^S_{p-e} holds for $X \subset \mathbb{P}(W)$, our assertion comes from Theorem 3. □

Remark 2 Property N^S_0 is closely related to the generating structure of the Hartshorne–Rao module. Let S be the homogeneous coordinate ring of $\mathbb{P}^r$. Then a non-degenerate projective variety $X \subset \mathbb{P}^r$ satisfies property N^S_0 if and only if the Hartshorne–Rao module

$$M = \bigoplus_{k \in \mathbb{Z}} H^1\left(\mathbb{P}^r, \mathcal{I}_X(k)\right)$$

is generated by $H^1(\mathbb{P}^r, \mathcal{I}_X(1))$ as a graded S-module. More precisely, consider the commutative diagram

$$\begin{array}{ccc}
H^0\big(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)\big) \otimes H^0\big(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)\big) & \to & H^0\big(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k+1)\big) \to 0 \\
\downarrow & & \downarrow \\
H^0\big(X, \mathcal{O}_X(1)\big) \otimes H^0\big(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)\big) & \stackrel{\alpha_k}{\to} & H^0\big(X, \mathcal{O}_X(k+1)\big) \\
\downarrow & & \downarrow \\
H^1\big(\mathbb{P}^r, \mathcal{I}_X(1)\big) \otimes H^0\big(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)\big) & \stackrel{\beta_k}{\to} & H^1\big(\mathbb{P}^r, \mathcal{I}_X(k+1)\big) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}.$$

Note that α_k is surjective if and only if β_k is surjective. Clearly the Hartshorne–Rao module M is generated by $H^1(\mathbb{P}^r, \mathcal{I}_X(1))$ if and only if β_k is surjective for all $k \geq 1$. Also from Lemma 3.2 in [9] one can check that property N_0^S holds if and only if α_k is surjective for all $k \geq 1$.

5 Hyperplane sections

In his paper [8], Green showed that the graded betti numbers of homogeneous coordinate rings $S(X)$ and $S(Y)$ are same for a general hyperplane section $Y = X \cap H \subset \mathbb{P}^{r-1}$ if $X \subset \mathbb{P}^r$ is projectively normal and $H^1(X, \mathcal{O}_X(j)) = 0$ for all $j \geq 0$. In particular, $(X, \mathcal{O}_X(1))$ satisfies property N_p if and only if $(Y, \mathcal{O}_Y(1))$ satisfies property N_p. We generalize this to (1) non-complete embedding of projective varieties, and (2) irregular varieties, i.e., $H^1(X, \mathcal{O}_X) \neq 0$. Note that in this generalization the converse is not true as the following Remark 3 shows.

Theorem 4 *Let X be a projective variety of* dim ≥ 2 *and let $\mathcal{L} \in PicX$ be very ample with $H^1(X, \mathcal{L}^j) = 0$ for all $j \geq 1$. For an $(r+1)$-dimensional subspace $V \subset H^0(X, \mathcal{L})$, if $X \subset \mathbb{P}(V) = \mathbb{P}^r$ satisfies property N_p^S, then a general hyperplane section $Y \subset \mathbb{P}^{r-1}$ also satisfies property N_p^S.*

Proof Let $W = H^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(1))$ and let S be the homogeneous coordinate ring of $\mathbb{P}(V)$. From the short exact sequence $0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$, we have the following diagram

$$\begin{array}{c}
0 \\
\downarrow \\
0 \to E_X(-1) \to E_X \to F \to 0 \\
\downarrow \\
E_Y \\
\downarrow \\
G \\
\downarrow \\
0
\end{array}$$

of graded S-modules with maps preserving the gradings where

$$E_X = \oplus_{k\in\mathbb{Z}} H^0\big(X, \mathcal{O}_X(k)\big) \quad E_Y = \oplus_{k\in\mathbb{Z}} H^0\big(Y, \mathcal{O}_Y(k)\big), \quad \text{and } G = 0 \oplus H^1(X, \mathcal{O}_X) \oplus 0 \oplus \cdots .$$

For the long exact sequence

$$\cdots \to \mathcal{K}_{i,j}(E_X, V) \to \mathcal{K}_{i,j}(F, V) \to \mathcal{K}_{i-1,j+1}\big(E_X(-1), V\big) \to \cdots ,$$

$\mathcal{K}_{i,j}(E_X, V) = 0$ if $0 \leq i \leq p$ and $j \geq 2$, and $\mathcal{K}_{i-1,j+1}(E_X(-1), V) = \mathcal{K}_{i-1,j}(E_X, V) = 0$ for $1 \leq i \leq p+1$ and all $j \geq 2$.

Thus $\mathcal{K}_{i,j}(F, V) = 0$ if $0 \leq i \leq p$ and $j \geq 2$. Also for the long exact sequence

$$\cdots \to \mathcal{K}_{i,j}(F, V) \to \mathcal{K}_{i,j}(E_Y, V) \to \mathcal{K}_{i,j}(G, V) \to \cdots ,$$

$\mathcal{K}_{i,j}(G, V) = 0$ for all $j \geq 2$. Thus we have $\mathcal{K}_{i,j}(E_Y, V) = 0$ if $0 \leq i \leq p$ and $j \geq 2$. Since $\mathcal{K}_{i,j}(E_Y, V) = \mathcal{K}_{i,j}(E_Y, W) \oplus \mathcal{K}_{i-1,j}(E_Y, W)$, $Y \subset \mathbb{P}^{r-1}$ satisfies property N_p^S. □

Remark 3 Unfortunately the converse of Theorem 4 does not hold. Let X be a ruled surface over an elliptic curve C with the numerical invariant e and let $L \equiv C_0 + bf$ with $b \geq 3 + e$. Then $X \subset \mathbb{P}H^0(X, L) \cong \mathbb{P}^{2b-e-1}$ satisfies property N_p if and only if $p \leq b - e - 3$. See Theorem 1.4 in [12]. Now let $Y \subset X$ be a hyperplane section. Then $Y \cong C$ and $\deg(L|_Y) = 2b - e$. Thus $Y \subset \mathbb{P}H^0(Y, L|_Y) \cong \mathbb{P}^{2b-e-1}$ satisfies property N_p if and only if $p \leq 2b - e - 3$. That is, property N_p of $(Y, L|_Y)$ does not imply property N_p of X. Also $Y \subset \mathbb{P}^{2b-e-2}$, an hyperplane section of $X \subset \mathbb{P}^{2b-e-1}$, satisfies property N^S_{2b-e-4} by Theorem 2. Therefore property N_p^S of a hyperplane section does not imply property N_p of X.

Corollary 5 *Let X be as above Theorem 4 and $t = h^0(X, \mathcal{L}) - \dim V$. If $t = 0$, then a general hyperplane section $Y \subset \mathbb{P}^{r-1}$ satisfies property $N_{2,p}$. If $t \geq 1$, then a general hyperplane section $Y \subset \mathbb{P}^{r-1}$ satisfies property $N_{t+2,p}$.*

Proof Assume $t = 0$, i.e. $V = H^0(\mathcal{L})$. Since X is projectively normal, the betti numbers of I_X and I_Y are all the same ([10] Theorem 1.3.6). Therefore, $Y \subset \mathbb{P}^{r-1}$ satisfies $N_{2,p}$.

For $t \geq 1$, Theorem 2 gives that X is k-normal for all $k \geq t+1$, whence Y is k-normal for all $k \geq t+1$ by the exact sequence $0 \to \mathcal{I}_{X/\mathbb{P}^r} \to \mathcal{I}_{Y/\mathbb{P}^r} \to \mathcal{O}_X(-1) \to 0$. Furthermore, $Y \subset \mathbb{P}^{r-1}$ satisfies property N_p^S by Theorem 4. Therefore, Y satisfies $N_{t+2,p}$ by Theorem 3. □

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References

1. Alzati, A., Russo, F.: On the k-normality of projected algebraic varieties. Bull. Braz. Math. Soc. (N.S.) **33**(1), 27–48 (2002)
2. Birkenhake, C.: Noncomplete linear systems on abelian varieties. Trans. Am. Math. Soc. **348**(5), 1885–1908 (1996)
3. Castelnuovo, G.: Sui multipli di une serie lineare di gruppi di punti appartenente ad une curva algebraic. Rend. Circ. Mat. Palermo **7**(2), 89–110 (1893)
4. Eisenbud, D.: The Geometry of Syzygies, vol. 229. Springer, New York (2005)
5. Eisenbud, D., Goto, S.: Linear free resolutions and minimal multiplicity. J. Algebra **88**, 89–133 (1984)
6. Eisenbud, D., Green, M., Hulek, K., Popescu, S.: Restriction linear syzygies: algebra and geometry. Compos. Math. **141**, 1460–1478 (2005)

7. Ein, L., Lazarsfeld, R.: Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension. Invent. Math. **111**, 51–67 (1993)
8. Green, M.: Koszul cohomology and the geometry of projective varieties. J. Differ. Geom. **19**, 125–171 (1984)
9. Kwak, S., Park, E.: Some effects of property N_p on the higher normality and defining equations of nonlinearly normal varieties. J. Reine Angew. Math. **582**, 87–105 (2005)
10. Migliore, J.: Introduction to liasion theory and deficiency modules. Birkhäuser (1998)
11. Mumford, D.: Lectures on curves on an algebraic surface. Ann. Math. Stud. **59**, (1966)
12. Park, E.: On higher syzygies of ruled surfaces. Trans. Am. Math. Soc. **358**, 3733–3749 (2006)
13. Park, E.: Projectivce curves of degree=codimension+2. Math. Zeit. **256**, 685–697 (2007)
14. Park, E.: Smooth varieties of almost minimal degree. J. Algebra (2007), doi:10.1016/j.jalgebra.2007.02.027