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# Some effects of property $N_p$ on the higher normality and defining equations of nonlinearly normal varieties

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**Abstract.** For a smooth projective variety $X \subset \mathbb{P}^r$ embedded by the complete linear system, Property $N_p$ has been studied for a long time ([5], [11], [12], [7] etc.). On the other hand, Castelnuovo-Mumford regularity conjecture and related problems have been focused for a projective variety which is not necessarily linearly normal ([2], [13], [15], [17], [20] etc.). This paper aims to explain the influence of Property $N_p$ on higher normality and defining equations of a smooth variety embedded by a sub-linear system. Also we prove a claim about Property $N_p$ of surface scrolls which is a generalization of Green's work in [11] about Property $N_p$ for curves.

## 1. Introduction

When a projective variety $X$ is embedded in a projective space $\mathbb{P}^r$, there are various natural interesting questions regarding the syzygy modules of the saturated ideal $I_X$ and the finitely generated graded module $\bigoplus_{\ell \in \mathbb{Z}} H^0\big(X, \mathcal{O}_X(\ell)\big)$. For a linearly normal variety $X \subset \mathbb{P}^r$, finding conditions to guarantee that $X \hookrightarrow \mathbb{P}H^0(X, L)$ is projectively normal or cut out by quadrics has expanded to the condition for numerical type of higher syzygies of the homogeneous coordinate ring $S(X)$, i.e. Property $N_p$ (Castelnuovo [6], Mumford [19], Green [11], Ein-Lazarsfeld [7], etc.). On the other hand, for a projective variety $X \subset \mathbb{P}^r$ which is not necessarily linearly normal, the classical problem of finding an upper bound of $n$ satisfying

(1) hypersurfaces of degree $n$ cut out a complete linear system on $X$,

(2) $X$ is cut out in $\mathbb{P}^r$ by hypersurfaces of degree $n$, and the homogeneous ideal of $X$ is generated in degrees $\geqq n$ by its component of degree $n$,

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has been reformulated as finding the upper bound of the regularity of the ideal sheaf $\mathscr{I}_X$ of $X \subset \mathbb{P}^r$ in terms of degree, codimension and other projective invariants (Castelnuovo [6], Mumford [18], Eisenbud-Goto [10], etc.).

For precise statements, we give notations and recall definitions. Let $X$ be a complex projective variety of dimension $n$ and let $\mathscr{L}$ be a very ample line bundle on $X$. Consider an embedding $X \subset \mathbb{P}(V)$ where $V \subset H^0(X, \mathscr{L})$ is a subsystem of dimension $r+1$. We use the following usual notations:

- $S = \operatorname{Sym}^\bullet V$: the homogeneous coordinate ring of $\mathbb{P}(V) = \mathbb{P}^r$,
- $I_X \subset S$: the homogeneous ideal of $X$,
- $S(X) = S/I_X$: the homogeneous coordinate ring of $X$,
- $\mathscr{O}_{\mathbb{P}^r}$, $\mathscr{O}_X$: the structure sheaves of $\mathbb{P}^r$ and $X$, respectively,
- $\mathscr{I}_X \subset \mathscr{O}_{\mathbb{P}^r}$: the sheaf of ideals of $X$.

Now let us consider the minimal free resolution of $S(X)$:

$$0 \to L_{r+1} \to \cdots \to L_i \overset{\varphi_i}{\to} L_{i-1} \to \cdots \to L_1 \overset{\varphi_1}{\to} L_0 \overset{\varphi_0}{\to} S(X) \to 0$$

where $L_i$, as free graded $S$-module, can be written as

$$L_i = \bigoplus_j S^{k_{i,j}}(-i-j).$$

Note that $L_0 = S$ and the image of $\varphi_1$ is $I_X$. First recall the definitions of Property $N_p$ and Castelnuovo-Mumford regularity:

**Definition 1.1.** (1) For a positive integer $p \geqq 1$, $\mathscr{L}$ is said to satisfy Property $N_p$ if for $V = H^0(X, \mathscr{L})$, a linearly normal embedding $X \subset \mathbb{P}(V)$ is projectively normal, i.e., $S(X) = \bigoplus_{\ell \in \mathbb{Z}} H^0(X, \mathscr{L}^\ell)$, and

$$k_{i,j} = 0 \quad \text{for all } 0 \leqq i \leqq p \text{ and } j \geqq 2.$$

(2) $X \subset \mathbb{P}(V)$, is said to be $m$-regular if we have $k_{i,j} = 0$ for all $j \geqq m$, that is, one can find a basis of $L_i$ with elements of degree at most $m + i - 1$.

Property $N_p$ means that the first $(p+1)$-th modules of syzygies of $S(X)$ are as simple as possible or more precisely the resolution is linear until the $p$-th stage. So, remark that $\mathscr{L}$ satisfies Property $N_0$ if and only if $\mathscr{L}$ is normally generated (i.e., projectively normal), $\mathscr{L}$ satisfies Property $N_1$ if and only if $\mathscr{L}$ satisfies Property $N_0$ and the homogeneous ideal is generated by quadrics, $\mathscr{L}$ satisfies Property $N_2$ if and only if $\mathscr{L}$ satisfies Property $N_1$ and the relations among the quadrics are generated by the linear relations and so on.

For Castelnuovo-Mumford regularity, there is another definition using vanishing of certain cohomology groups due to D. Mumford. In fact, as D. Mumford defined, $X$ is said to be $m$-regular if $H^i\big(\mathbb{P}^r, \mathscr{I}_X(m-i)\big) = 0$ for every $i \geqq 1$. Recall ([18], Lecture 14) that if $X$ is $m$-regular, then $X \subset \mathbb{P}^r$ is $k$-normal for all $k \geqq m-1$ and $\mathscr{I}_X(m)$ is globally generated. Later D. Eisenbud and S. Goto gave the definition of regularity for graded $S$-modules as above and proved the equivalence of them [10].

Now we generalize Property $N_p$ to non-complete embedding. Let $X$ be a complex projective variety of dimension $n$ and let $\mathscr{L}$ be a very ample line bundle on $X$. For an isomorphic embedding

$$X \hookrightarrow \mathbb{P}(V)$$

where $V \subset H^0(X, \mathscr{L})$ a subsystem of codimension $t$, let $S$ be the homogeneous coordinate ring of $\mathbb{P}(V)$. In order to adjust the Koszul cohomology methods to noncomplete embeddings, let us consider the graded $S$-module $E = \bigoplus_{\ell \in \mathbb{Z}} H^0(X, \mathscr{L}^\ell)$. We consider the situations in which the first few modules of syzygies of $E$ are as simple as possible:

**Definition 1.2.** Assume that $E$ has a minimal free resolution of the form

$$\cdots \xrightarrow{\varphi_{i+1}} \bigoplus_j S^{k_{i,j}}(-i-j) \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_1} \bigoplus_j S^{k_{0,j}}(-j) \xrightarrow{\varphi_0} E \longrightarrow 0.$$

For an integer $p \geqq 0$, the embedding $X \hookrightarrow \mathbb{P}(V)$ is said to satisfy Property $N_p^S$ if $k_{i,j} = 0$ for $0 \leqq i \leqq p$ and $j \geqq 2$.

Property $N_p^S$ means that $E$ admits a minimal free resolution of the form

$$\cdots \to S^{m_p}(-p-1) \to \cdots \to S^{m_2}(-3) \to S^{m_1}(-2) \to S \oplus S^t(-1) \to E \to 0.$$

See also the following table of Betti numbers:

<table>
<tr><td rowspan="5">$k_{i,j} = 0$<br>for all $j \leqq -1$</td><td>$k_{0,0} = 1$</td><td>$k_{0,1}$</td><td rowspan="4">$k_{i,j} = 0$<br>for all<br>$0 \leqq i \leqq p$<br>and $j \geqq 2$</td></tr>
<tr><td rowspan="4">$k_{i,0} = 0$<br>for all $i \geqq 1$</td><td>$k_{1,1}$</td></tr>
<tr><td>$\vdots$</td></tr>
<tr><td>$k_{p,1}$</td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td></tr>
</table>

Table 1. Betti numbers when Property $N_p^S$ holds.

Here $k_{0,1} = t$ and hence for linearly normal embedding, Property $N_p^S$ is equal to Property $N_p$. This paper is intended to investigate the following three problems:

(1) the influence of Property $N_p$ on higher syzygies of $X$ embedded by subsystems (i.e., Property $N_q^S$ for $q \leqq p-1$),

(2) the influence of Property $N_1^S$ on higher normality and defining equations,

(3) Property $N_p$ for surface scrolls (which is a generalization of Green's "$2g + 1 + p$" theorem for curves).

Precisely, we give answers for these problems in Theorem 1.1, Theorem 1.2 and Theorem 1.3, respectively.

**Theorem 1.1.** *Let $X$ be a smooth complex projective variety and let $\mathscr{L} \in \mathrm{Pic}\, X$ be a very ample line bundle. For an embedding $X \hookrightarrow \mathbb{P}(V)$ given by a subsystem $V \subset H^0(X, \mathscr{L})$ of codimension $t$, if it satisfies Property $N_1^S$, then $X \hookrightarrow \mathbb{P}(V)$ is*

(1) *$k$-normal for all $k \geqq t + 1$,*

(2) $\max\{m + 1, t + 2\}$*-regular and*

(3) *cut out by hypersurfaces of degree $t + 2$, and the homogeneous ideal of $X$ is generated in degrees $\geqq t + 2$ by its component of degree $t + 2$,*

*where $m$ is the regularity of $\mathcal{O}_X$ with respect to $\mathscr{L}$.*

**Theorem 1.2.** *Let $X$ be a smooth complex projective variety and let $\mathscr{L} \in \mathrm{Pic}\, X$ be a very ample line bundle such that*

$$H^1(X, \mathscr{L}^j) = 0 \quad \textit{for all } j \geqq 2.$$

*If $\mathscr{L}$ satisfies Property $N_p$, then for every embedding $X \hookrightarrow \mathbb{P}(V)$ given by a subsystem $V \subset H^0(X, \mathscr{L})$ of codimension $0 \leqq t \leqq p$, Property $N_{p-t}^S$ holds. In particular, for $0 \leqq t \leqq p - 1$, $X \hookrightarrow \mathbb{P}(V)$ is*

(1) *$k$-normal for all $k \geqq t + 1$,*

(2) $\max\{m + 1, t + 2\}$*-regular and*

(3) *cut out by hypersurfaces of degree $t + 2$, and the homogeneous ideal of $X \hookrightarrow \mathbb{P}(V)$ is generated in degrees $\geqq t + 2$ by its component of degree $t + 2$,*

*where $m$ is the regularity of $\mathcal{O}_X$ with respect to $\mathscr{L}$.*

In brief, our main Theorems 1.1 and 1.2 imply that Property $N_p$ for some $p \geqq 1$ guarantees a good behavior of higher normality, Castelnuovo-Mumford regularity and optimal upper bound of the degree of polynomials generating $I_X$ for an embedding by the subsystem of codimension less than $p$. Furthermore we can apply our main theorems to almost all known smooth projective varieties with Property $N_p$ because the cohomological condition in Theorem 1.2 ($H^1(X, \mathscr{L}^\ell) = 0$ for $\ell \geqq 2$) is satisfied for all those cases (for details, see §5). We also remark that Theorem 1.2 can be applied to almost all special curves since we do not assume $H^1(X, \mathscr{L}) = 0$.

To get the upper bound of degree of minimal generators for the homogeneous ideal, it has been canonical to investigate Castelnuovo-Mumford regularity of the ideal sheaf. Along this point of view, Theorem 1.1 implies surprisingly that the degree bound can be much smaller than the optimal regularity bound. In particular, this degree bound is also optimal since there exists a $(t+2)$-secant line to $X$ obtained from linear projections with special centers. Also the bound of higher normality and regularity can not be refined since in the case of ruled scrolls over curves, the higher normality is equivalent to regularity. In general, regularity or normality is worse as $t$ is getting larger which is geometrically related to the existence of a higher multisecant line. Remark that $X$ fails to be $k$-regular if it admits the $(k+1)$-secant line.

**Remark 1.** In [4], Christina Birkenhake generalizes Property $N_p$ in a different way. Let $\mathscr{L}$ be a very ample line bundle on a smooth projective variety $X$ and $V \subset H^0(X, \mathscr{L})$ a subsystem defining an embedding $X \hookrightarrow \mathbb{P}(V)$. For the homogeneous coordinate ring $S = \mathrm{Sym}^\bullet V$ of $\mathbb{P}(V)$, define the graded $S$-module

$$R := k \oplus V \oplus H^0(X, \mathscr{L}^2) \oplus H^0(X, \mathscr{L}^3) \oplus \cdots.$$

Then $X \hookrightarrow \mathbb{P}(V)$ is said to satisfy Property $\tilde{N}_p$ if the minimal free resolution of $R$ is linear until the $p$-th stage. That is, $R$ admits a minimal free resolution of the form

$$\cdots \to S^{m_p}(-p-1) \to \cdots \to S^{m_2}(-3) \to S^{m_1}(-2) \to S \to R \to 0.$$

Therefore Property $\tilde{N}_0$ holds if and only if $k$-normality holds for all $k \geqq 2$, Property $\tilde{N}_1$ holds if and only if Property $\tilde{N}_0$ holds and the homogenous ideal is generated by quadrics, Property $\tilde{N}_2$ holds if and only if Property $\tilde{N}_1$ holds and the relations among the quadrics are generated by the linear relations and so on. Clearly $S(X) \subseteqq R \subseteqq E$ and in particular $R = E$ if and only if $V = H^0(X, L)$. However, the Koszul cohomological method developed in [11] does not work well in the case of Property $\tilde{N}_p$ as Alberto Alzati and Francesco Russo mentioned in their paper [1]. To our knowledge, this is because the graded module $R$ is not saturated. Indeed, for a very ample line bundle $\mathscr{L}$ such that $H^1(X, \mathscr{L}^j) = 0$ for all $j \geqq 2$ and Property $N_2$ holds, every isomorphic one point projection satisfies Property $N_1^S$ and $k$-normality for all $k \geqq 2$. But, there exists one point projection admitting a trisecant line in many cases and hence its image cannot be cut out by quadrics ideal-theoretically, which also supports the argument in [1], Example 4.4.

**Remark 2.** In [1], Alberto Alzati and Francesco Russo gave a necessary and sufficient condition for the isomorphic projection of a $k$-normal variety to remain $k$-normal. As an application, they also proved that if $\mathscr{L}$ satisfies Property $N_2$, then every isomorphic one point projection satisfies $k$-normality for all $k \geqq 2$ (without the assumption that $H^1(X, \mathscr{L}^j) = 0$ for all $j \geqq 2$). For details, see Theorem 3.2 and Corollary 3.3 in [1].

In §4 we study higher syzygies of linear series on surface scrolls. Indeed we prove a criterion for Property $N_p$ of surface scrolls and then apply Theorem 1.2 to investigate the geometry of subsystems of very ample line bundles giving scroll embedding. We will follow the notation and terminology of R. Hartshorne's book [14], V, §2. Let $C$ be a smooth projective curve of genus $g$ and let $\mathscr{E}$ be a vector bundle of rank 2 on $C$ which is normalized, i.e., $H^0(C, \mathscr{E}) \neq 0$ while $H^0(C, \mathscr{E} \otimes \mathscr{O}_C(D)) = 0$ for every divisor $D$ of negative degree. We set

$$\mathfrak{e} = \bigwedge^2 \mathscr{E} \quad \text{and} \quad e = -\deg(\mathfrak{e}).$$

Let $X = \mathbb{P}_C(\mathscr{E})$ be the associated ruled surface with projection morphism $\pi : X \to C$. We fix a minimal section $C_0$ such that $\mathscr{O}_X(C_0) = \mathscr{O}_{\mathbb{P}_C(\mathscr{E})}(1)$. For $\mathfrak{b} \in \operatorname{Pic} C$, $\mathfrak{b}f$ denotes the pull-back of $\mathfrak{b}$ by $\pi$. Thus any element of $\operatorname{Pic} X$ can be written $aC_0 + \mathfrak{b}f$ with $a \in \mathbb{Z}$ and $\mathfrak{b} \in \operatorname{Pic} C$ and any element of $\operatorname{Num} X$ can be written $aC_0 + bf$ with $a, b \in \mathbb{Z}$ and scroll means that $a = 1$. Precisely our result is the following:

**Theorem 1.3.** *Let $\mathscr{L} = C_0 + \mathfrak{b}f \in \operatorname{Pic}(X)$ be a line bundle in the numerical class of $C_0 + bf$ such that*

$$b = \max\{3g, 3g + e\} + p \quad \text{for some } p \geqq 0.$$

*Then $\mathscr{L}$ has Property $N_p$.*

It is easily checked that $\mathscr{O}_X$ is 2-regular with respect to $\mathscr{L}$. So by applying Theorem 1.2 to this result, we obtain the following:

**Corollary 1.4.** *Under the same situation as in Theorem* 1.3, *consider the embedding $X \hookrightarrow \mathbb{P}(V)$ given by a subsystem $V \subset H^0(X, \mathscr{L})$ of codimension t. If $0 \leqq t \leqq p - 1$, then*

$$X \hookrightarrow \mathbb{P}(V) \text{ is } \begin{cases} k\text{-normal for all } k \geqq t + 1 \text{ and} \\ \max\{3, t + 2\}\text{-regular.} \end{cases}$$

*In particular, $X \hookrightarrow \mathbb{P}(V)$ is cut out by hypersurfaces of degree $t + 2$, and the homogeneous ideal of $X$ is generated in degrees $\geqq t + 2$ by its component of degree $t + 2$.*

The organization of this paper is as follows. In §3, we investigate the geometric meaning of syzygies of the module $E$. Precisely we prove Theorem 1.1 and Theorem 1.2. In §4, we give a concrete statement and proof of Property $N_p$ for surface scrolls. §5 is devoted to apply Theorem 1.1 and Theorem 1.2. In particular we reprove some old results about regularity and Property $N_p$.

## 2. Notations and conventions

Throughout this paper the following is assumed.

(1) All varieties are defined over the complex number field $\mathbb{C}$.

(2) For a finite dimensional $\mathbb{C}$-vector space $V$, $\mathbb{P}(V)$ is the projective space of one-dimensional quotients of $V$.

(3) When a variety $X$ is embedded in a projective space, we always assume that it is non-degenerate, i.e. it does not lie in any hyperplane.

(4) When a projective variety $X$ is embedded in a projective space $\mathbb{P}^r$ by a very ample line bundle $\mathscr{L} \in \operatorname{Pic} X$, we may write $\mathscr{O}_X(1)$ instead of $\mathscr{L}$ so long as no confusion arises.

(5) For arbitrary nonzero coherent sheaf $\mathscr{F}$ on $\mathbb{P}^r$, $\operatorname{Reg}(\mathscr{F})$ is defined to be

$$\min\{m \mid H^i(\mathbb{P}^r, \mathscr{F}(m-i)) = 0 \text{ for all } i \geqq 1\}.$$

Also it is well known that for an exact sequence $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$,

(i) $\mathrm{Reg}(\mathscr{G}) \geqq \max\{\mathrm{Reg}(\mathscr{E}) - 1, \mathrm{Reg}(\mathscr{F})\}$ and

(ii) $\mathrm{Reg}(\mathscr{E}) \geqq \max\{\mathrm{Reg}(\mathscr{F}), \mathrm{Reg}(\mathscr{G}) + 1\}$ provided that $H^1(\mathbb{P}^r, \mathscr{E}(j)) = 0$ for all $j \in \mathbb{Z}$.

This will be freely used without explicit mention.

## 3. Regularity criterion and effects of Property $N_p$

We begin with proving Theorem 1.1.

*Proof of Theorem* 1.1. Let $S$ be the homogeneous coordinate ring of $\mathbb{P}(V)$. Since $X \hookrightarrow \mathbb{P}(V)$ satisfies Property $N_1^S$, the graded $S$-module $E$ admits a minimal graded free resolution with the following numerical type in the first two terms:

$$\cdots \to S^{n_1}(-2) \to S \oplus S^t(-1) \to E \to 0.$$

Then we can prove our theorem by using the surprising technique used in [13]. From the above resolution, $\mathscr{O}_X$ admits a resolution as follows:

$$\cdots \to \mathscr{O}_{\mathbb{P}^r}^{n_1}(-2) \xrightarrow{v} \mathscr{O}_{\mathbb{P}^r} \oplus \mathscr{O}_{\mathbb{P}^r}^t(-1) \to \mathscr{O}_X \to 0.$$

As the proof of [13], Theorem 2.1, we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & 0 & & & & \mathscr{O}_{\mathbb{P}^r} & \longrightarrow & \mathscr{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & K & \longrightarrow & \mathscr{O}_{\mathbb{P}^r}^{n_1}(-2) & \xrightarrow{v} & \mathscr{O}_{\mathbb{P}^r} \oplus \mathscr{O}_{\mathbb{P}^r}^t(-1) & \longrightarrow & \mathscr{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & \mathscr{O}_{\mathbb{P}^r}^{n_1}(-2) & \xrightarrow{u} & \mathscr{O}_{\mathbb{P}^r}^t(-1) & \longrightarrow & 0 & & \\
 & & & & & & \downarrow & & & & \\
 & & & & & & 0 & & & &
\end{array}$$

where $K$ and $N$ are kernels of $v$ and $u$, respectively. Now let $L$ be the kernel of

$$\mathscr{O}_{\mathbb{P}^r} \oplus \mathscr{O}_{\mathbb{P}^r}^{n_0}(-1) \to \mathscr{O}_X \to 0.$$

Then we can separate the above diagram as follows:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathscr{I}_X & \longrightarrow & \mathcal{O}_{\mathbb{P}^r} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L & \longrightarrow & \mathcal{O}_{\mathbb{P}^r} \oplus \mathcal{O}^t_{\mathbb{P}^r}(-1) & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}^t_{\mathbb{P}^r}(-1) & = & \mathcal{O}^t_{\mathbb{P}^r}(-1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \mathscr{I}_X & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & \mathcal{O}^{n_1}_{\mathbb{P}^r}(-2) & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & \mathcal{O}^{n_1}_{\mathbb{P}^r}(-2) & \longrightarrow & \mathcal{O}^t_{\mathbb{P}^r}(-1) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & \mathscr{I}_X & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

Each diagram is directly obtained by Snake Lemma. We observe the following facts:

(1) $H^1(\mathbb{P}^r, K(n)) = H^1(\mathbb{P}^r, L(n)) = 0$ for all $n \in \mathbb{Z}$ by the minimality of the resolution.

(2) $H^2(\mathbb{P}^r, K(n)) = 0$ for all $n \in \mathbb{Z}$ by the exact sequence

$$0 \to K \to \mathcal{O}^{n_1}_{\mathbb{P}^r}(-2) \to L \to 0.$$

(3) $N$ is $(t+2)$-regular by the following Lemma 3.1.

Now consider $0 \to K \to N \to \mathscr{I}_X \to 0$. In the exact sequence

$$H^1(\mathbb{P}^r, N(t+j)) \to H^1(\mathbb{P}^r, \mathscr{I}_X(t+j)) \to H^2(\mathbb{P}^r, K(t+j)),$$

$H^1(\mathbb{P}^r, N(t+j)) = 0$ for all $j \geqq 1$ and $H^2(\mathbb{P}^r, K(t+j)) = 0$ for all $j \in \mathbb{Z}$ by the above observation. Therefore $H^1(\mathbb{P}^r, \mathcal{I}_X(t+j)) = 0$ for all $j \geqq 1$.

For the regularity, the third condition implies that $L$ is $(m+1)$-regular and $K$ is $(m+2)$-regular since their first cohomology groups vanish always. So by the exact sequence $0 \to K \to N \to \mathcal{I}_X \to 0$, we obtain that $\mathcal{I}_X$ is $\max\{m+1, t+2\}$-regular.

For the third statement, consider the following commutative diagram:

$$\begin{array}{ccccc}
H^0(\mathbb{P}^r, N(t+2)) \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) & \xrightarrow{\alpha} & H^0(\mathbb{P}^r, N(t+2+k)) & \longrightarrow & 0 \\
\gamma \big\downarrow & & \big\downarrow \delta & & \\
H^0(\mathbb{P}^r, \mathcal{I}_X(t+2)) \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) & \xrightarrow{\beta} & H^0(\mathbb{P}^r, \mathcal{I}_X(t+2+k)) & & \\
\big\downarrow & & \big\downarrow & & \\
H^1(\mathbb{P}^r, K(t+2)) \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) & & H^1(\mathbb{P}^r, K(t+2+k)) & &
\end{array}$$

Since $N$ is $(t+2)$-regular, $\alpha$ is surjective for all $k \geqq 0$. Since $H^1(\mathbb{P}^r, K(n)) = 0$ for all $n \in \mathbb{Z}$, $\gamma$ and $\delta$ are surjective. Therefore $\beta$ is surjective for all $k \geqq 0$ which completes the proof. $\square$

**Lemma 3.1.** *Let $\mathcal{E} = \mathcal{O}^e_{\mathbb{P}^r}(-2) \xrightarrow{u} \mathcal{F} = \mathcal{O}^f_{\mathbb{P}^r}(-1)$ be a generically surjective homomorphism such that* $\operatorname{Supp}(\operatorname{Coker}(u))$ *is a finite set. Then*

$$\operatorname{Reg}(\operatorname{Ker}(u)) \leqq f + 2.$$

*Proof.* This can be proved by the Eagon-Northcott complex and diagram chasing. For details, see [20], Lemma 5. $\square$

From now on, we proceed to prove Theorem 1.2. Let $X$ be a smooth complex projective variety of dimension $n$ and let $\mathcal{L} \in \operatorname{Pic} X$ be a very ample line bundle such that

$$H^1(X, \mathcal{L}^j) = 0 \quad \text{for all } j \geqq 2.$$

First we give a criterion for Property $N^S_p$.

**Lemma 3.2.** *Under the hypothesis just stated, let $X \hookrightarrow \mathbb{P}(V)$ be an isomorphic embedding where $V \subset H^0(X, \mathcal{L})$ and consider the canonical exact sequence*

$$0 \to \mathcal{M}_V \to V \otimes \mathcal{O}_X \to \mathcal{L} \to 0.$$

*For $p \geqq 0$, the embedding $X \hookrightarrow \mathbb{P}(V)$ satisfies Property $N^S_p$ if and only if for $0 \leqq i \leqq p$*

(1) $\bigwedge^{i+1} V \otimes H^0(X, \mathcal{L}) \to H^0\left(X, \bigwedge^{i} \mathcal{M}_V \otimes \mathcal{L}^2\right)$ *is surjective and*

(2) $H^1\left(X, \bigwedge^{i+1} \mathcal{M}_V \otimes \mathcal{L}^j\right) = 0$ *for all $j \geqq 2$.*

*Proof.* This can be proved using a canonical method computing Betti numbers. Indeed assume that $E$ has a minimal free resolution of the form

$$\cdots \xrightarrow{\varphi_{i+1}} \bigoplus_j S^{k_{i,j}}(-i-j) \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_1} \bigoplus_j S^{k_{0,j}}(-j) \xrightarrow{\varphi_0} E \longrightarrow 0.$$

As explained in [9] or [11], $k_{i,j} = \dim_k \operatorname{Coker}(\alpha_{i,j})$ where $\alpha_{i,j}$ is the map contained in the exact sequence

$$\bigwedge^{i+1} V \otimes H^0(X, \mathscr{L}^{j-1}) \xrightarrow{\alpha_{i,j}} H^0\Big(X, \bigwedge^{i} \mathscr{M}_V \otimes \mathscr{L}^j\Big)$$
$$\longrightarrow H^1\Big(X, \bigwedge^{i+1} \mathscr{M}_V \otimes \mathscr{L}^{j-1}\Big) \longrightarrow \bigwedge^{i+1} V \otimes H^1(X, \mathscr{L}^{j-1}).$$

Therefore (1) is equivalent to the fact that $k_{i,2} = 0$ for $0 \leqq i \leqq p$. Also for $j \geqq 2$,

$$k_{i,j} = h^1\Big(X, \bigwedge^{i+1} \mathscr{M}_V \otimes \mathscr{L}^{j-1}\Big)$$

since $H^1(X, \mathscr{L}^{j-1}) = 0$ by assumption. So (2) is equivalent to the fact that $k_{i,j} = 0$ for $0 \leqq i \leqq p$ and $j \geqq 2$. $\square$

**Proposition 3.3.** *Let $X \subset \mathbb{P}(V)$ be a complex projective variety and let $\mathscr{L}$ be a very ample line bundle on $X$. Consider subspaces $W \subset V \subset H^0(X, \mathscr{L})$ such that $\operatorname{codim}(W, V) = 1$ and $X \to \mathbb{P}(W)$ is an isomorphic embedding. Assume that $H^1(X, \mathscr{L}^j) = 0$ for all $j \geqq 2$. If $X \subset \mathbb{P}(V)$ satisfies Property $N_p^S$ for some $p \geqq 1$, then $X \subset \mathbb{P}(W)$ satisfies Property $N_{p-1}^S$.*

*Proof.* Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathscr{M}_W & \longrightarrow & W \otimes \mathscr{O}_X & \longrightarrow & \mathscr{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathscr{M}_V & \longrightarrow & V \otimes \mathscr{O}_X & \longrightarrow & \mathscr{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathscr{O}_X & = & \mathscr{O}_X & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\mathscr{M}_W$ and $\mathscr{M}_V$ are kernels of the above evaluation maps. Since $X \subset \mathbb{P}(V)$ satisfies Property $N_p^S$, for $0 \leqq i \leqq p$,

(1) $\bigwedge^{i+1} V \otimes H^0(X, \mathscr{L}) \to H^0\Big(X, \bigwedge^{i} \mathscr{M}_V \otimes \mathscr{L}^2\Big)$ is surjective and

(2) $H^1\Big(X, \bigwedge^{i+1} \mathscr{M}_V \otimes \mathscr{L}^j\Big) = 0$ for all $j \geqq 2$,

by Lemma 3.2. Also what we must show is that for $0 \leqq i \leqq p-1$,

$$(*) = \begin{cases} \bigwedge^{i+1} W \otimes H^0(X, \mathscr{L}) \to H^0\Big(X, \bigwedge^{i} \mathscr{M}_W \otimes \mathscr{L}^2\Big) & \text{and} \\ H^1\Big(X, \bigwedge^{i+1} \mathscr{M}_W \otimes \mathscr{L}^j\Big) = 0 & \text{for all } j \geqq 2. \end{cases}$$

Now consider the following commutative diagram induced from the above one:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigwedge^{i+1} \mathscr{M}_W & \longrightarrow & \bigwedge^{i+1} \mathscr{M}_V & \longrightarrow & \bigwedge^{i} \mathscr{M}_W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigwedge^{i+1} W \otimes \mathscr{O}_X & \longrightarrow & \bigwedge^{i+1} V \otimes \mathscr{O}_X & \longrightarrow & \bigwedge^{i} W \otimes \mathscr{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigwedge^{i} \mathscr{M}_W \otimes \mathscr{L} & \longrightarrow & \bigwedge^{i} \mathscr{M}_V \otimes \mathscr{L} & \longrightarrow & \bigwedge^{i-1} \mathscr{M}_W \otimes \mathscr{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

This gives the following commutative diagram of cohomological groups:

$$\begin{array}{ccccc}
\bigwedge^{i+1} V \otimes H^0(X, \mathscr{L}^{j-1}) & \xrightarrow{\gamma_{i,j}} & \bigwedge^{i} W \otimes H^0(X, \mathscr{L}^{j-1}) & & \\
\Big\downarrow{\scriptstyle \alpha_{i,j}} & & \Big\downarrow{\scriptstyle \beta_{i,j}} & & \\
H^0\Big(X, \bigwedge^{i} \mathscr{M}_V \otimes \mathscr{L}^j\Big) & \xrightarrow{\delta_{i,j}} & H^0\Big(X, \bigwedge^{i-1} \mathscr{M}_W \otimes \mathscr{L}^j\Big) & \longrightarrow & H^1\Big(X, \bigwedge^{i} \mathscr{M}_W \otimes \mathscr{L}^j\Big) \\
\Big\downarrow & & \Big\downarrow & & \\
H^1\Big(X, \bigwedge^{i+1} \mathscr{M}_V \otimes \mathscr{L}^{j-1}\Big) & & H^1\Big(X, \bigwedge^{i} \mathscr{M}_W \otimes \mathscr{L}^{j-1}\Big) & & \\
 & & \Big\downarrow & & \\
 & & \bigwedge^{i} W \otimes H^1(X, \mathscr{L}^{j-1}) & &
\end{array}$$

Note that $\alpha_{i,j}$ and $\gamma_{i,j}$ are surjective for $j \geqq 2$ and $0 \leqq i \leqq p$. So if $\delta_{i,j}$ is surjective then so is $\beta_{i,j}$. The trick is that in this range,

$$\text{if } H^1\Bigl(X, \bigwedge^i \mathscr{M}_W \otimes \mathscr{L}^j\Bigr) = 0, \quad \text{then } H^1\Bigl(X, \bigwedge^i \mathscr{M}_W \otimes \mathscr{L}^{j-1}\Bigr) = 0$$

for all $j \geqq 3$. Let us remind that $H^1(X, \mathscr{L}^\ell) = 0$ for $\ell \geqq 2$. This guarantees that

$$H^1\Bigl(X, \bigwedge^i \mathscr{M}_W \otimes \mathscr{L}^j\Bigr) = 0 \quad \text{for all } j \geqq 2 \text{ and } 0 \leqq i \leqq p$$

because $H^1\Bigl(X, \bigwedge^i \mathscr{M}_W \otimes \mathscr{L}^j\Bigr) = 0$ for sufficiently large $j$. That is, $\beta_{i,j}$ is surjective for all $j \geqq 3$ and $0 \leqq i \leqq p-1$. Finally $\alpha_{i,2}$ is surjective by assumption and $\delta_{i,2}$ is surjective since it is proved that $H^1\Bigl(X, \bigwedge^i \mathscr{M}_W \otimes \mathscr{L}^2\Bigr) = 0$. So $\beta_{i,2}$ is surjective for $0 \leqq i \leqq p-1$ which completes the proof. □

*Proof of Theorem* 1.2. For a given $V \subset H^0(X, \mathscr{L})$, fix a filtration

$$V = V_t \subset V_{t-1} \subset \cdots \subset V_0 = H^0(X, \mathscr{L})$$

by subspaces each having codimension one in the next. Then the first statement is proved by applying Proposition 3.3 repeatedly. Also the second statement is induced directly by Theorem 1.1. □

## 4. Higher syzygies of surface scrolls over a curve

In this section we investigate Property $N_p$ of surface scrolls over a curve and generalize Green's result about Property $N_p$ for curves. For the convenience of the reader, we recall notations in §1:

- $C$: smooth curve of genus $g$,
- $\mathscr{E}$: normalized vector bundle of rank 2 over $C$,
- $\mathfrak{e} = \bigwedge^2 \mathscr{E}$ and $e = -\deg(\mathfrak{e})$,
- $X = \mathbb{P}_C(\mathscr{E})$: the associated ruled surface with the projection map $\pi : X \to C$,
- $C_0$: a minimal section such that $\mathscr{O}_X(C_0) = \mathscr{O}_{\mathbb{P}_C(\mathscr{E})}(1)$.

Throughout this section we consider line bundles of the form

$$\mathscr{L} = H + \pi^* B \quad \text{with } \deg(B) \geqq \max\{3g, 3g+e\} + 1.$$

Note that this line bundle is very ample ([14], Ex. 5.2.11) and satisfies $H^1(X, \mathscr{L}^j) = 0$ for all $j \geqq 1$. Consider the short exact sequence

$$0 \to \mathscr{M}_{\mathscr{L}} \to H^0(X, \mathscr{L}) \otimes \mathcal{O}_X \to \mathscr{L} \to 0.$$

Remark that $(X, L)$ satisfies Property $N_p$ if and only if $H^1\Big(X, \overset{p+1}{\bigwedge} \mathscr{M}_{\mathscr{L}} \otimes \mathscr{L}\Big) = 0$ ([12]).

*Proof of Theorem* 1.3. Let $h^0(X, \mathscr{L}) = r + 1$. For a smooth hyperplane section $C_1 (\cong C) \in H^0(X, \mathscr{L})$, $\mathscr{L}|_{C_1} = \mathcal{O}_C(2\mathfrak{b} + \mathfrak{e})$ and hence we have

$$0 \to \mathscr{M}_C \to \mathcal{O}_C^r \to \mathcal{O}_C(2\mathfrak{b} + \mathfrak{e}) \to 0.$$

If we pull back this sequence by $\pi$, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & -C_0 + (\mathfrak{b} + \mathfrak{e})f & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \pi^*\mathscr{M}_C & \longrightarrow & \mathcal{O}_X^r & \longrightarrow & (2\mathfrak{b} + \mathfrak{e})f & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathscr{M}_{\mathscr{L}} & \longrightarrow & \mathcal{O}_X^r & \longrightarrow & \mathcal{O}_C(2\mathfrak{b} + \mathfrak{e}) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & -C_0 + (\mathfrak{b} + \mathfrak{e})f & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

by Snake Lemma where the second row is induced by the following Lemma 4.1. From

$$0 \to \pi^*\mathscr{M}_C \to \mathscr{M}_{\mathscr{L}} \to \mathcal{O}_X\big(-C_0 + (\mathfrak{b} + \mathfrak{e})f\big) \to 0,$$

we have

$$0 \to \overset{p+1}{\bigwedge} \pi^*\mathscr{M}_C \to \overset{p+1}{\bigwedge} \mathscr{M} \to \overset{p}{\bigwedge} \pi^*\mathscr{M}_C \otimes \mathcal{O}_X\big(-C_0 + (\mathfrak{b} + \mathfrak{e})f\big) \to 0,$$

and hence it is enough to check that

$$H^1\Big(X, \overset{p+1}{\bigwedge} \pi^*\mathscr{M}_C \otimes \mathscr{L}\Big) = H^1\Big(X, \overset{p}{\bigwedge} \pi^*\mathscr{M}_C \otimes (2\mathfrak{b} + \mathfrak{e})f\Big) = 0,$$

or equivalently

$$H^1\Big(C, \overset{p+1}{\bigwedge} \mathscr{M}_C \otimes \mathscr{E} \otimes \mathfrak{b}\Big) = H^1\Big(C, \overset{p}{\bigwedge} \mathscr{M}_C \otimes (2\mathfrak{b} + \mathfrak{e})\Big) = 0.$$

Note that by using the short exact sequence $0 \to \mathcal{O}_C \to \mathscr{E} \to \mathfrak{e} \to 0$ corresponding to $C_0$, the first one is guaranteed if we show $H^1\Big(C, \overset{p+1}{\bigwedge} \mathscr{M}_C \otimes B\Big) = 0$ and

$H^1\left(C, \bigwedge^{p+1} \mathscr{M}_C \otimes \mathfrak{e} \otimes B\right) = 0$. To prove the desired vanishing we use the filtration of $\mathscr{M}_C$. As [20], Lemma 6, for generic $x_0, \dots, x_{r-2} \in C_1 \cong C$, $\mathscr{M}_C$ admits a filtration by vector bundles

$$0 = \mathscr{F}^0 \subset \mathscr{F}^1 \subset \cdots \subset \mathscr{F}^{r-1} = \mathscr{M}_C$$

such that

$$\mathscr{F}^{r-j-1}/\mathscr{F}^{r-j-2} \cong \mathcal{O}_C(-B_j) \quad (j = 0, \dots, r-2)$$

for effective divisors $B_j$ on $C$ whose support $\mathrm{Supp}(B_j)$ contains $x_j$. Note that

$$B_0 + \cdots + B_{r-2} = 2\mathfrak{b} + \mathfrak{e} \quad \text{and so} \quad \deg(B_0) + \cdots + \deg(B_{r-2}) = 2b - e.$$

Therefore it suffices to show that for every distinct $0 \leqq i_1, \dots, i_{p+1} \leqq r-2$,

$$\begin{cases} H^1\big(C, \mathcal{O}_C(-B_{i_1} - \cdots - B_{i_{p+1}} + \mathfrak{b})\big) = 0, \\ H^1\big(C, \mathcal{O}_C(-B_{i_1} - \cdots - B_{i_{p+1}} + \mathfrak{b} + \mathfrak{e})\big) = 0, \end{cases}$$

or equivalently, for every distinct $0 \leqq j_1, \dots, j_{r-2-p} \leqq r-2$,

$$\begin{cases} H^1\big(C, \mathcal{O}_C(B_{j_1} + \cdots + B_{j_{r-2-p}} - \mathfrak{b} - \mathfrak{e})\big) = 0, \\ H^1\big(C, \mathcal{O}_C(B_{j_1} + \cdots + B_{j_{r-2-p}} - \mathfrak{b})\big) = 0. \end{cases}$$

By Serre duality, we show

$$\begin{cases} H^0\big(C, \mathcal{O}_C(\mathfrak{b} + \mathfrak{e} + K_C - B_{j_1} - \cdots - B_{j_{r-2-p}})\big) = 0, \\ H^0\big(C, \mathcal{O}_C(\mathfrak{b} + K_C - B_{j_1} - \cdots - B_{j_{r-2-p}})\big) = 0. \end{cases}$$

Since $B_i$ is an effective divisor containing $x_i$, it is enough to prove

$$\begin{cases} H^0\big(C, \mathcal{O}_C(\mathfrak{b} + \mathfrak{e} + K_C - x_{j_1} - \cdots - x_{j_{r-2-p}})\big) = 0, \\ H^0\big(C, \mathcal{O}_C(\mathfrak{b} + K_C - x_{j_1} - \cdots - x_{j_{r-2-p}})\big) = 0. \end{cases}$$

This is true for generic $x_0, \dots, x_{r-2}$ since

$$\begin{cases} h^0\big(C, \mathcal{O}_C(\mathfrak{b} + \mathfrak{e} + K_C)\big) = b - e + g - 1 \leqq r - 2 - p = 2b - e - 2g - p - 1, \\ h^0\big(C, \mathcal{O}_C(\mathfrak{b} + K_C)\big) = b + g - 1 \leqq r - 2 - p = 2b - e - 2g - p - 1, \end{cases}$$

which is equivalent to our assumption

$$b \geqq \max\{3g, 3g + e\} + p \quad (p \geqq 0). \quad \square$$

**Lemma 4.1.** *Let $Z$ be a projective variety embedded in $\mathbb{P}^r$ and let $\mathscr{M} = \Omega_{\mathbb{P}^r}(1)|_Z$. Then for a hyperplane $H$ of $\mathbb{P}^r$, there is the following exact sequence*:

$$0 \to \mathscr{M} \to \mathcal{O}_Z^r \to \mathcal{O}_{Z \cap H}(1) \to 0.$$

*Proof.* Let us consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_Z & = & \mathcal{O}_Z & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{O}_Z^{r+1} & \longrightarrow & \mathcal{O}_Z(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O}_Z^{r} & \longrightarrow & \mathcal{O}_{Z\cap H}(1) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $\mathcal{F}$ is just the kernel of $\mathcal{O}_Z^r \to \mathcal{O}_{Z\cap H}(1)$. By Snake Lemma, $\mathcal{F} \cong \mathcal{M}$. □

## 5. Applications and examples

**5.1. Applications.** Now we apply Theorem 1.2. First let us recall the following works of M. Green [11], David C. Butler [5] and L. Ein and R. Lazarsfeld [7] about Property $N_p$:

- (M. Green, [11]) Let $X$ be a smooth projective curve of genus $g$ and let $\mathcal{L}$ be a line bundle on $X$ of degree $d$. If $d \geqq 2g + 1 + p$ for $p \geqq 0$, $(X, \mathcal{L})$ satisfies $N_p$.

- (David C. Butler, [5]) Let $C$ be a smooth projective curve of genus $g$. For a vector bundle $\mathbb{E}$ of rank $n$ over $C$, let $X = \mathbb{P}(\mathbb{E})$ be the associated ruled variety with tautological line bundle $H$ and projection map $\pi : X \to C$. For a line bundle $\mathcal{L} = aH + \pi^* B$ on $X$ with $a \geqq 1$ and $B \in \operatorname{Pic} C$, assume that $\mu^-(\pi_* \mathcal{L}) \geqq 2g + 2p$ for some $1 \leqq p \leqq a$. Then $(X, \mathcal{L})$ satisfies $N_p$.

- (L. Ein and R. Lazarsfeld, [7]) Let $X$ be a smooth complex projective variety of dimension $n$ with the canonical sheaf $K_X$. For $A, B \in \operatorname{Pic} X$, assume that $A$ is very ample and $B$ is numerically effective. Then for $p \geqq 0$

$$K_X + (n + p)A + B \text{ satisfies Property } N_p$$

except the case $(X, A, B) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n})$ and $p = 0$.

Note that in these results the line bundle $\mathcal{L}$ satisfies $H^i(X, \mathcal{O}_X(j)) = 0$ for all $i, j \geqq 1$. Therefore Theorem 1.2 can be applied to these results. Recently, A. Noma obtained a very sharp regularity bound for curves:

$(*)$ (Noma, [20]) Let $X \subset \mathbb{P}^r$ $(r \geqq 3)$ be an integral projective curve of degree $d$ and arithmetic genus $\rho_a$. Let $l \leqq \min\{r - 2, \rho_a\}$ be a nonnegative integer. Then $X$ is $(d - r + 2 - l)$-regular and $d \geqq r + l + 1$ unless $X$ is a curve embedded by a complete linear system of degree $d \geqq 2\rho_a + 2$ and $l = \rho_a$.

From this result, when $d = 2g + 1 + p$ for some $p \geqq 1$ and $X \subset \mathbb{P}^r$ by a subsystem of codimension $t \geqq p - 1$, then $X$ is $(t+2)$-regular and hence $(t+1)$-normal since we can take $l = g$ and in this case $d - r + 2 - g = t + 2$. This is the same as information obtained by applying Theorem 1.1 and Theorem 1.2 to Green's "$2g + 1 + p$" theorem [11]. In fact, since

(∗∗) (Green-Lazarsfeld, [12]) Let $\mathscr{L}$ be a line bundle of degree $2g + p$ $(p \geqq 1)$ on a smooth projective curve $X$ of genus $g$, defining an embedding $X \hookrightarrow \mathbb{P}(H^0(X, \mathscr{L})) = \mathbb{P}^{g+p}$. Then $\mathscr{L}$ fails to satisfy $N_p$ if and only if either
(i) $X$ is hyperelliptic;
or
(ii) $X \hookrightarrow \mathbb{P}^{g+p}$ has a $(p+2)$-secant $p$-plane, i.e., $H^0(X, \mathscr{L} \otimes K_X^*) \neq 0$.

is known, Theorem 1.2 refines Noma's bound a little. We remark that in the above fact, our result gives a new proof of the relevant part:

**Theorem 5.1.** *Let $C$ be a smooth curve of genus $g \geqq 2$, and let $\mathscr{L} \in \operatorname{Pic} C$ be a line bundle of degree $d = 2g + p$. If $C \hookrightarrow \mathbb{P}^{g+p}$ has a $(p+2)$-secant $p$-plane, then it does not satisfy Property $N_p$.*

*Proof.* We prove that there is a subsystem $V \subset H^0(X, \mathscr{L})$ such that the $(p+2)$-secant $p$-plane gives a $(p+2)$-secant line in the embedding $C \hookrightarrow \mathbb{P}(V) = \mathbb{P}^{g+1}$ and hence Theorem 1.2 says that $N_p$ does not hold. Indeed $\dim \operatorname{Sec} C = 3$ and $\operatorname{Sec} C \subset \mathbb{P}^{g+p}$ is non-degenerate. So we can take a center of dimension $(p-1)$ in the $p$-plane. □

In the same reason, Noma's bound can be refined if the following conjecture introduced by Green and Lazarsfeld [12] is solved:

(⋆) Let $C$ be a smooth curve of genus $g$ and let $\mathscr{L}$ be a very ample line bundle on $C$ with $\deg(\mathscr{L}) \geqq 2g + 1 + p - 2h^1(C, \mathscr{L}) - \operatorname{Cliff}(C)$. Then $\mathscr{L}$ satisfies Property $N_p$ unless $C \subset \mathbb{P}(H^0(C, \mathscr{L}))$ has a $(p+2)$-secant $p$-plane.

Finally in the case of rational normal scrolls including rational normal curves, Property $N_p$ holds for all $p \geqq 0$ because these varieties are always 2-regular. In fact, the following is known:

(∗∗∗) (Ottaviani-Paoletti, [21]) The only smooth varieties in $\mathbb{P}^r$ such that Property $N_p$ holds for every $p \geqq 0$ are the quadrics, the rational normal scrolls and the Veronese surface in $\mathbb{P}^5$.

Therefore Theorem 1.2 implies that the Regularity Conjecture is true for smooth rational scrolls:

**Theorem 5.2.** *For a rational scroll $X \subset \mathbb{P}^r$ of dimension $n$ and degree $d$,*

$$\operatorname{reg}(X) \leqq d - (r - n) + 1.$$

*Proof.* Let $\mathscr{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$ be such that $a_i \geqq 1$ and let $X = \mathbb{P}_{\mathbb{P}^1}(\mathscr{E})$ with the tautological line bundle $\mathscr{L}$. Recall that the degree $d$ of $\mathscr{L}$ is equal to

$a_1 + \cdots + a_n$, $\mathcal{O}_X$ is 1-regular, and $h^0(X, \mathcal{L}) = a_1 + \cdots + a_n + n$. When $X \subset \mathbb{P}^r$ by $\mathcal{L}$, let $t = h^0(X, \mathcal{L}) - r - 1$. Then $X$ satisfies $\max\{2, t+2\}$-regular. But $t + 2 = d - (r - n) + 1$. Also if $t = 0$, then clearly $X$ is 2-regular since the genus of $\mathbb{P}^1$ is 0. □

Remark that Bertin [3] proved the Regularity Conjecture for smooth ruled scrolls over a curve of arbitrary genus using a different method.

**5.2. Examples.** Here we give concrete examples which say that the range of $t$ cannot be refined. As these examples show, we can also know that the failure of higher order normality of noncomplete embeddings gives some obstruction to Property $N_p$ for linearly normal embeddings.

**Example 1.** Let $C$ be a smooth projective curve of genus 2 and $B \in \operatorname{Pic} C$ a line bundle of degree 6. Now consider $C \subset \mathbb{P}^3$ given by a subsystem of $H^0(C, B)$. Then $C$ does not satisfy 2-normality since $h^0\big(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)\big) = 10 < h^0(C, 2B) = 11$. So it fails to be 3-regular. In fact it is exactly 4-regular by [20], Theorem 1. Note that $B$ satisfies Property $N_1$ by [11] and fails to satisfy $N_2$ by Theorem 1.2 or by [12] since $C$ is hyperelliptic.

**Example 2.** For $C$ and $B$ in Example 1, let $\mathcal{E} = \mathcal{O}_C(B) \oplus \mathcal{O}_C(B)$ and $X = \mathbb{P}_C(\mathcal{E})$ with the tautological line bundle $\mathcal{O}_X(1)$. Observe the following:

(1) For the Segre embedding $Y = \mathbb{P}^4 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^9$, choose

$$P = (P_1, P_2), \quad Q = (Q_1, Q_2) \in Y$$

such that $P_i \neq Q_i$ for $i = 1, 2$. Then $\overline{PQ} \cap Y = \{P, Q\}$.

(2) Since $X \cong C \times \mathbb{P}^1$, $C \xrightarrow{|B|} \mathbb{P}^4$ gives $X \hookrightarrow Y \hookrightarrow \mathbb{P}^9$ which is just the embedding by $\mathcal{O}_X(1)$. In this case, $\operatorname{Sec}(X) \neq Y$ by (1). So we can choose a point $\alpha \in Y - \operatorname{Sec}(X)$. Let $f : X \to \mathbb{P}^8$ be the linear projection from $\alpha$.

Now we claim that $f(X) \subsetneq \mathbb{P}^8$ fails to be 3-regular. Let $\Delta \subset \mathbb{P}^9$ be the fibre of $\mathbb{P}^4 \times \mathbb{P}^1 \to \mathbb{P}^1$ containing $\alpha$. Then $f$ maps $\Delta$ to a 3-dimensional linear subspace $\Lambda \subset \mathbb{P}^8$. Let $f(X \cap \Delta)$ be $C_1 \cong C$. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_X \cap \mathcal{I}_\Lambda & \longrightarrow & \mathcal{I}_\Lambda & \longrightarrow & \mathcal{O}_X(-C_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_X & \longrightarrow & \mathcal{O}_{\mathbb{P}^8} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{C_1} & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_{C_1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathscr{I}_X$, $\mathscr{I}_\Lambda$ and $\mathscr{I}_{C_1}$ are the ideal sheaves of $X, \Lambda \subset \mathbb{P}^8$ and $C_1 \subset \Lambda \cong \mathbb{P}^3$, respectively. Note that in general $\mathscr{I}_X \to \mathscr{I}_{C_1}$ is not surjective. But in our case, this is surjective since $X$ is a product of other varieties and the embedding is just a Segre embedding. From this diagram, if $h^1(\mathbb{P}^8, \mathscr{I}_X(2)) = 0$, then $h^1(\mathbb{P}^3, \mathscr{I}_{C_1}(2)) = 0$ since $h^2(\Lambda, \mathscr{I}_X \cap \mathscr{I}_\Lambda(2)) = 0$. But as explained in Example 1, $C_1 \subset \mathbb{P}^3$ can never be 2-normal. Therefore $h^1(\mathbb{P}^8, \mathscr{I}_X(2)) \neq 0$ and the regularity of $X \subseteqq \mathbb{P}^8$ is at least 4. Note that since $\mu^-(\mathscr{E}) = 6$, $\mathscr{O}_X(1)$ has Property $N_1$ by [5]. On the other hand Theorem 1.2 implies the failure of Property $N_2$.

**Example 3.** It is well known and can be seen in [21] that

$$(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(d)) \text{ has Property } N_p \begin{cases} \text{for all } p \geqq 0 \text{ when } d = 2, \text{ and} \\ \text{if and only if } p \leqq 3d-3 \text{ when } d \geqq 3. \end{cases}$$

In particular for $(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(3))$, our results about Property $N_p^S$ can be applied to every subsystem giving embedding. For instance, subsystems of $H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(3))$ can be understood as the following table. Note that this is optimal since there exists a center such that $\mathbb{P}^2 \to \mathbb{P}^{9-t}$ has a $(t+1)$-secant line.

| $t$ | embedding | higher normality | regularity |
|---|---|---|---|
| 0 | $\mathbb{P}^2 \to \mathbb{P}^9$ | projectively normal | 2-regular |
| 1 | $\mathbb{P}^2 \to \mathbb{P}^8$ | $k$-normal for all $k \geqq 2$ | 3-regular |
| 2 | $\mathbb{P}^2 \to \mathbb{P}^7$ | $k$-normal for all $k \geqq 3$ | 4-regular |
| 3 | $\mathbb{P}^2 \to \mathbb{P}^6$ | $k$-normal for all $k \geqq 4$ | 5-regular |
| 4 | $\mathbb{P}^2 \to \mathbb{P}^5$ | $k$-normal for all $k \geqq 5$ | 6-regular |

Table 2. The case of $(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(3))$.

**Remark 1.** For a smooth projective variety $X$, assume that $H^i(X, \mathscr{L}^j) = 0$ for all $i, j \geqq 1$ and $H^2(X, \mathscr{O}_X) = H^n(X, \mathscr{O}_X(2-n)) = 0$ (e.g., $X$ is a curve or a ruled scroll over a curve etc.). Then $\mathscr{O}_X$ is 2-regular thanks to the Kodaira Vanishing Theorem. Property $N_p$ for those varieties implies that $X \subset \mathbb{P}(V_t)$ is $\max\{3, t+2\}$-regular for $0 \leqq t \leqq p-1$.

**Remark 2.** It seems very interesting to find a range of $t$ such that Property $N_1^S$ holds since this guarantees that the regularity or higher order normality changes as expected under linear projections without depending on the center. For ruled varieties, it looks possible to investigate this kind of expected good behaviors by analyzing the vector bundle $\mathscr{M}_V$ [16]. This leads to the connection between geometry of higher secant varieties and higher linear syzygies of a smooth projective variety in some sense.

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