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Smooth threefolds in $\mathbb{P}^5$ without apparent triple or quadruple points and a quadruple-point formula*

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Abstract. For a projective variety X of codimension 2 in $\mathbb{P}^{n+2}$ defined over the complex number field $\mathbb{C}$, it is traditionally said that X has no apparent $(k+1)$-ple points if the $(k+1)$-secant lines of X do not fill up the ambient projective space $\mathbb{P}^{n+2}$, equivalently, the locus of $(k+1)$-ple points of a generic projection of X to $\mathbb{P}^{n+1}$ is empty. We show that a smooth threefold in $\mathbb{P}^5$ has no apparent triple points if and only if it is contained in a quadric hypersurface. We also obtain an enumerative formula counting the quadrisecant lines of X passing through a general point of $\mathbb{P}^5$ and give necessary cohomological conditions for smooth threefolds in $\mathbb{P}^5$ without apparent quadruple points. This work is intended to generalize the work of F. Severi [fSe] and A. Aure [Au], where it was shown that a smooth surface in $\mathbb{P}^4$ has no triple points if and only if it is either a quintic elliptic scroll or contained in a hyperquadric. Furthermore we give open questions along these lines.

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1. Introduction

Let X be a nondegenerate smooth projective subvariety of degree d and codimension two in $\mathbb{P}^{n+2}$ defined over the complex number field $\mathbb{C}$. Denote by $S_k(X)$ the closure of a union of all k-secant lines of X in $\mathbb{P}^{n+2}$. Then one can consider the following descending filtration associated to X

$$\mathbb{P}^{n+2} = S_2(X) \supseteq S_3(X) \supseteq \cdots \supseteq S_{t-1}(X) \supseteq S_t(X) = \cdots = S_\infty(X),$$

where $S_\infty(X)$ is the subvariety swept out by the lines contained in X. It is clear that the first number t for which $S_t(X) = S_\infty(X)$ satisfies the inequality $t \leq d$. Furthermore, if X is contained in a hypersurface F of degree m, then $S_k(X) \subset F$ for all $k > m$ by Bezout theorem.

As it was shown by Z. Ran, $S_{n+2}(X) \subsetneq \mathbb{P}^{n+2}$, i.e. the $(n+2)$-secant lines of X do not fill up the whole space $\mathbb{P}^{n+2}$ for $n \geq 1$, which can be viewed as a generalization of the classical "trisecant lemma" for curves in $\mathbb{P}^3$; cf. [R3] for details.

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For a smooth surface in $\mathbb{P}^4$, F. Severi showed that if $S_3(X) \neq \mathbb{P}^4$, then either X is contained in a quadric hypersurface or $S_3(X)$ is a hypersurface ruled in planes, each plane intersecting the surface in a curve of degree at least 3 [fSe]. It was conjectured by C. Peskine and proved by A. Aure that the quintic elliptic scroll (which is cut out by cubic hypersurfaces) is the only surface in $\mathbb{P}^4$ such that $S_3(X) \neq \mathbb{P}^4$ and $H^0(\mathbb{P}^4, \mathcal{I}_X(2)) = 0$ [Au].

On the other hand, in [R1] it was proved by using vector bundle techniques that for the variety X^n defined by a section of rank two vector bundle over $\mathbb{P}^{n+2}$, the following statements are equivalent for $m \leq n$:

(1) $S_{m+1}(X) \neq \mathbb{P}^{n+2}$;
(2) X is contained in a hypersurface of degree m.

Note that if $\dim(X) \geq 4$, then, by Serre's construction, a smooth variety of codimension two in $\mathbb{P}^{n+2}$ is always defined by a section of rank two vector bundle over $\mathbb{P}^{n+2}$ because it is necessarily subcanonical.

One can also show that the above equivalence holds for arbitrary arithmetically Cohen-Macaulay codimension two subvarieties of dimension ≥ 2 (cf. Proposition 3.7).

On the other hand, smooth threefolds of codimension two need not be subcanonical, but one always has the following inclusions:

$$\mathbb{P}^5 \neq S_5(X) \subseteq S_4(X) \subseteq S_3(X) \subseteq S_2(X) = \mathbb{P}^5.$$

Thus it is natural to ask whether there exist smooth threefolds with $S_{k+1}(X) \neq \mathbb{P}^5$ and $H^0(\mathcal{I}_X(k)) = 0$, $k = 2, 3$, thus extending the work of F. Severi [fSe] and A. Aure [Au] to the case of smooth threefolds in $\mathbb{P}^5$. Recently, there are some related works about threefolds in $\mathbb{P}^5$ due to Mezzetti and Portelli [MP1], [MP2].

In the present paper we show that a smooth threefold in $\mathbb{P}^5$ has no apparent triple points (i.e. $S_3(X) \neq \mathbb{P}^5$) if and only if it is contained in a quadric hypersurface and give some necessary cohomological conditions for nontrivial smooth threefolds without apparent quadruple points (cf. Theorem 3.9). We note that such nontrivial examples (if exist) are also on the boundary of the Peskine-Zak conjecture (cf. Remark 3.10 (b)). Here, threefolds without apparent quadruple points are called 'nontrivial' if they are not contained in a cubic hypersurface.

The method we use to prove the main theorems (Theorem 3.4, Theorem 3.9, Theorem 4.2) is to check how the absence of apparent quadruple points affects Castelnuovo's regularity of the vector bundle $\mathcal{E}_X$ in the exact sequence (3.0) and to apply the Kodaira-Le Potier vanishing theorem for ample vector bundles.

Furthermore, in Theorem 4.2 of Sect. 4 we obtain an enumerative formula for the quadrisecant lines passing through a general point of $\mathbb{P}^5$ by using the Giambelli-Thom-Porteous formula for the degeneracy locus of a morphism between vector bundles.

One more motivation to study the family of quadrisecant lines of a smooth threefold X comes from the following. Let X be a smooth threefold of arbitrary codimension. If $\dim S_4(X) \leq 4$ then the Castelnuovo-Eisenbud-Goto regularity conjecture is true for any codimension [K2], [R2], i.e.

$$\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1.$$

In fact, Z. Ran showed that for any smooth threefold in $\mathbb{P}^n$, $n \geq 9$ one has $\dim S_4(X) \leq 4$ and, consequently, the regularity conjecture is true in this case [R2]. See also [K2] for connections between the Castelnuovo-Eisenbud-Goto regularity of smooth threefolds and fourfolds and their multisecant loci.

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2. Basic background

In this section we recall the definitions and basic results which will be used in subsequent sections. We work over an algebraically closed field of characteristic zero. For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^N$ it is said to be m-regular if $H^i(\mathbb{P}^N, \mathcal{F}(m - i)) = 0$ for all $i > 0$, and the regularity of $\mathcal{F}$ is defined by the formula

$$\operatorname{reg}\mathcal{F} = \min\{m \in \mathbb{Z} : \mathcal{F} \text{ is } m\text{-regular}\}.$$

In particular, for a projective subscheme X, $\operatorname{reg} X$ is defined as $\operatorname{reg}\mathcal{I}_X$. In general, $\operatorname{reg}\mathcal{F}$ may be negative; however, it is not hard to show that $\operatorname{reg} X \geq 2$ and X is 2-regular if and only if X is of minimal degree.

Lemma 2.1.

(a) If $\mathcal{E}$ is a m-regular coherent sheaf over $\mathbb{P}^N$, then $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^N}(k)$ is generated by global sections for all $k \geq m$.

(b) Let $\mathcal{F}$ be a p-regular vector bundle and $\mathcal{G}$ be a q-regular vector bundle over $\mathbb{P}^N$. Then $\mathcal{F} \otimes \mathcal{G}$ is $(p + q)$-regular, and $S^k(\mathcal{F})$, $\Lambda^k(\mathcal{F})$ are (kp)-regular.

Proof. See [L], p. 428. □

Definition 2.2. *Suppose that X is a smooth projective variety, and $\mathcal{E}$ is a vector bundle over X. one defines $\mathcal{E}$ is ample if the Serre line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E}^*)}(1)$ on a projective bundle $\mathbb{P}(\mathcal{E}^*)$ is an ample line bundle.*

Remark 2.3. Let $\mathcal{E}$ be a vector bundle of rank e over a smooth projective variety X, and L is an ample line bundle such that $\mathcal{E} \otimes L^*$ is globally generated then $\mathcal{E}$ is an ample vector bundle.

Proposition 2.4 (Le Potier). *Assume that X is a smooth variety of dimension n, and $\mathcal{E}$ is an ample vector bundle on X of rank e. Then, $H^i(X, \mathcal{E}^*) = 0$ for $i \leq n - e$.*

Proof. This is a generalization of Kodaira's vanishing theorem for ample line bundles to the case of ample vector bundles, see [Po]. □

Theorem 2.5 (The (dimension+2)-secant Lemma). *Let $X \subset \mathbb{P}^N$ be a smooth n-dimensional subvariety and let Y be an irreducible subvariety parameterizing a family $\{L_y\}$ of lines in $\mathbb{P}^N$. Assume that for a general L_y, the length of a scheme-theoretic intersection $L_y \cap X$ is at least $(n+2)$. Then we have*

$$dim(\cup_{y \in Y} L_y) \leq n+1.$$

Proof. See [R3].

It is also quite useful to know the following result due to B. Segre:

Theorem 2.6 (B. Segre, [bSe]). *Let $X^n \subset \mathbb{P}^N$ be an irreducible variety of dimension n. Let $\Sigma_\infty(X) \subset \mathbb{G}(1, N)$ be a component of maximal dimension of the variety of lines contained in X. Then,*

(a) if $dim \Sigma_\infty(X) = 2n-2$, then $X = \mathbb{P}^n$.
(b) if $dim \Sigma_\infty(X) = 2n-3$, then X is either a quadric or a scroll in $\mathbb{P}^2$'s over a curve.

On the other hand, for any smooth curve C in $\mathbb{P}^3$, there exists a smooth surface S containing C, see [Ha, Ch IV exercise 6.9]. However, in case $X^n \subset \mathbb{P}^{n+2}$, $n \geq 2$, this condition is very strong and we have the following equivalent conditions by Grothendieck's Lefschetz theorem:

(a) X is a complete intersection.
(b) There exists a *smooth* hypersurface $Y \subset \mathbb{P}^{n+2}$ containing X.

3. Remarks on smooth threefolds of codimension two without apparent triple or 4-ple points

Let's consider the monoidal construction for a smooth threefold X in $\mathbb{P}^5$ defined over the complex number field. Let $\pi_p : X \to \mathbb{P}^4$ be a generic projection from a general pont $p = (0,0,0,0,0,1) = Z(T_0, T_1, T_2, T_3, T_4) \notin X$ with homogeneous coordinates $(T_0 : T_1 : T_2 : T_3 : T_4 : T_5)$ in $\mathbb{P}^5$. we have the following exact sequence:

$$(3.0)\quad 0 \to \mathcal{E}_X \to \mathcal{O}_{\mathbb{P}^4}(-3) \oplus \mathcal{O}_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4} \xrightarrow{\omega_3} \pi_{p_*}\mathcal{O}_X \to 0$$

where $\mathcal{E}_X = \mathrm{Ker}(\omega_3)$ is a vector bundle of rank 4 over $\mathbb{P}^4$. It follows that every fiber of $\pi_p : X \to \mathbb{P}^4$ has length at most 4 by Theorem 2.5, which implies ω_3 is surjective (see [L], [K1] for details).

Lemma 3.1. *Let X be a smooth threefold in $\mathbb{P}^5$ and $\mathcal{E}_X{}^*$ be the dual of $\mathcal{E}_X$. One gets the following:*

(a) ${\mathcal{E}_X}^*$ *is* (-3)*-regular if and only if* X *is not contained in a hyperquadric.*
(b) ${\mathcal{E}_X}^*$ *is* (-4)*-regular if and only if* X *is not contained in a cubic hypersurface and* $H^1(\mathcal{I}_X(2)) = H^1(\mathcal{O}_X(1)) = H^2(\mathcal{O}_X) = 0$.
(c) *Suppose that* X *is arithmetically Cohen-Macaulay. if* X *is not contained in a hypersurface of degree* k *then* ${\mathcal{E}_X}^*$ *is* $-(k+1)$*-regular.*

Proof. (a) Our argument is similar to that in [Al]. By definition, ${\mathcal{E}_X}^*$ is (-3)-regular iff $H^i(\mathbb{P}^4, {\mathcal{E}_X}^*(-3-i)) = 0$ for $i > 0$. By Serre's duality, this is equivalent to $H^j(\mathbb{P}^4, \mathcal{E}_X(2-j)) = 0$ for $0 \leq j \leq 3$. For $j = 3$, this follows from Kodaira's vanishing theorem, i.e. $H^3(\mathcal{E}_X(-1)) = H^2(\mathcal{O}_X(-1)) = 0$. From the exact cohomology sequence associated to (3.0), it follows that

$$j = 0,\ \ H^0(\mathbb{P}^4, \mathcal{E}_X(2)) = 0 \text{ if and only if } H^0(\mathcal{I}_X(2)) = 0,$$

$$j = 1,\ \ H^1(\mathcal{E}_X(1)) = 0 \text{ iff } X \text{ is linearly normal (Zak's theorem)},$$

$$j = 2,\ \ H^2(\mathcal{E}_X) = 0 \text{ if and only if } H^1(\mathcal{O}_X) = 0 \text{ (Barth's theorem)}.$$

So, we are done. (b) and (c) are straightforward by the same argument used in (a). Note that X is arithmetically Cohen-Macaulay if and only if $H^i(\mathcal{I}_X(j)) = 0$ for all $j \in \mathbb{Z}$ and $1 \leq i \leq \dim(X)$. □

Remark 3.2.

(a) In the proof of Lemma 3.1. (a),(b), smoothness of X is needed but we do not have to assume smoothness of X in Lemma 3.1.(c).
(b) For a smooth threefold X in $\mathbb{P}^5$ which is contained in either a hyperquadric or an irreducible cubic hypersurface, it is known [DP] that X is arithmetically Cohen-Macaulay.

For a smooth threefold X in $\mathbb{P}^5$, if X is either a cubic scroll or a complete intersection of two quadrics, then there is no trisecant line of X. However, we get the following general fact.

Lemma 3.3. *Let* X *be a smooth threefold of degree* d *in* $\mathbb{P}^5$*. Let* $\Sigma_3(X)$ *be the locus of all trisecant lines of* X *in the grassmannian* $\mathbb{G}(1,5)$*. Then,*

$$dim\Sigma_3(X) = 5$$

unless X *is either a cubic scroll or a complete intersection of two quadrics.*

Proof. This actually comes from the "classical trisecant lemma " for curves in $\mathbb{P}^3$, i.e. let $\Sigma_3(C)$ be the locus of all trisecant lines of a curve $C \subset \mathbb{P}^3$ in the grassmannian $\mathbb{G}(1,3)$. Then, $\dim\Sigma_3(C) \leq 1$ and furthermore, by Castelnuovo's genus bound, $\dim\Sigma_3(C) = 1$ unless it is either a twisted cubic curve or a complete intersection of two quadrics.

Consider the incidence correspondence

$$\Phi_3 = \{(\ell, H) : \ell \subset H\} \subset \Sigma_3(X) \times \mathbb{P}^{5^*}$$

with two projections π_1 and π_2 to $\Sigma_3(X)$ and $\mathbb{P}^{5^*}$ respectively. Note that the fibers of π_1 are all irreducible of dimension 3. So $\dim\Phi_3 = \dim\Sigma_3(X) + 3$. If we can show that all generic fibers of π_2, ${\pi_2}^{-1}(H) = \{(\ell, H) : \ell \subset H\} = \Sigma_3(X \cap H)$ are 3-dimensional, then $\dim \Phi_3 = 8$ and $\dim \Sigma_3(X) = 5$.

Claim. *$\dim \Sigma_3(X \cap H) = 3$*

For a smooth surface $S = X \cap H$ in $\mathbb{P}^4$, consider similarly

$$\begin{array}{ccc} \boldsymbol{\Phi_2 = \{(\ell, H) : \ell \subset H\} \subset \Sigma_3(S) \times \mathbb{P}^{4*}} & \xrightarrow{\pi_2} & \mathbb{P}^{4*} \\ \downarrow {\scriptstyle \pi_1} & & \\ \Sigma_3(S) \subset \mathbb{G}(1, 4) & & \end{array}$$

Then, the fibers of π_1 are all irreducible of dimension 2. So $\dim \Phi_2 = \dim \Sigma_3(S) + 2$. By the classical “trisecant lemma" for curves in $\mathbb{P}^3$, all generic fibers of π_2, ${\pi_2}^{-1}(H) = \{(\ell, H) : \ell \subset H\} = \Sigma_3(S \cap H)$ are exactly 1-dimensional because it is neither a twisted cubic curve nor a complete intersection of two quadrics, so $\dim \Phi_2 = 5$ and $\dim \Sigma_3(S) = 3$ and we are done. □

On the other hand, a smooth threefold $X \subset \mathbb{P}^5$ is said to be without apparent quadruple points if the quadrisecant lines of X do not fill up $\mathbb{P}^5$ (i.e. $\dim S_4(X) \leq 4$), equivalently a general projection of X to $\mathbb{P}^4$ has no quadruple points.

Next, we check how the absence of apparent quadruple points of a smooth threefold affects its cohomological properties and defining equations.

Theorem 3.4. *Let X be a smooth threefold of degree d in $\mathbb{P}^5$.*

(a) $\dim S_3(X) \leq 4$ if and only if X is contained in a hyperquadric;
(b) If $\dim S_4(X) \leq 4$ and X is not lying on a cubic hypersurface, then either $\dim H^1(X, \mathcal{O}_X(1)) \neq 0$ or $\dim H^2(X, \mathcal{O}_X) \neq 0$.

Proof. For a proof of (a), ‘if’ part is trivial by Bezout theorem. Now, suppose X is not contained in a hyperquadric Q and $\dim S_3(X) \leq 4$. Therefore, we have an exact sequence

$$0 \to \mathcal{E}_X \to \mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4} \xrightarrow{\omega_1 = [T_5, 1]} \pi_{p_*}\mathcal{O}_X \to 0$$

where $p = (0, 0, 0, 0, 0, 1) = Z(T_0, T_1, T_2, T_3, T_4) \notin X$, $\pi_p : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ is a generic projection and $\mathcal{E}_X$ is a vector bundle of rank 2 ([L], [K2]). Then, by Lemma 3.1.(a), ${\mathcal{E}_X}^*$ is (-3)-regular and consequently, ${\mathcal{E}_X}^*(-3) = {\mathcal{E}_X}^*(-2) \otimes \mathcal{O}_{\mathbb{P}^4}(-1)$ is globally generated (see Lemma 2.1.(a)) and ${\mathcal{E}_X}^*(-2)$ is an ample vector bundle over $\mathbb{P}^4$ (see Remark 2.3). Therefore Le Potier’s vanishing theorem implies $H^i(\mathbb{P}^4, \mathcal{E}_X(2)) = 0$ for $0 \leq i \leq 2$. So, X is 2-normal and the following morphism

$$H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \oplus H^0(\mathcal{O}_{\mathbb{P}^4}(2)) \xrightarrow{H^0(\omega_1)} H^0(\mathbb{P}^4, \pi_{p_*}\mathcal{O}_X(2)) \simeq H^0(X, \mathcal{O}_X(2))$$

is an isomorphism. By dimension counting, $\dim H^0(\mathbb{P}^5, \mathcal{I}_{X/\mathbb{P}^5}(2)) = 1$ and X is contained in a hyperquadric Q whose quadratic equation is

$$T_5^2 + a_1 T_5(b_0T_0 + b_1T_1 + b_2T_2 + b_3T_3 + b_4T_4) + \sum_{0\le i,j\le 4} c_{ij}T_iT_j = 0.$$

However, this contradicts our assumption.

For a proof of (b), suppose X is not contained in a cubic hypersurface and $\dim S_4(X) \le 4$. Therefore, we have an exact sequence

$$0 \to \mathcal{E}_X \to \mathcal{O}_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4} \xrightarrow{\omega_2=[T_5^2,T_5,1]} \pi_{p_*}\mathcal{O}_X \to 0$$

and $\mathrm{reg}\mathcal{E}_X{}^* \le (-3)$ as in proof of (a). Assume that $\mathcal{E}_X{}^*$ is (-4)-regular. Then, $\mathcal{E}_X{}^*(-4) = \mathcal{E}_X{}^*(-3) \otimes \mathcal{O}_{\mathbb{P}^4}(-1)$ is globally generated and $\mathcal{E}_X{}^*(-3)$ is an ample vector bundle. By Le Potier's vanishing theorem, $H^i(\mathbb{P}^4, \mathcal{E}_X(3)) = 0$ for all $i = 0, 1$. Therefore, X is 3-normal and the following morphism

$$H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \oplus H^0(\mathcal{O}_{\mathbb{P}^4}(2)) \oplus H^0(\mathcal{O}_{\mathbb{P}^4}(3)) \xrightarrow{H^0(\omega_2)} H^0(\pi_{p_*}\mathcal{O}_X(3))$$

is an isomorphism. So, by dimension counting, $\dim H^0(\mathcal{I}_X(3)) = 1$ and X lies on an irreducible cubic hypersurface which contradicts our assumption. Therefore, $\mathcal{E}_X{}^*$ is not (-4)-regular.

Thus, If $\dim S_4(X) \le 4$ and $\dim H^0(\mathcal{I}_X(3)) = 0$ then X should satisfy $\dim H^1(X, \mathcal{O}_X(1)) \ne 0$ or $\dim H^2(X, \mathcal{O}_X) \ne 0$ by Lemma 3.1.(b). □

Corollary 3.5. *Let X be a smooth threefold of degree d in $\mathbb{P}^5$.*

(a) Suppose that X is neither a cubic scroll nor a complete intersection of two quadrics and $S_3(X) \ne \mathbb{P}^5$, then $S_3(X)$ is the unique quadric hypersurface containing X.

(b) *If X is not 2-normal, then $S_4(X) = \mathbb{P}^5$.*

(c) *Suppose X be arithmetically Cohen-Macaulay, not necessary smooth. Then we have $\dim S_4(X) \le 4$ if and only if it is contained in a cubic hypersurface.*

(d) *For a smooth surface in $\mathbb{P}^4$, we can similarly show that if $S_3(X) \ne \mathbb{P}^4$ and $H^0(\mathcal{I}_X(2)) = 0$ then $H^1(X, \mathcal{O}_X) \ne 0$.*

Proof. For a proof of (a), by Lemma 3.3, X has always a 5-dimensional family of trisecant lines unless X is either a cubic scroll or a complete intersection of two quadrics. By Theorem 3.4.(a), if $\dim S_3(X) \le 4$ then X is contained in a hyperquadric Q and consequently, $S_3(X) \subset Q$.

By the way, since any projective variety of dimension 3 or less can not contain 5-dimensional family of lines,(see also Theorem 2.6), we have $S_3(X) = Q$ which is the *unique* quadric hypersurface containing X. (b),(c) and (d) easily follow from the proof of Theorem 3.4.(b). For (d), note also that the only irregular smooth surface of codimension two such that $S_3(X) \ne \mathbb{P}^4$ and $H^0(\mathcal{I}_X(2)) = 0$ is a quintic elliptic scroll which is cut out by cubic hypersurfaces, see [Au] for a proof. □

Remark 3.6. Fyodor L. Zak pointed out that Theorem 3.4.(a) can be shown from Aure's work, by taking a generic hyperplane section and using the inextensibility of a quintic elliptic scroll. Also, one can prove an analog of Severi's theorem for varieties of codimension two with arbitrary singularities. To wit, if $X^n \subset \mathbb{P}^{n+2}$ is such a variety, then there are the following possibilities:

(a) $\dim S_3(X) \leq n$. In this case X is either a complete intersection of two quadrics or a cone over a twisted cubic curve in $\mathbb{P}^3$ or a cubic scroll in $\mathbb{P}^4$ (that is, the image of $\mathbb{P}^2$ under the map defined by the linear system of conics passing through a given point) or the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$;
(b) $\dim S_3(X) = n + 1$. In this case there are two possibilities:
  (i) $S_3(X)$ is a quadric. Conversely, if $X \subset Q$, where $Q^{n+1} \subset \mathbb{P}^{n+2}$ is a quadric, then $S_3(X) \subset Q$ and $S_3(X) = Q$ except in the cases listed in (a);
  (ii) $S_3(X)$ is a "scroll", i.e. a union of the members of a one-dimensional family of $\mathbb{P}^n$ in $\mathbb{P}^{n+2}$. In this case X contains a family of (degenerate) hypersurfaces of degree at least three in $\mathbb{P}^n$'s;
(c) $S_3(X) = \mathbb{P}^{n+2}$.

To prove the above it suffices to use the well known and easy fact that an m-dimensional variety containing a $(2m-3)$-dimensional family of lines is either a quadric or a "scroll", see Theorem 2.6. □

According to Ch. Peskine, the following proposition is known, but, since we do not know of any reference, we give its proof based on the techniques used in the proof of Theorem 3.4.

Proposition 3.7. *Let $X \subset \mathbb{P}^{n+2}$ be an arithmetically Cohen-Macaulay (not necessary smooth) projective subvariety of codimension two. Then for all $m \leq n, 2 \leq n$, we have the following equivalence:*

(a) $S_{m+1}(X) \neq \mathbb{P}^{n+2}$
(b) X is not contained in a hypersurface of degee m.

Proof. Let's assume that X is not contained in a hypersurface of degree m and $S_{m+1}(X) \neq \mathbb{P}^{n+2}$. Therefore, we have an exact sequence

$$0 \to \mathcal{E}_X \to \mathcal{O}_{\mathbb{P}^{n+1}}(-m+1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{n+1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n+1}} \to \pi_{p_*}\mathcal{O}_X \to 0$$

where $\mathcal{E}_X$ is a vector bundle of rank m because X is arithmetically Cohen-Macaulay. Note that $\mathrm{reg}\mathcal{E}_X{}^*$ is $-(m+1)$-regular by Lemma 3.1.(c) and $\mathcal{E}_X{}^*(-m)$ is an ample vector bundle. Similarly, we get $H^i(\mathbb{P}^{m+1}, \mathcal{E}_X(m)) = 0$ for $i = 0, 1$ by Le Potier's vanishing theorem. Furthermore, X is m-normal and the following morphism

$$\bigoplus_{i=1}^{m} H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(i)) \longrightarrow H^0(\pi_{p_*}\mathcal{O}_X(m))$$

is an isomorphism. However, by this isomorphism we have $\dim H^0(\mathcal{I}_X(m)) = 1$ and X lies on an irreducible hypersurface of degree m which contradicts our assumption. □

We remark that in the proof of Theorem 3.4, the global generation of $\mathcal{E}(-1)$ for some vector bundle $\mathcal{E}$ is required for showing ampleness of $\mathcal{E}$ and in order to show the globally generatedness of $\mathcal{E}(-1)$, Castelnuovo-Mumford regularity of $\mathcal{E}$ is used in the proof of Theorem 3.4.

The following generalized Castelnuovo-Mumford criterion for vector bundles to be globally generated is useful to weaken the assumption of Theorem 3.4.

Lemma 3.8 (Castelnuovo-Mumford). *Let $\mathcal{M}$ be a coherent sheaf on $\mathbb{P}^n = \mathbb{P}(V^*)$ with no nonzero skyscraper subsheaves, and let $0 \le q \le n-1$ and m be integers. Suppose that $\mathcal{M}$ is $(m+1)$-regular, that $H^i(\mathcal{M}(m-i)) = 0$ for $i \ge n-q$, and that the comultiplication maps*

$$(**) \qquad H^i(\mathcal{M}(m-1-i)) \to V \otimes H^i(\mathcal{M}(m-i))$$

are surjective for $1 \le i \le n-q-1$. Then for any $0 \le j \le q$, the module of j-th syzygies of $H^0_*(\mathcal{M})$ is generated in degrees $\le m+j$.

Proof. This is a strict application of the proof of proposition given in lecture 14, [Mu]. See also lemma 8.8 in [EPW]. □

The following Theorem is a slight generalization of Theorem 3.4.(b) by using the above Lemma 3.6.

Theorem 3.9. *Let X be a smooth threefold in $\mathbb{P}^5$. Suppose that X is not contained in a cubic hypersurface. If the following two multiplicative maps*

$$H^0(\mathcal{O}_{\mathbb{P}^5}(1)) \otimes H^1(\mathcal{O}_X(1)) \to H^1(\mathcal{O}_X(2))$$
$$H^0(\mathcal{O}_{\mathbb{P}^5}(1)) \otimes H^2(\mathcal{O}_X) \to H^2(\mathcal{O}_X(1))$$

are injective, then $S_4(X) = \mathbb{P}^5$.

Proof. Suppose X is not contained in a cubic hypersurface and $S_4(X) \neq \mathbb{P}^5$. Then, we have an exact sequence

$$0 \to \mathcal{E}_X \to \mathcal{O}_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4} \xrightarrow{\omega_2=[T_5^2, T_5, 1]} \pi_{p_*}\mathcal{O}_X \to 0$$

and reg${\mathcal{E}_X}^*$ is at least (-3)-regular as shown in Theorem 3.4. It is easy to check that the surjective condition on the comultiplicative maps in $(**)$ is equivalent to the injective condition on the following multiplicative maps

$$H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes H^j(\mathcal{M}^*(-m-1-j)) \to H^j(\mathcal{M}^*(-m-j))$$

for $q+1 \le j \le n-1$. Note also that X is 2-normal, $H^0(\mathcal{I}_X(3)) = 0$ and $\mathcal{M} = {\mathcal{E}_X}^*(-4)$ is 1-regular by Lemma 2.1.(b). Thus, $\mathcal{M} = {\mathcal{E}_X}^*(-4)$ satisfies the assumption in Lemma 3.7 for $m = 0$, and $q = 1$.

Therefore, $\mathcal{M} = {\mathcal{E}_X}^*(-4)$ is generated by global sections. As in the proof of Theorem 3.4.(b), by Le Potier's vanishing theorem for an ample vector bundle ${\mathcal{E}_X}^*(-3)$, $H^i(\mathbb{P}^4, \mathcal{E}_X(3)) = 0$ for all $i = 0, 1$.

Similarly, by dimension counting, $\dim H^0(\mathcal{I}_X(3)) = 1$ and X lies on an irreducible cubic hypersurface which contradicts our assumption. □

Remark 3.10.

(a) If X is not contained in a quadric hypersurface Q then the locus of triple points under the generic projection of X into $\mathbb{P}^4$ is a curve by Theorem 3.4.(a) and Lemma 3.3. It is still open that this triple curve is irreducible except the Palatini scroll of degree 7.

(b) (**Peskine-Zak Conjecture**). Let $X \subset \mathbb{P}^5$ be a nondegenerate (not necessary smooth) threefold and let $\mathcal{I}_X$ be its sheaf of ideals. For $i \geq 1$, $j \geq 0$, $i+j = 3$, it is possible to describe all varieties for which $H^i(\mathbb{P}^N, \mathcal{I}_X(j)) \neq 0$. In particular, for $i = 1$, $j = 2$, it is conjectured by Peskine and Van de Ven that all threefolds are quadratically normal except the Palatini scroll of degree 7. So, we might guess there are a few examples such that $\dim H^0(\mathbb{P}^5, \mathcal{I}_{X/\mathbb{P}^5}(3)) = 0$, $\dim S_4(X) \leq 4$ and either $\dim H^1(X, \mathcal{O}_X(1)) \neq 0$ or $\dim H^2(X, \mathcal{O}_X) \neq 0$.

(c) It is useful to remark that for any integer $d \geq 7$ with exception $d = 8, 10$, there exist smooth threefolds in $\mathbb{P}^5$ which are not arithmetically Cohen-Macaulay, see [Mi, Corollary 1.2.].

4. The quadruple-point formula and open questions

Let X be a smooth threefold of degree d in $\mathbb{P}^5$ and S be a general hyperplane section surface. For a generic point $p \in \mathbb{P}^5$, we know that there are at most finitely many quadrisecant lines of X passing through p. Thus, denote by $q_4(X)$ the number of all quadrisecant lines of X through p. It would be interesting to compute such a number $q_4(X)$ in terms of basic invariants of X, i.e. d, $\chi(\mathcal{O}_X)$, $\chi(\mathcal{O}_S)$ and the sectional genus π as like the double-point formula for space curves, see Remark 4.4.

Consider the following exact sequence mentioned in Sect. 3, twisted by $\mathcal{O}_{\mathbb{P}^4}(3)$;

$$(4.0) \quad 0 \to \mathcal{E}_X(3) \to \mathcal{O}_{\mathbb{P}^4} \oplus \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(3) \xrightarrow{\omega_3} \pi_{p_*}\mathcal{O}_X(3) \to 0.$$

Let $Y_k = \{y \in Y = \pi_p(X) |\ \pi_p^{-1}(y)$ has length exactly $k\}$. Let $\ell_{p,y}$ be a line joining two points p and y. Since the finite subscheme $\pi_p^{-1}(y) \subset \ell_{p,y}$ is of length 4 for all $y \in Y_4$, it is 3-normal and the following commutative diagram

$$\begin{array}{ccc} \bigoplus_{i=0}^{3} \mathcal{O}_{\mathbb{P}^4}(i) \otimes \mathbb{C}(y) & \xrightarrow{\omega_3 \otimes \mathbb{C}(y)} & \pi_{p_*}\mathcal{O}_X(3) \otimes \mathbb{C}(y) \\ \simeq\downarrow & & \simeq\downarrow \\ H^0(\ell_{py}, \mathcal{O}_{\ell_{py}}(3)) & \xrightarrow{\simeq} & H^0(\ell_{py}, \mathcal{O}_{\pi_p^{-1}(y)}(3)) \end{array}$$

is an isomorphism. As a result, the natural morphism in the left side of (4.0)

$$\mathcal{E}_X(3) \to \mathcal{O}_{\mathbb{P}^4} \oplus \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(3)$$

is a zero map for all $y \in Y_4$. By the way, for all $y \in Y_3$

$$\bigoplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}^4}(i) \otimes \mathbb{C}(y) \xrightarrow{\omega_2 \otimes \mathbb{C}(y)} \pi_{p_*}\mathcal{O}_X(2) \otimes \mathbb{C}(y)$$

is isomorphic and the morphism $\mathcal{E}_X(3) \to \mathcal{O}_{\mathbb{P}^4} \oplus \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(3)$ has rank 1. Therefore,

$$Y_4 = \{y \in \mathbb{P}^4 | \operatorname{rank}\big(\mathcal{E}_X(3) \to \mathcal{O}_{\mathbb{P}^4}\big) = \operatorname{rank}\big(\mathcal{E}_X \to \mathcal{O}_{\mathbb{P}^4}(-3)\big) = 0\}$$

and so, $q_4(X) = \deg Y_4$ can be computed using the Giambelli-Thom-Porteous formula for the degeneracy locus of a morphism between vector bundles, see [Fu, Sect. 6].

Lemma 4.1. *For a vector bundle $\mathcal{E}_X$ in the exact sequence* (4.0)

$$\begin{aligned}
c_1(\mathcal{E}_X) &= -d - 6, \quad c_2(\mathcal{E}_X) = \frac{1}{2}(d^2 + 9d) + 12 - \pi, \\
c_3(\mathcal{E}_X) &= -\frac{1}{6}(d^3 + 9d^2 + 32d) + \pi(d+2) - 2\chi(\mathcal{O}_S) - 8, \\
c_4(\mathcal{E}_X) &= \frac{1}{24}(d^4 + 6d^3 + 11d^2 + 6d) + \frac{\pi^2}{2} - \frac{\pi}{2}(d^2 + d + 1) \\
&\quad + \chi(\mathcal{O}_S)(2d - 3) + 6\chi(\mathcal{O}_X).
\end{aligned}$$

Proof. If we can compute $c_i(\pi_{p_*}\mathcal{O}_X)$ for $i = 1, 2, 3, 4$, then by additivity of Chern polynomials we can also get $c_i(\mathcal{E}_X)$. By the Grothendieck-Riemann-Roch theorem, putting $\operatorname{td}(\mathcal{T}_X)$ to be the Todd class of the tangent bundle $\mathcal{T}_X$ of X,

$$\text{(4.1.1)} \qquad \operatorname{ch}(\pi_{p_*}\mathcal{O}_X) \cdot \operatorname{td}(\mathbb{P}^4) = \pi_{p_*}\big(\operatorname{ch}(\mathcal{O}_X) \cdot \operatorname{td}(\mathcal{T}_X)\big).$$

By direct computations,

$$\begin{aligned}
&\operatorname{ch}(\pi_{p_*}\mathcal{O}_X) \cdot \operatorname{td}(\mathbb{P}^4) \\
&= c_1 + \frac{1}{2}(c_1^2 - 2c_2 + 5c_1) + \frac{1}{12}(2c_1^3 - 6c_1c_2 + 6c_3 + 15c_1^2 - 30c_2 + 35c_1) \\
&\quad + \frac{1}{24}\big(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4 + 10(c_1^3 - 3c_1c_2 + 3c_3) \\
&\quad + 35(c_1^2 - 2c_2) + 50c_1\big),
\end{aligned}$$

where $c_i = c_i(\pi_{p_*}\mathcal{O}_X)$. On the other hand,

$$\begin{aligned}
&\pi_{p_*}(\operatorname{ch}(\mathcal{O}_X) \cdot \operatorname{td}(\mathcal{T}_X)) \\
&\quad = \pi_{p_*}\Big(1 + \frac{c_1(X)}{2} + \frac{c_1(X)^2 + c_2(X)}{12} + \frac{c_1(X)c_2(X)}{24}\Big)
\end{aligned}$$

where $c_i(X)$ is the i-th Chern class of X.

Remark the following identities for any smooth projective threefold X;

(4.1.2) $K_X \cdot H^2 = 2\pi - 2 - 2d$ (adjunction formula), (4.1.3) $\frac{1}{24}c_1(X)c_2(X) = \chi(\mathcal{O}_X)$ (Noether formula).

and in particular, we have also the identities for a smooth threefold X in $\mathbb{P}^5$ as follows;

(4.1.4) $c_2(X) = (15-d)H^2 + 6HK_X + K_X^2$ (example 4.1.3 [Ha, p 433]),
(4.1.5) $K_X^2 \cdot H = \frac{1}{2}(d^2+d) - 9(\pi-1) + 6\chi(\mathcal{O}_S)$ (double-point formula).

By using the above identities, we get

$$\begin{aligned}&\pi_{p_*}(\mathrm{ch}(\mathcal{O}_X)\cdot \mathrm{td}(\mathcal{T}_X))\\ &\quad = dH + (d+1-\pi)H^2 + \Big(\frac{d}{3} - \frac{\pi-1}{2} + \chi(\mathcal{O}_S)\Big)H^3 + \chi(\mathcal{O}_X)H^4.\end{aligned}$$

Therefore, we have the following equalities by comparing both sides of (4.1.1):

$$\begin{aligned}c_1\big(\pi_{p_*}\mathcal{O}_X\big) &= d, \quad c_2\big(\pi_{p_*}\mathcal{O}_X\big) = \frac{1}{2}(d^2+3d) + \pi - 1,\\ c_3\big(\pi_{p_*}\mathcal{O}_X\big) &= \frac{1}{6}d(d+1)(d+8) + \pi(d+4) + 2(\chi(\mathcal{O}_S)-2),\\ c_4\big(\pi_{p_*}\mathcal{O}_X\big) &= \frac{1}{24}(d^4+18d^3+71d^2-42d) + \frac{\pi^2}{2} - \frac{\pi}{2}(d^2+11d+23)\\ &\quad +\chi(\mathcal{O}_S)(2d+15) - 6\chi(\mathcal{O}_X) - 12.\end{aligned}$$

Consequently, by additivity of Chern polynomials, we have

$$c_t(\mathcal{E}_X)\cdot c_t(\pi_{p_*}\mathcal{O}_X) = (1-t)(1-2t)(1-3t).$$

Therefore, we can get $c_i(\mathcal{E}_X)$ as described above. □

Theorem 4.2. *Let X be a smooth threefold of degree d in $\mathbb{P}^5$ with the Euler characteristics $\chi(\mathcal{O}_X)$, $\chi(\mathcal{O}_S)$, and the sectional genus π. Then, the number of all quadrisecant lines of X through a general point in $\mathbb{P}^5$, $q_4(X)$ is given as follows:*

$$\begin{aligned}q_4(X) &= \frac{1}{24}(d^4 - 6d^3 + 11d^2 - 54d + 72) + \frac{\pi^2}{2} - \frac{\pi}{2}(d^2-5d+7)\\ &\quad +2\chi(\mathcal{O}_S)d + 6\chi(\mathcal{O}_X) - 9\chi(\mathcal{O}_S).\end{aligned}$$

Proof. As we mentioned before, Y_4 is defined as the degeneracy locus of the following morphism $\varphi : \mathcal{E}_X \to \mathcal{O}_{\mathbb{P}^4}(-3)$. So, by the Giambelli-Thom-Porteous formula [Fu, Sect. 6]

$$\begin{aligned}q_4(X) &= \deg Y_4 = \Delta_1^{(4)}(c(\mathcal{O}_{\mathbb{P}^4}(-3) - \mathcal{E}_X))\\ &= \det\begin{pmatrix} c_1 & c_2 & c_3 & c_4\\ 1 & c_1 & c_2 & c_3\\ 0 & 1 & c_1 & c_2\\ 0 & 0 & 1 & c_1\end{pmatrix} = c_1^4 - 3c_1^2c_2 + 2c_1c_3 + c_2^2 - c_4,\end{aligned}$$

where $c_i = c_i(\mathcal{O}_{\mathbb{P}^4}(-3) - \mathcal{E}_X)$ is the i-th Chern class of $\mathcal{O}_{\mathbb{P}^4}(-3) - \mathcal{E}_X$.

Let $s_i = s_i(\mathcal{E}_X)$ be the i-th Segre class. In general, the Chern classes and Segre classes of a vector bundle are formally related as follows:

$$(4.2.1) \qquad (1 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4) = (1 + s_1 t + s_2 t^2 + s_3 t^3 + \cdots)^{-1}.$$

So,

$$\begin{aligned}
&c_t(\mathcal{O}_{\mathbb{P}^4}(-3) - \mathcal{E}_X) \\
&\quad = c_t(\mathcal{O}_{\mathbb{P}^4}(-3)) s_t(\mathcal{E}_X) \\
&\quad = (1 - 3t)(1 + s_1 t + s_2 t^2 + s_3 t^3 + s_4 t^4) \\
&\quad = 1 + (s_1 - 3)t + (s_2 - 3s_1)t^2 + (s_3 - 3s_2)t^3 + (s_4 - 3s_3)t^4.
\end{aligned}$$

Thus, we can compute all Segre classes of $\mathcal{E}_X$ by using Lemma 4.1 and (4.2.1). As a result, we get the $c_i(\mathcal{O}_{\mathbb{P}^4}(-3) - \mathcal{E}_X)$ as follows:

$$\begin{aligned}
c_1(\mathcal{O}_{\mathbb{P}^4}(-3) - \mathcal{E}_X) &= d + 3, \\
c_2(\mathcal{O}_{\mathbb{P}^4}(-3) - \mathcal{E}_X) &= \frac{1}{2}(d^2 + 9d) + \pi + 6, \\
c_3(\mathcal{O}_{\mathbb{P}^4}(-3) - \mathcal{E}_X) &= \frac{1}{6}(d^3 + 18d^2 + 77d) + \pi(d + 7) + 2(\chi(\mathcal{O}_S) + 8), \\
c_4(\mathcal{O}_{\mathbb{P}^4}(-3) - \mathcal{E}_X) &= \frac{1}{24}(d^4 + 30d^3 + 263d^2 + 666d) + \frac{\pi^2}{2} \\
&\quad + \frac{\pi}{2}(d^2 + 17d + 61) + \chi(\mathcal{O}_S)(2d + 21) - 6\chi(\mathcal{O}_X).
\end{aligned}$$

Finally, by plugging $c_i(\mathcal{O}_{\mathbb{P}^4}(-3) - \mathcal{E}_X)$ into (4.3) we obtain the desired formula on $q_4(X)$. □

Corollary 4.3. *Let X be a smooth threefold of degree d in $\mathbb{P}^5$ with the Euler characteristics $\chi(\mathcal{O}_X)$, $\chi(\mathcal{O}_S)$, and the sectional genus π. Suppose $q_4(X) = 0$. Then the sectional genus π is given as follows:*

$$\begin{aligned}
\pi = &\frac{1}{2}(d^2 - 5d + 7) - \\
&\frac{1}{6}\sqrt{6d^4 - 72d^3 + 318d^2 - 468d - 144\chi(\mathcal{O}_S)d + 225 + 648\chi(\mathcal{O}_S) - 432\chi(\mathcal{O}_X)}.
\end{aligned}$$

Proof. It is obtained by solving the equation $q_4(X) = 0$ in Theorem 4.2 and Castelnuovo's genus bound for curves in $\mathbb{P}^3$. □

Remark 4.4. We are also able to compute by the same method used in the proof of Theorem 4.2 the well known formulas enumerating secant lines of a smooth curve C in $\mathbb{P}^3$ and trisecant lines of a smooth surface S in $\mathbb{P}^4$ through a general point. For examples, let us denote such numbers by $q_2(C)$ and $q_3(S)$ respectively. Then,

$$q_2(C) = \deg Y_2 = \frac{(d-1)(d-2)}{2} - g(C)$$
$$q_3(S) = \deg Y_3 = \frac{1}{6}(d-1)(d-2)(d-3) - \pi(d-3) + 2\chi(\mathcal{O}_S) - 2$$

where $Y_2 = \{y \in \mathbb{P}^2 | \operatorname{rank}\big(\mathcal{E}_C \to \mathcal{O}_{\mathbb{P}^2}(-1)\big) = 0\}$, $Y_3 = \{y \in \mathbb{P}^3 | \operatorname{rank}\big(\mathcal{E}_S \to \mathcal{O}_{\mathbb{P}^3}(-2)\big) = 0\}$ are the degeneracy loci induced by the following two monoidal morphisms respectively:

$$0 \to \mathcal{E}_C \to \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\omega_1} \pi_{p_*}\mathcal{O}_C \to 0,$$
$$0 \to \mathcal{E}_S \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\omega_2} \pi_{p_*}\mathcal{O}_S \to 0.$$

Furthermore, we'd like to mention here that for a smooth threefold of arbitrary codimension, the quadruple-point formula is given in terms of algebraic cycles, see [PR], [Kl, p 389].

Examples 4.5. Let's compute $q_4(X)$ for well known threefolds in $\mathbb{P}^5$. First note that if X is contained in a cubic hypersurface then $q_4(X) = 0$. Put $h^0(\mathcal{I}_X(3)) = \dim H^0(\mathcal{I}_X(3))$.

Table 4.6.

	$q_4(X)$	d	π	$\chi(\mathcal{O}_X)$	$\chi(\mathcal{O}_S)$	$h^0(\mathcal{I}_X(3))$
Castelnuovo 3-fold	0	5	2	1	1	nonzero
Bordiga 3-fold	0	6	3	1	1	nonzero
Palatini scroll	1	7	4	1	1	0
K3 scroll	1	9	8	2	2	0

Open questions 4.7.

(a) Classify all smooth threefolds in $\mathbb{P}^5$ with $q_4(X) = 0$, $h^0(\mathcal{I}_X(3)) = 0$.
(b) Fyodor L. Zak asked what in general can be said about geometric and cohomological properties of a (smooth) projective variety $X^n \subset \mathbb{P}^N$ such that $S_{m+1}(X) \neq \mathbb{P}^N$ and $H^0(\mathcal{I}_X(m)) = 0$ for some $m \leq \frac{n}{N-n-1}$.

Remark 4.8 (F. Zak).

(a) It should be pointed out that, in accordance with 4.7 (b), a true generalization of Aure's work would be to classify all nonsingular $X^n \subset \mathbb{P}^N$ with $n = \frac{2}{3}(N-1)$ and $S_3(X) \neq \mathbb{P}^N$ (the first really interesting case being fourfolds in $\mathbb{P}^7$).
(b) Let $G(N, 1)$ be the Grassmann variety of lines in $\mathbb{P}^N$, let Σ^{N-3} be its section by a general linear subspace of codimension $N+1$ in $\langle G(N, 1)\rangle$, and let V^{N-2} be the variety in $\mathbb{P}^N$ swept out by the lines from Σ (thus, for $N = 4$

one gets a quintic elliptic scroll and for $N = 5$ a scroll over a K3 surface). Then, denoting by $q_{N-1}(V)$ the number of all $(N-1)$-secant lines of V through a general point in $\mathbb{P}^N$, it can be shown that $q_{N-1}(V) = N \pmod 2$. The difference between the even- and odd-dimensional cases reduces to the fact that a general linear complex has a center iff N is even. $\square$

Now, let us consider the structure of the quadrisecant locus $S_4(X) \subset \mathbb{P}^5$ if the quadrisecant lines of X do not fill up the ambient space $\mathbb{P}^5$.

Proposition 4.9. *Suppose X be a smooth threefold of degree $d \geq 9$ without apparent 4-uple points in $\mathbb{P}^5$. If X is not contained in a cubic hypersurface, then $S_4(X)$ is a hypersurface in $\mathbb{P}^5$.*

Proof. Let $\Sigma_4(X)$ be the locus of 4-secant lines in the $\mathbb{G}(1,5)$ as before. Remark that $\dim \Sigma_4(X) \leq 5$. Since $\dim S_4(X) \leq 4$, we get $\dim \Sigma_4(X) \neq 5$ by Theorem 2.6 and so $\dim \Sigma_4(X) \leq 4$. Next, if we assume $\dim \Sigma_4(X) \leq 3$ then for a smooth sectional curve $C = X \cap H_1 \cap H_2 \subset \mathbb{P}^3$, we can easily show $\dim \Sigma_4(C) = \varnothing$(see Lemma 3.3) which means C has no 4-secant lines. On the other hand, X is 2-normal by Corollary 3.5.(b). Hence,

$$H^0(\mathcal{I}_X(3)) \to H^0(\mathcal{I}_{X\cap H_1}(3))$$

is surjective and furthermore $H^0(\mathcal{I}_C(3)) \neq 0$ implies $H^0(\mathcal{I}_{X\cap H_1}(3)) \neq 0$ for $d \geq 9$ by Roth-type lifting theorem (see theorem 0.1 in [Me]). Therefore, our assumption $H^0(\mathcal{I}_X(3)) = 0$ implies $H^0(\mathcal{I}_C(3)) = 0$. By the way, from Lemma 2 in [Ma] a smooth curve C with $H^0(\mathcal{I}_{C/\mathbb{P}^3}(3)) = 0$ has always 4-secant lines, and we reach a contradiction. Now we assume $\dim \Sigma_4(X) = 4$. By Theorem 2.6 again, $S_4(X)$ is just 4-dimensional and we are done. $\square$

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