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Higher linear syzygies of inner projections ☆

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Abstract

Let $X \subset \mathbb{P}^r$ be a smooth projective variety embedded by the complete linear system associated to a very ample line bundle $\mathcal{L}$ on X. In this paper, we prove that if X satisfies property N_p for $p \geqslant 1$, then every isomorphic inner projection $\tilde{X} \subset \mathbb{P}^{r-1}$ of X satisfies N_{p-1}. We also give some applications of our results to various examples.

Keywords: Syzygies; Property N_p; Inner projections

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1. Introduction

For a smooth projective variety $X \subset \mathbb{P}(H^0(\mathcal{L}))$ embedded by the complete linear system of a very ample line bundle $\mathcal{L}$ on X, one can ask more detailed information about defining equations of X, i.e., the syzygies of X. As M. Green defined [11], we can say that $\mathcal{L}$ satisfies property N_0 if it gives the projectively normal embedding and $\mathcal{L}$ satisfies property N_1 if property N_0 holds and X is cut out by quadrics. In general, $\mathcal{L}$ satisfies property N_p, $p \geqslant 1$, if X is projectively normal and the projective coordinate ring $S(X)$ of X has the following minimal free resolution of the simplest type up to pth steps as a graded S-module:

$$\to \cdots \to E_p \to E_{p-1} \to \cdots \to E_1 \to S \to S(X) \to 0,$$

where $S = \mathrm{Sym}(H^0(\mathcal{L}))$ and $E_i = \bigoplus^{\beta_{i,1}} S(-i-1)$ for all $1 \leqslant i \leqslant p$. In other words, property N_p means the minimal free resolution of the homogeneous coordinate ring $S(X)$ of the projectively normal embedding of X is linear until the pth step.

In general, we can consider two kinds of subsystems of $H^0(\mathcal{L})$ with respect to projections. One gives an isomorphic outer projection of $X \subset \mathbb{P}(H^0(\mathcal{L}))$ with the center outside $\mathrm{Sec}(X)$ which is not linearly normal and the other is a subsystem with base points that is the very ample complete linear system of the blow-up of X and gives an inner projection of X. Recently, we have been interested in the geometric and syzygetic effects of property N_p of $X \subset \mathbb{P}(H^0(\mathcal{L}))$ to the isomorphic projection of X in $\mathbb{P}(W)$ by a very ample subsystem $W \subset H^0(\mathcal{L})$.

More precisely, a generalization of property N_p to a smooth nonlinearly normal variety $X \subset \mathbb{P}(W)$ can be made as follows [13]: X satisfies property N_p^S if for $R = \bigoplus_{\ell \in \mathbb{Z}} H^0(X, \mathcal{L}^\ell)$, it has the following minimal free resolution of the simplest type as a graded S_W-module:

$$\to \cdots \to E_p \to E_{p-1} \to \cdots \to E_1 \to S_W \oplus S_W(-1)^t \to R \to 0,$$

where $S_W = \mathrm{Sym}(W)$, $t = \mathrm{codim}(W, H^0(\mathcal{L}))$ and $E_i = S_W(-i-1)^{\oplus \beta_{i,1}}$ for all $1 \leqslant i \leqslant p$. In other words, property N_p^S means the minimal free resolution of R as a graded S_W-module is linear until the pth step. Generally speaking, for a nonlinearly normal embedding of X, it is hard to find defining equations or to control the degree bound of defining equations of X in $\mathbb{P}(W)$. However, it can be shown that if $X \subset \mathbb{P}(H^0(\mathcal{L}))$ satisfies property N_p then the isomorphic projection of X in $\mathbb{P}(W)$ satisfies property N_{p-t}^S for $t = \mathrm{codim}(W, H^0(\mathcal{L})) \leqslant p$. Furthermore, if X satisfies property N_1^S, then defining equations of X in $\mathbb{P}(W)$ have degree at most $(t+2)$, see [13] for details.

In the present paper, we are mainly interested in an inner projection of X which is given by subsystem W of $H^0(\mathcal{L})$ with given base points in X. Let X be a smooth projective variety in $\mathbb{P}(H^0(\mathcal{L}))$. For a closed point $q \in X$, the inner projection $\pi_q : X \dashrightarrow \mathbb{P}(W)$ defined by $\pi_q(p) = \overline{qp} \cap \mathbb{P}(W)$, $p \neq q$, is a rational map. We can understand this situation in the following diagram; for the blow-up $\mathrm{Bl}_q(X)$ of X at q, one has the regular morphism $\tilde{\pi}_q : \mathrm{Bl}_q(X) \to \mathbb{P}(W)$ with the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Bl}_q(X) & & \\ {\scriptstyle\sigma}\downarrow & \searrow^{\tilde{\pi}_q} & \\ X \subset P\big(H^0(\mathcal{L})\big) & \overset{\pi_q}{\dashrightarrow} & X' = \overline{\pi_q\big(X \setminus \{q\}\big)} \subset \mathbb{P}(W) \end{array}$$

It is said that X admits an inner projection at a point $q \in X$ if the morphism $\tilde{\pi}_q : \mathrm{Bl}_q(X) \to \overline{\pi_q(X \setminus \{q\})}$ is an embedding, i.e., $\sigma^*\mathcal{L} - E$ is very ample. When one considers this situation in the projective embedding, we have a nice criterion for $\sigma^*\mathcal{L} - E$ to be very ample. Note that the inner projection $\tilde{\pi}_q : \mathrm{Bl}_q(X) \to \mathbb{P}(W)$ with center $q \in X$ is a closed embedding if and only if $q \notin \mathrm{Trisec}(X)$ where $\mathrm{Trisec}(X)$ is the union of all trisecant lines to X and all lines contained in X [8, pp. 268–275]. In this case, $\tilde{\pi}_q(\mathrm{Bl}_q(X))$ is equal to $\overline{\pi_q(X \setminus \{q\})}$, the Zariski closure of $\pi_q(X \setminus \{q\})$.

With these in mind, we get the following theorem without any assumption on a very ample line bundle $\mathcal{L}$.

Theorem 1.1. *Let $X \subset \mathbb{P}(H^0(\mathcal{L}))$ be a smooth irreducible projective variety. Suppose $\mathcal{L}$ satisfies property N_p for $p \geqslant 1$. For any $q \in X \setminus \mathrm{Trisec}(X)$, $\tilde{\pi}_q(\mathrm{Bl}_q(X)) = \overline{\pi_q(X \setminus \{q\})}$ in $\mathbb{P}(W)$ is smooth and satisfies property N_{p-1}, i.e., property N_{p-1} holds for $(\mathrm{Bl}_q(X), \sigma^*\mathcal{L} - E)$.*

Let $\sigma : \tilde{X} \to X$ be the blowing-up at distinct s points $q_1, q_2, \ldots, q_s$ in X with the exceptional divisors $E_1, \ldots, E_s$. If $\mathcal{L}' := \sigma^*\mathcal{L}(-E_1 - \cdots - E_s)$ on $\tilde{X}$ is very ample and $\mathcal{L}$ satisfies N_p for $p \geqslant 1$, then we can check from Theorem 1.1 that $\mathcal{L}'$ satisfies N_{p-s} for $p \geqslant s$.

As another simple corollary, we have the following:

Corollary 1.2. *Let $X \subset \mathbb{P}(W)$ be a smooth variety with property N_p^S and $t = \mathrm{codim}(W, H^0(\mathcal{L}))$. Then, for any $q \in X \setminus \mathrm{Trisec}(X)$ we have*:

(1) *An inner projection $\overline{\pi_q(X \setminus \{q\})} \subset \mathbb{P}^{N-1}$ satisfies property N_{p-1}^S.*
(2) *The defining equations of $\overline{\pi_q(X \setminus \{q\})}$ in $\mathbb{P}^{N-1}$ have degree at most $p+2$ for $p \geqslant 2$.*

By using Koszul cohomology technique due to M. Green and the vanishing of higher cohomology groups of related vector bundles, we can show *inductively* $\mathrm{Tor}_i^{S_W}(R', \mathbb{C})_{i+j} = 0, 0 \leqslant i \leqslant p-1$ and $j \geqslant 2$, for a finitely generated graded S_W module $R' = \bigoplus_{\ell \in \mathbb{Z}} H^0((\sigma^*\mathcal{L} - E)^{\otimes \ell})$. In Section 3, we prove the main Theorem 1.1 for curves and then, we deal with the higher dimensional case in Section 4.

As a typical example of our main Theorem 1.1, it is well known that a Del Pezzo surface $\upsilon_3(\mathbb{P}^2)$ of degree 9 in $\mathbb{P}^9$ satisfies property N_6 (and fails to hold N_7) and by taking successive inner projections at general points $\{q_1, q_2, \ldots, q_i,\ i \leqslant 6\}$, we get smooth Del Pezzo surfaces of degree $9-i$ in $\mathbb{P}^{9-i}$ with property N_{6-i}. In general, it would be very interesting to know syzygies of the form $\sigma^*\mathcal{L}(-m_1E_1 - m_2E_2 - \cdots - m_\alpha E_\alpha)$ if it is very ample where $\sigma : \tilde{X} \to X$ is the blowing-up with exceptional divisors E_i, $i = 1, 2, \ldots, \alpha$.

In Section 5, we give some applications of our results to various varieties and their embeddings, such as the inner projections of Veronese embeddings, Calabi–Yau manifolds, rational surfaces, and adjoint linear series of projective varieties. We also relate multisecant spaces of X to syzygies of X.

2. Notations and preliminaries

2.1. Notations

Throughout this paper the following are assumed.

(1) We work throughout over the complex numbers.

(2) For a finite dimensional vector space V, $\mathbb{P}(V)$ is the projective space of one-dimensional quotients of V.
(3) When a projective variety X is embedded in a projective space $\mathbb{P}^r$, we always assume that it is nondegenerate, i.e., does not lie on any hyperplane in $\mathbb{P}^r$.
(4) For a smooth projective variety $X \subset \mathbb{P}^r$ embedded by the complete linear system associated to very ample line bundle $\mathcal{L}$ on X and a closed point $q \in X$, we use the following notations:
- $V = H^0(X, \mathcal{L}) = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$.
- $S = \bigoplus_{\ell\in\mathbb{Z}} \operatorname{Sym}^\ell(V) = \bigoplus_{\ell\in\mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(\ell))$: the homogeneous coordinate ring of $\mathbb{P}^r$.
- $R = \bigoplus_{\ell\in\mathbb{Z}} H^0(X, \mathcal{L}^\ell) = \bigoplus_{\ell\in\mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{O}_X(\ell))$: the associated graded S-module of $\mathcal{O}_X$.
- $\tilde{X} = \operatorname{Bl}_q(X)$: a blowing-up of X at a point q with a morphism $\sigma : \tilde{X} \to X$.
- E: the exceptional divisor of $\tilde{X}$.
- $W = H^0(\tilde{X}, \sigma^*\mathcal{L}(-E)) = H^0(X, \mathcal{L}(-q))$.
- $S_W = \bigoplus_{\ell\in\mathbb{Z}} \operatorname{Sym}^\ell(W)$: the homogeneous coordinate ring of $\mathbb{P}(W) = \mathbb{P}^{r-1}$.
- $R' = \bigoplus_{\ell\in\mathbb{Z}} H^0(\tilde{X}, (\sigma^*\mathcal{L} - E)^\ell)$: the associated graded S_W-module.

2.2. *Criteria for N_p property*

Suppose that $X \subset \mathbb{P}(V)$ is defined by the complete linear system associated to very ample line bundle $\mathcal{L}$ on X and consider the natural exact sequence

$$0 \to \mathcal{M}_V \to H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \xrightarrow{\varphi} \mathcal{L} \to 0,$$

where $\mathcal{M}_V$ is the kernel of the surjective map φ. Taking $(i+1)$st exterior powers and twisting by $\mathcal{L}^{j-1}$ yield

$$0 \to \wedge^{i+1}\mathcal{M}_V \otimes \mathcal{L}^{j-1} \to \wedge^{i+1}H^0(X, \mathcal{L}) \otimes \mathcal{L}^{j-1} \to \wedge^i \mathcal{M}_V \otimes \mathcal{L}^j \to 0.$$

Let $R = \bigoplus_{\ell\in\mathbb{Z}} H^0(X, \mathcal{L}^{\otimes\ell})$ and $V = H^0(X, \mathcal{L})$. Then $\operatorname{Tor}_i^S(R, \mathbb{C})_{i+j}$ fits in the following exact sequence:

$$\begin{aligned} 0 \to \operatorname{Tor}_i^S(R, \mathbb{C})_{i+j} &\to H^1\big(X, \wedge^{i+1}\mathcal{M}_V \otimes \mathcal{L}^{j-1}\big) \to \wedge^{i+1}V \otimes H^1\big(X, \mathcal{L}^{j-1}\big) \\ &\to H^1\big(X, \wedge^i \mathcal{M}_V \otimes \mathcal{L}^j\big) \to \cdots. \end{aligned}$$

Note that $\beta_{i,j} = \dim(\operatorname{Tor})_i^S(R, \mathbb{C})_{i+j}$ where $\beta_{i,j}$ is the Betti number of the minimal free resolution of a graded S-module R.

Lemma 2.1. *A smooth variety X satisfies property N_p if and only if the homomorphism*

$$\wedge^{i+1}V \otimes H^0\big(X, \mathcal{L}^{j-1}\big) \to H^0\big(X, \wedge^i \mathcal{M}_V \otimes \mathcal{L}^j\big)$$

is surjective, equivalently the homomorphism

$$H^1\big(X, \wedge^{i+1}\mathcal{M}_V \otimes \mathcal{L}^{j-1}\big) \to \wedge^{i+1}V \otimes H^1\big(X, \mathcal{L}^{j-1}\big)$$

is injective for $0 \leqslant i \leqslant p$ and $j \geqslant 2$.

Proof. See [12, Section 1]. □

2.3. Blow-up and regularity

Let $\mathcal{F}$ be a coherent sheaf on a smooth projective variety X, and let $\mathcal{L}$ be a very ample line bundle on X. A coherent sheaf $\mathcal{F}$ is said to be m-regular with respect to $\mathcal{L}$ in the sense of Castelnuovo–Mumford if $H^i(X, \mathcal{F} \otimes \mathcal{L}^{m-i}) = 0$ for $i \geqslant 1$. It was shown in [15, Lecture 14], [14] that if $\mathcal{F}$ is m-regular, then $\mathcal{F}$ is $(m+1)$-regular. So one can define the regularity of $\mathcal{F}$ with respect to $\mathcal{L}$ to be the least integer m such that $\mathcal{F}$ is m-regular, i.e.,

$$\operatorname{reg}(\mathcal{F}, \mathcal{L}) = \min\{m \in \mathbb{Z} \mid \mathcal{F} \text{ is } m\text{-regular with respect to } \mathcal{L}\}.$$

On the other hand, if a graded S-module M has the following minimal free resolution

$$\cdots \to F_i \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where $F_i = \bigoplus S(-i-j)^{\beta_{i,j}}$, then regularity of M is defined by $\operatorname{reg}(M) = \max\{j \mid \beta_{i,j} \neq 0 \text{ for all } i \geqslant 0\}$. Let $\mathcal{F}$ be a coherent sheaf on the projective space $\mathbb{P}^r$, and let $F = \bigoplus_{\ell \in \mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{F}(\ell))$ be the corresponding graded S-module of twisted global sections. If F is finitely generated as a graded S-module, it is proved in [5, Exercise 20.20] that $\operatorname{reg}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^r}(1)) = \operatorname{reg}(F)$.

We have the following well-known cohomological properties between a smooth variety X and its blow-up $\tilde{X}$.

Lemma 2.2. *Let $Z \subset X$ be a smooth codimension e subvariety of a smooth variety X, let $\sigma : \tilde{X} = \operatorname{Bl}_Z(X) \to X$ be the blowing-up of X along Z, and let $E \subset \tilde{X}$ be the exceptional divisor.*

(1) *If $0 \leqslant t \leqslant e-1$, then*

$$H^i\big(\tilde{X}, \sigma^*\mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(tE)\big) = H^i(X, \mathcal{F}) \quad \text{for } i \geqslant 0,$$

for any locally free sheaf $\mathcal{F}$ on X.

(2) *If $t > 0$, then*

$$H^i\big(\tilde{X}, \sigma^*\mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(-tE)\big) = H^i\big(X, \mathcal{F} \otimes \mathcal{I}_Z^t\big) \quad \text{for } i \geqslant 0,$$

for any locally free sheaf $\mathcal{F}$ on X where $\mathcal{I}_Z$ is the sheaf of ideals defining Z.

Proof. See [1, p. 592]. □

When $\sigma^*\mathcal{L} - E$ is very ample, $R' = \bigoplus_{\ell \in \mathbb{Z}} H^0(\tilde{X}, (\sigma^*\mathcal{L} - E)^{\otimes \ell})$ is a finitely generated S_W-module. In particular, we have the following fact about the regularity of R'.

Proposition 2.3. *Assume that $\mathcal{L}$ and $(\sigma^*\mathcal{L} - E)$ are very ample line bundles on varieties X and $\tilde{X} = \operatorname{Bl}_q(X)$, respectively. Then $\operatorname{reg}(\mathcal{O}_X, \mathcal{L}) = m$ if and only if $\operatorname{reg}(\mathcal{O}_{\tilde{X}}, (\sigma^*\mathcal{L} - E)) = m$.*

Proof. It is enough to show that $\mathcal{O}_X$ is m-regular with respect to $\mathcal{L}$ if and only if $\mathcal{O}_{\tilde{X}}$ is m-regular with respect to $\sigma^*\mathcal{L}(-E)$, i.e., for all $i \geqslant 1$,

$$H^i\big(X, \mathcal{L}^{m-i}\big) = 0 \quad \text{if and only if} \quad H^i\big(\tilde{X}, (\sigma^*\mathcal{L} - E)^{m-i}\big) = 0.$$

(1) $1 \leqslant i < m$. By the above Lemma 2.2, $H^i(\tilde{X}, (\sigma^*\mathcal{L} - E)^{m-i}) = H^i(X, \mathcal{L}^{m-i} \otimes m_q^{m-i})$ where m_q is the sheaf of ideals corresponding to the point $q \in X$.

For $2 \leqslant i < m$, from the following exact sequence

$$0 \to \mathcal{L}^{m-i} \otimes m_q^{m-i} \to \mathcal{L}^{m-i} \to \mathcal{O}_X/m_q^{m-i} \to 0$$

and $\dim \operatorname{Spec}(\mathcal{O}_X/m_q^{m-i}) = 0$, we get

$$H^i\big(X, \mathcal{L}^{m-i}\big) \simeq H^i\big(\mathcal{L}^{m-i} \otimes m_q^{m-i}\big) \simeq H^i\big(\tilde{X}, (\sigma^*\mathcal{L} - E)^{m-i}\big).$$

For $i = 1$, notice that the natural morphism $H^0(\mathcal{L}^{m-1}) \to H^0(\mathcal{O}_X/m_q^{m-1})$ is surjective. In fact, we have the following natural isomorphism as a $\mathbb{C} \simeq k(q) = \mathcal{O}_X/m_q$ vector spaces:

$$\mathcal{O}_X/m_q^{m-1} \simeq \mathcal{O}_X/m_q \oplus m_q/m_q^2 \oplus m_q^2/m_q^3 \oplus \cdots \oplus m_q^{m-2}/m_q^{m-1},$$

and $H^0(\mathcal{L}^{m-1}) \to O_X/m_q^{m-1} \simeq \bigoplus_{i=0}^{m-2} m_q^i/m_q^{i+1}$ is surjective since $\mathcal{L}$ is very ample and $m_q^i/m_q^{i+1} \simeq S^i(m_q/m_q^2)$ for a smooth point $q \in X$ where S^i denotes the ith symmetric power. Thus,

$$H^1\big(X, \mathcal{L}^{m-1}\big) \simeq H^1\big(\tilde{X}, (\sigma^*\mathcal{L} - E)^{m-1}\big).$$

(2) $m \leqslant i \leqslant \dim(X)$. Since $m \geqslant 1$, we have $0 \leqslant i - m \leqslant \dim(X) - m \leqslant \dim(X) - 1 = e - 1$ where e is the codimension of q in X. By Lemma 2.2,

$$H^i\big(X, \mathcal{L}^{m-i}\big) \simeq H^i\big(\tilde{X}, (\sigma^*\mathcal{L} - E)^{m-i}\big), \quad i \geqslant m. \qquad \square$$

We have the following easy corollary which we need in the subsequent sections.

Corollary 2.4. *Under the same conditions as in Proposition* 2.3*, we have* $H^i(\tilde{X}, (\sigma^*\mathcal{L} - E)^j) = H^i(\tilde{X}, \sigma^*\mathcal{L}^j(-kE))$ *for* $i \geqslant 1$ *and* $0 \leqslant k \leqslant j$.

Proof. In Proposition 2.3, we showed that $H^i(\tilde{X}, (\sigma^*\mathcal{L} - E)^j) \simeq H^i(X, \mathcal{L}^j)$, $i \geqslant 1$. From the short exact sequence $0 \to \mathcal{L}^j \otimes m_q^k \to \mathcal{L}^j \to \mathcal{O}_X/m_q^k \to 0$ and the same argument as in Proposition 2.3, we are done. $\square$

3. Syzygies of inner projections of curves

In this section, we assume that $X \subset \mathbb{P}(H^0(\mathcal{L}))$ is a smooth projective curve embedded by complete linear system of a very ample line bundle $\mathcal{L}$ on X. If X has no trisecant line, then the inner projection $\pi_q(X)$ for a point $q \in X$ is embedded in $\mathbb{P}(W)$ where $W = H^0(\mathcal{L}(-q))$. Let S_W be the coordinate ring of $\mathbb{P}(W)$ and $R' = \bigoplus_{\ell \in \mathbb{Z}} H^0(X, \mathcal{L}(-q)^{\otimes \ell})$ a finite S_W-module. The following theorem is a generalization of Theorem 1 in [4].

Theorem 3.1. *Let* $X \subset \mathbb{P}(H^0(\mathcal{L}))$ *be a smooth irreducible curve with property* N_p*,* $p \geqslant 1$*. For every point* $q \in X$*, the Zariski closure of the linear projection* $\overline{\pi_q(X \setminus \{q\})}$ *has property* N_{p-1}*. In other words,* $\mathcal{L}(-q)$ *satisfies* N_{p-1} *for every point* $q \in X$.

Proof. Note that $\mathcal{L}(-q)$ is very ample for any point $q \in X$ since $\mathcal{L}$ satisfies property N_1, i.e., X has no trisecant line. Consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}_W & \longrightarrow & W\otimes\mathcal{O}_X & \longrightarrow & \mathcal{L}(-q) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}_V & \longrightarrow & V\otimes\mathcal{O}_X & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-q) & \longrightarrow & \mathcal{O}_X & \longrightarrow & k(q) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The above commutative diagram induces the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \wedge^{i+1}\mathcal{M}_W & \longrightarrow & \wedge^{i+1}W\otimes\mathcal{O}_X & \longrightarrow & \wedge^{i}\mathcal{M}_W\otimes\mathcal{L}(-q) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & \wedge^{i+1}\mathcal{M}_V & \longrightarrow & \wedge^{i+1}V\otimes\mathcal{O}_X & \longrightarrow & \wedge^{i}\mathcal{M}_V\otimes\mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \wedge^{i}\mathcal{M}_W\otimes\mathcal{O}_X(-q) & \longrightarrow & \wedge^{i}W\otimes\mathcal{O}_X & \longrightarrow & \operatorname{coker}\alpha & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.1}$$

Here, α is a composition map $\wedge^i\mathcal{M}_W\otimes\mathcal{L}(-q)\to\wedge^i\mathcal{M}_V\otimes\mathcal{L}(-q)\to\wedge^i\mathcal{M}_V\otimes\mathcal{L}$. Twisting through by $\mathcal{L}^{\otimes(j-1)}$ in diagram (3.1) and taking cohomology, we have the following diagram:

$$\begin{array}{ccccccc}
\wedge^{i+1}W\otimes H^0(\mathcal{L}^{j-1}) & \xrightarrow{\widetilde{\gamma_{i,j}}} & H^0(\wedge^i\mathcal{M}_W\otimes\mathcal{L}^j(-q)) & \longrightarrow & H^1(\wedge^{i+1}\mathcal{M}_W\otimes\mathcal{L}^{j-1}) & \xrightarrow{\widetilde{\delta_{i,j}}} & \wedge^{i+1}W\otimes H^1(\mathcal{L}^{j-1}) \\
\downarrow & & \downarrow & & \downarrow & & \\
\wedge^{i+1}V\otimes H^0(\mathcal{L}^{j-1}) & \xrightarrow{\alpha_{i,j}} & H^0(\wedge^i\mathcal{M}_V\otimes\mathcal{L}^j) & \longrightarrow & H^1(\wedge^{i+1}\mathcal{M}_V\otimes\mathcal{L}^{j-1}) & \xrightarrow{\beta_{i,j}} & \wedge^{i+1}V\otimes H^1(\mathcal{L}^{j-1}) \\
\downarrow{\scriptstyle\mu_{i,j}} & & \downarrow{\scriptstyle\nu_{i,j}} & & \downarrow{\scriptstyle\xi_{i,j}} & & \\
\wedge^{i}W\otimes H^0(\mathcal{L}^{j-1}) & \xrightarrow{\gamma_{i,j}} & H^0(\operatorname{coker}\alpha\otimes\mathcal{L}^{j-1}) & \longrightarrow & H^1(\wedge^i\mathcal{M}_W\otimes\mathcal{L}^{j-1}(-q)) & \xrightarrow{\delta_{i,j}} & \wedge^{i}W\otimes H^1(\mathcal{L}^{j-1}) \\
\downarrow & & \downarrow & & & & \\
0 & & H^1(\wedge^i\mathcal{M}_W\otimes\mathcal{L}^j(-q)) & \xrightarrow{\delta_{i,j+1}} & \wedge^iW\otimes H^1(\mathcal{L}^j) & & \\
 & & \downarrow{\scriptstyle\omega_{i,j}} & & \downarrow{\scriptstyle\rho_{i,j}} & & \\
 & & H^1(\wedge^i\mathcal{M}_V\otimes\mathcal{L}^j) & \xrightarrow{\beta_{i-1,j+1}} & \wedge^iV\otimes H^1(\mathcal{L}^j) & &
\end{array} \tag{3.2}$$

Observe in the above diagram that $\mu_{i,j}$ is surjective and $\rho_{i,j}$ is injective for all $i\geqslant 0$ and $j\geqslant 0$.

Now suppose that N_p holds for $(X, \mathcal{L})$, i.e., $\alpha_{i,j}$ is surjective for $0 \leqslant i \leqslant p$ and $j \geqslant 2$. Then we claim that $\delta_{i,j}$ is injective for all $1 \leqslant i \leqslant p$ and $j \geqslant 2$. Since $H^1(\wedge^i \mathcal{M}_W \otimes \mathcal{L}^j(-q)) = 0$ for all $j \gg 0$, $\delta_{i,j+1}$ is injective for all sufficiently large j. Then from the previous diagram, we have, for $1 \leqslant i \leqslant p$,

$$\begin{aligned} \delta_{i,j+1} \text{ is injective} &\iff \omega_{i,j} \text{ is injective} \\ &\iff \nu_{i,j} \text{ is surjective} \\ &\iff \gamma_{i,j} \text{ is surjective} \\ &\iff \delta_{i,j} \text{ is injective} \end{aligned}$$

until $j \geqslant 2$. Therefore, $\delta_{i,j}$ is injective for $1 \leqslant i \leqslant p$ and $j \geqslant 2$.

To show that $(X, \mathcal{L}(-q))$ has property N_{p-1}, it suffices to show that, for all $0 \leqslant i \leqslant p-1$ and $j \geqslant 2$,

$$\wedge^{i+1} W \otimes H^0\big(\mathcal{L}(-q)^{\otimes(j-1)}\big) \to H^0\big(\wedge^i \mathcal{M}_W \otimes \mathcal{L}(-q)^{\otimes j}\big)$$

is surjective, or

$$H^1\big(\wedge^{i+1} \mathcal{M}_W \otimes \mathcal{L}(-q)^{\otimes j-1}\big) \xrightarrow{\widetilde{\beta_{i,j}}} \wedge^{i+1} W \otimes H^1(\mathcal{L}(-q)^{\otimes(j-1)})$$

is injective.

Now, consider the following diagram for $0 \leqslant i \leqslant p-1$ and $j \geqslant 2$:

$$\begin{array}{ccc} H^1\big(\wedge^{i+1} \mathcal{M}_W \otimes \mathcal{L}^{j-1}\big(-(j-1)q\big)\big) & \xrightarrow{\widetilde{\beta_{i,j}}} & \wedge^{i+1} W \otimes H^1\big(\mathcal{L}^{j-1}\big(-(j-1)q\big)\big) \\ \downarrow{\scriptstyle (*)_{i,j}} & & \downarrow \\ H^1\big(\wedge^{i+1} \mathcal{M}_W \otimes \mathcal{L}^{j-1}(-q)\big) & \xrightarrow{\delta_{i+1,j}} & \wedge^{i+1} W \otimes H^1\big(\mathcal{L}^{j-1}\big) \end{array}$$

where $(*)_{i,j}$ is the identity map when $j = 2$.

By the below Lemma 3.2, $(*)_{i,j}$ is injective for $j \geqslant 2$ and for all i. Therefore, $\widetilde{\beta_{i,j}}$ is injective for all $0 \leqslant i \leqslant p-1$ and $j \geqslant 2$ and $(X, \mathcal{L}(-q))$ has property N_{p-1}. □

Lemma 3.2. *Let* $\mathcal{E} = \wedge^{i+1} \mathcal{M}_W \otimes \mathcal{L}^{j-1}((-k+1)q)$. *From the exact sequence* $0 \to \mathcal{O}_X(-q) \to \mathcal{O}_X \to k(q) \to 0$, *we have the following exact sequence*:

$$0 \to \mathcal{E}(-q) \to \mathcal{E} \to \mathcal{E} \otimes k(q) \to 0.$$

Then $H^1(X, \mathcal{E}(-q)) \xrightarrow{(*)_{i,j,k}} H^1(X, \mathcal{E})$ *is an isomorphism for* $j \geqslant 2$ *and* $2 \leqslant k \leqslant j$.

Proof. It is enough to show that the morphism $H^0(\mathcal{E}) \to H^0(\mathcal{E} \otimes k(q))$ is surjective. This surjectivity is equivalent to the fact that $\mathcal{E}$ is globally generated at $q \in X$. From the exact sequence

$$0 \to \wedge^{i+2} \mathcal{M}_W \to \wedge^{i+2} W \otimes \mathcal{O}_X \to \wedge^{i+1} \mathcal{M}_W \otimes \mathcal{L}(-q) \to 0,$$

we know that $\wedge^{i+1}\mathcal{M}_W \otimes \mathcal{L}(-q)$ is globally generated. Now

$$\begin{aligned}\wedge^{i+1}\mathcal{M}_W \otimes \mathcal{L}^{j-1}\big((-k+1)q\big) &= \wedge^{i+1}\mathcal{M}_W \otimes \big(\mathcal{L}(-q)\big)^{k-1} \otimes \mathcal{L}^{j-k} \\ &= \wedge^{i+1}\mathcal{M}_W \otimes \mathcal{L}(-q) \otimes \big(\mathcal{L}(-q)\big)^{k-2} \otimes \mathcal{L}^{j-k}.\end{aligned}$$

Since $\wedge^{i+1}\mathcal{M}_W \otimes \mathcal{L}(-q)$ is globally generated and $\mathcal{L}(-q)^{k-2} \otimes \mathcal{L}^{j-k}$ is very ample for $j \geqslant 2$ and $2 \leqslant k \leqslant j$ except for $j = k = 2$ (in this case, it is trivial), $\mathcal{E} = \wedge^{i+1}\mathcal{M}_W \otimes \mathcal{L}^{j-1}((-k+1)q)$ is globally generated. □

Corollary 3.3. *Let X be a smooth curve and $\mathcal{L}$ a very ample line bundle on X satisfying property N_p for $p \geqslant 1$. For any s points (including infinitely near points) $\{q_1, q_2, \ldots, q_s\}$ in X, if $s \leqslant p$, then $\mathcal{L}(-q_1 - q_2 - \cdots - q_s)$ satisfies N_{p-s}. In particular, $\mathcal{L}(-q_1 - q_2 - \cdots - q_p)$ is always very ample for any points $q_1, q_2, \ldots, q_p \in X$ and normally generated.*

Corollary 3.4. *Let $X \subset \mathbb{P}^N$ be a smooth linearly normal projective curve. If X has a $(p+2)$-secant p-plane, then property N_p fails for X.*

Proof. Assume that X has property N_p. Then, by Corollary 3.3, $\mathcal{L}(-q_1 - q_2 - \cdots - q_{p-1})$ satisfies property N_1 for any points $q_1, q_2, \ldots, q_{p-1} \in X$. But if X has a $(p+2)$-secant p-plane, the projection at $(p-1)$ points has a trisecant line. This gives that the projection of X at these $(p-1)$ points fails to hold property N_1, which is a contradiction. □

4. Inner projections of higher dimensional projective varieties

For a higher dimensional smooth variety X, we need many technical lemmas to prove the main theorem in this section.

Let X be a smooth projective variety of $\mathbb{P}^r$ and q a closed point of X. The inner projection $\pi_q : X \dashrightarrow \mathbb{P}^{r-1}$ is a rational map which is well defined outside q. Let $\sigma : \mathrm{Bl}_q(X) \to X$ be a blowing-up of X at q. Then one has the regular morphism $\tilde{\pi}_q : \mathrm{Bl}_q(X) \to \mathbb{P}^{r-1}$ with the following diagram:

$$\begin{array}{ccc} \tilde{X} = \mathrm{Bl}_q(X) & & \\ \Big\downarrow{\scriptstyle \sigma} & \searrow^{\tilde{\pi}_q} & \\ X \subset \mathbb{P}^r & \overset{\pi_q}{\cdots\cdots\succ} & X' = \overline{\pi_q\big(X \setminus \{q\}\big)} \subset \mathbb{P}^{r-1} \end{array}$$

If $\tilde{\pi}_q : \mathrm{Bl}_q(X) \to \mathbb{P}^{r-1}$ is an embedding, then the exceptional divisor E is linearly embedded via $\tilde{\pi}_q$ in $\mathbb{P}^{r-1}$, i.e., $\tilde{\pi}_q(E) = \mathbb{P}^{l-1} \subset \mathbb{P}^{r-1}$, $l = \dim(X)$.

Proposition 4.1. *Assume that X is a smooth nondegenerate projective variety in $\mathbb{P}^r$ and $q \in X$ is a closed point. Then, the morphism $\tilde{\pi}_q : \mathrm{Bl}_q(X) \to \mathbb{P}^{r-1}$ is a closed embedding if and only if $q \notin \mathrm{Trisec}(X)$ where $\mathrm{Trisec}(X)$ is the union of all lines ℓ with the property that either $\ell \subset X$ or $\ell \cap X$ is a subscheme of length $\geqslant 3$.*

Proof. For details, see [8, pp. 268–269]. □

Let $W = H^0(\tilde{X}, \sigma^*\mathcal{L} - E) \subset H^0(X, \mathcal{L}) = V$. If $\sigma^*\mathcal{L} - E$ is very ample on $\mathrm{Bl}_q(X)$, the morphism $\tilde{\pi}_q : \tilde{X} = \mathrm{Bl}_q(X) \to \mathbb{P}(W)$ is a closed embedding defined by the global sections of $\sigma^*\mathcal{L} - E$ and there is a natural exact sequence

$$0 \to \mathcal{M}_W \to W \otimes \mathcal{O}_{\tilde{X}} \xrightarrow{\phi} \sigma^*\mathcal{L} - E \to 0,$$

where $\mathcal{M}_W$ is the kernel of ϕ.

Lemma 4.2. *$R^1\sigma_*(\wedge^i \mathcal{M}_W \otimes (\sigma^*\mathcal{L} - E)^k) = 0$ for $k \geqslant 0$ and $i \geqslant 0$.*

Proof. Consider the following exact sequence:

$$0 \to \wedge^{i+1}\mathcal{M}_W \otimes (\sigma^*\mathcal{L} - E)^{k-1} \to \wedge^{i+1} W \otimes (\sigma^*\mathcal{L} - E)^{k-1} \to \wedge^i \mathcal{M}_W \otimes (\sigma^*\mathcal{L} - E)^k \to 0.$$

Since $R^i\sigma_*(\sigma^*\mathcal{L} \otimes \mathcal{O}_{\tilde{X}}(-tE)) = \mathcal{L} \otimes R^i\sigma_*(\mathcal{O}_{\tilde{X}}(-tE)) = 0$ for $i \geqslant 1$ and $t \geqslant 0$,

$$\begin{aligned} R^1\sigma_*\big(\wedge^i \mathcal{M}_W \otimes (\sigma^*\mathcal{L} - E)^k\big) = 0 \quad &\Longleftrightarrow \quad R^2\sigma_*\big(\wedge^{i+1}\mathcal{M}_W \otimes (\sigma^*\mathcal{L} - E)^{k-1}\big) = 0 \\ &\Longleftrightarrow \quad R^3\sigma_*\big(\wedge^{i+2}\mathcal{M}_W \otimes (\sigma^*\mathcal{L} - E)^{k-2}\big) = 0 \\ &\cdots \\ &\Longleftrightarrow \quad R^{k+1}\sigma_*\big(\wedge^{i+k}\mathcal{M}_W\big) = 0. \end{aligned}$$

Therefore it is enough to show that $R^i\sigma_*(\wedge^t \mathcal{M}_W) = 0$ for all positive integers t and i. Let $\mathcal{E} = \wedge^t \mathcal{M}_W$ and $\mathcal{E}^i = R^i\sigma_*(\mathcal{E})$. Since σ is an isomorphism of $\tilde{X} - E$ onto $X - q$, $R^i\sigma_*(\mathcal{E})|_{X-q} = 0$. Therefore, the sheaves $\mathcal{E}^i$ for $i > 0$ have support at q. By the theorem on formal functions [9, p. 277],

$$\hat{\mathcal{E}}^i \cong \varprojlim H^i(E_n, \mathcal{E}|_{E_n}),$$

where E_n is the closed subscheme of $\tilde{X}$ defined by $\mathcal{I}^n$, where $\mathcal{I}$ is the ideal sheaf of E in $\tilde{X}$. There are natural exact sequences

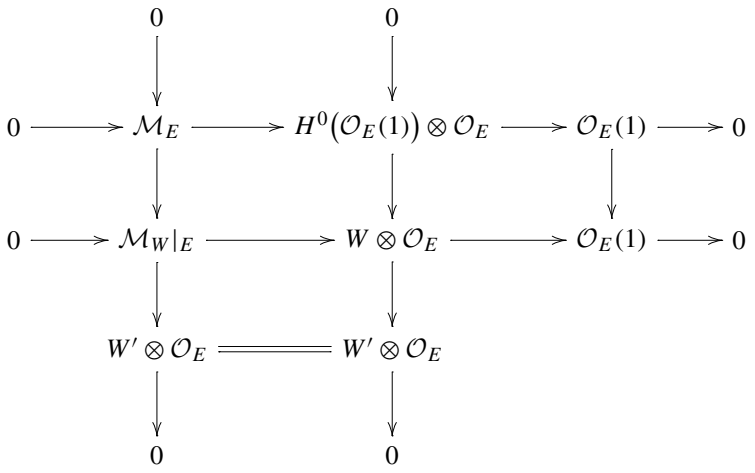
$$0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to \mathcal{O}_{E_{n+1}} \to \mathcal{O}_{E_n} \to 0$$

for each n. Since q is a smooth point, we have $\mathcal{I}/\mathcal{I}^2 = \mathcal{O}_E(1)$ and $\mathcal{I}^n/\mathcal{I}^{n+1} \cong S^n(\mathcal{I}/\mathcal{I}^2) \cong \mathcal{O}_E(n)$.

Claim. *$H^i(E, \mathcal{E} \otimes \mathcal{O}_E(n)) = 0$ for $i > 0$ and all $n \geqslant 0$.*

If the claim is true, then we conclude from the long exact sequence of cohomology, using induction on n, that $H^i(\mathcal{E}|_{E_n}) = 0$ for all $i > 0$ and all $n > 0$. It follows that $\hat{\mathcal{E}}^i = 0$ for $i > 0$. Since $\mathcal{E}^i$ is a coherent sheaf with support at q, so $\mathcal{E}^i = 0$. □

Proof of Claim. Since the exceptional divisor E is a projective space of dimension $l - 1$ where $l = \dim(X)$, we have the following commutative diagram:



where $\mathcal{M}_E = \Omega_{\mathbb{P}^{l-1}}(1)$ and W' is a quotient vector space of W by $H^0(\mathcal{O}_E(1))$ and $\dim(W') = r - l$.

From the exact sequence $0 \to \mathcal{M}_E \to \mathcal{M}_W|_E \to W' \otimes \mathcal{O}_E \to 0$, there is a finite filtration of $\wedge^t \mathcal{M}_W|_E$,

$$\wedge^t \mathcal{M}_W|_E = F^0 \supset F^1 \supset \cdots \supset F^t \supset F^{t+1} = 0$$

with quotients

$$F^p/F^{p+1} \cong \wedge^p(\mathcal{M}_E) \otimes \wedge^{t-p}(W' \otimes O_E)$$

for each p. Since $H^i(\wedge^j \Omega(k)) = 0$ unless $i \geqslant 1$ and $k \geqslant 1$ by Bott formula, we get $H^i(\wedge^t \mathcal{M}_W|_E) = 0$. For $H^i(\wedge^t \mathcal{M}_W \otimes \mathcal{O}_E(n)) = 0$, use induction on l. For $l = 2$, the vanishing is clear since $E = \mathbb{P}^1$ and $\wedge^t \mathcal{M}_W|_E$ is a sum of line bundles on E. For $l > 2$, the vanishing follows from the exact sequence:

$$0 \to \wedge^t \mathcal{M}_W \otimes \mathcal{O}_E(n) \to \wedge^t \mathcal{M}_W \otimes \mathcal{O}_E(n+1) \to \wedge^t \mathcal{M}_W \otimes \mathcal{O}_H(n+1) \to 0,$$

where H is a hyperplane in E. □

Lemma 4.3. *$\sigma_*(\wedge^i \mathcal{M}_W \otimes \sigma^* \mathcal{L}^j(-kE))$ is globally generated for $i \geqslant 0$, $j \geqslant 1$ and $1 \leqslant k \leqslant j$.*

Proof. Since

$$\begin{aligned}\sigma_*\big(\wedge^i \mathcal{M}_W \otimes \sigma^* \mathcal{L}^j(-kE)\big) &= \sigma_*\big(\wedge^i \mathcal{M}_W \otimes (\sigma^* \mathcal{L} - E)^k \otimes \sigma^* \mathcal{L}^{j-k}\big) \\ &= \sigma_*\big(\wedge^i \mathcal{M}_W \otimes (\sigma^* \mathcal{L} - E)^k\big) \otimes \mathcal{L}^{j-k},\end{aligned}$$

it is enough to show that $\sigma_*(\wedge^i \mathcal{M}_W \otimes (\sigma^* \mathcal{L} - E)^k)$ is globally generated.

By taking the pushforward σ_* to the exact sequence

$$0 \to \wedge^{i+1} \mathcal{M}_W \otimes (\sigma^* \mathcal{L} - E)^{k-1} \to \wedge^{i+1} W \otimes (\sigma^* \mathcal{L} - E)^{k-1} \to \wedge^i \mathcal{M}_W \otimes (\sigma^* \mathcal{L} - E)^k \to 0$$

we have

$$0 \to \sigma_*\big(\wedge^{i+1}\mathcal{M}_W \otimes (\sigma^*\mathcal{L} - E)^{k-1}\big) \to \sigma_*\big(\wedge^{i+1}W \otimes (\sigma^*\mathcal{L} - E)^{k-1}\big)$$
$$\to \sigma_*\big(\wedge^{i}\mathcal{M}_W \otimes (\sigma^*\mathcal{L} - E)^{k}\big) \to R^1\sigma_*\big(\wedge^{i+1}\mathcal{M}_W \otimes (\sigma^*\mathcal{L} - E)^{k-1}\big) \to \cdots.$$

By Lemma 4.2, $R^1\sigma_*(\wedge^{i+1}\mathcal{M}_W \otimes (\sigma^*\mathcal{L} - E)^{k-1}) = 0$ for $k \geqslant 1$. Since $\mathcal{L}$ is very ample, $\sigma_*(\wedge^{i+1}W \otimes (\sigma^*\mathcal{L} - E)^{k-1}) = \wedge^{i+1}W \otimes (\mathcal{L} \otimes m_q)^{k-1}$ is globally generated. From the following commutative diagram

$$\begin{array}{ccc} \wedge^{i+1}W \otimes H^0(\mathcal{L} \otimes m_q)^{(k-1)} \otimes \mathcal{O}_X & \longrightarrow & H^0\big(\wedge^i M_W \otimes (\sigma^*\mathcal{L} - E)^k\big) \otimes \mathcal{O}_X \\ \downarrow & & \downarrow \\ \wedge^{i+1}W \otimes (\mathcal{L} \otimes m_q)^{(k-1)} & \longrightarrow & \sigma_*\big(\wedge^i M_W \otimes (\sigma^*\mathcal{L} - E)^k\big) \end{array}$$

we are done. □

Lemma 4.4. *Assume that $\mathcal{L}$ and $\sigma^*\mathcal{L} - E$ are very ample line bundles on X and $\tilde{X} = \mathrm{Bl}_q(X)$, respectively. Then the natural morphism*

$$H^1\big(\wedge^i\mathcal{M}_W \otimes \sigma^*\mathcal{L}^j\big((-k-1)E\big)\big) \to H^1\big(\wedge^i\mathcal{M}_W \otimes \sigma^*\mathcal{L}^j(-kE)\big) \tag{4.1}$$

is injective for all $i \geqslant 0$, $j \geqslant 1$ and $1 \leqslant k \leqslant j$.

Proof. Let $\mathcal{E} = \wedge^i\mathcal{M}_W \otimes \sigma^*\mathcal{L}^j(-kE)$. From the natural exact sequence $0 \to \mathcal{E}(-E) \to \mathcal{E} \to \mathcal{E}|_E \to 0$, the injectivity of morphism (4.1) is equivalent to the surjectivity of the morphism $H^0(\tilde{X}, \mathcal{E}) \to H^0(\mathcal{E}|_E)$. Note again that by Lemma 4.2,

$$R^1\sigma_*\big(\wedge^i\mathcal{M}_W \otimes \sigma^*\mathcal{L}^j\big((-k-1)E\big)\big) = R^1\sigma_*\big(\wedge^i\mathcal{M}_W \otimes (\sigma^*\mathcal{L} - E)^{(k+1)}\big) \otimes \mathcal{L}^{j-k-1} = 0.$$

Thus, we have an exact sequence

$$0 \to \sigma_*\mathcal{E}(-E) \to \sigma_*\mathcal{E} \to \sigma_*(\mathcal{E}|_E) \to 0.$$

Since $\sigma_*(\mathcal{E})$ is globally generated by Lemma 4.3, we have surjective morphisms

$$H^0(\tilde{X}, \mathcal{E}) \otimes \mathcal{O}_X \to \sigma_*(\mathcal{E}) \to \sigma_*(\mathcal{E}|_E) = H^0(\mathcal{E}|_E). \qquad \square$$

Remark 4.5. In Lemma 4.4, it can be shown that the given morphism is not always injective for $k = j + 1$.

Theorem 4.6. *Let $X \subset \mathbb{P}(H^0(\mathcal{L}))$ be a smooth irreducible variety of $\dim(X) \geqslant 2$ with property N_p, $p \geqslant 1$. For any $q \in X \setminus \mathrm{Trisec}(X)$, $\tilde{\pi}_q(\mathrm{Bl}_q(X)) = \overline{\pi_q(X \setminus \{q\})}$ in $\mathbb{P}(W)$ is smooth and satisfies property N_{p-1}, i.e., property N_{p-1} holds for $(\mathrm{Bl}_q(X), \sigma^*\mathcal{L} - E)$.*

Proof. By Proposition 4.1, we know $\tilde{\pi}_q(\mathrm{Bl}_q(X)) = \overline{\pi_q(X \setminus \{q\})}$ in $\mathbb{P}(W)$ is smooth. From the restriction of the Euler sequence on $\mathbb{P}(W)$ to $\tilde{X} = \mathrm{Bl}_q(X)$, we have the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}_W & \longrightarrow & W \otimes \mathcal{O}_{\tilde{X}} & \longrightarrow & \sigma^*\mathcal{L}(-E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma^*\mathcal{M}_V & \longrightarrow & V \otimes \mathcal{O}_{\tilde{X}} & \longrightarrow & \sigma^*\mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\tilde{X}}(-E) & \longrightarrow & \mathcal{O}_{\tilde{X}} & \longrightarrow & \mathcal{O}_E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The above commutative diagram induces the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \wedge^{i+1}\mathcal{M}_W & \longrightarrow & \wedge^{i+1}W \otimes \mathcal{O}_{\tilde{X}} & \longrightarrow & \wedge^{i}\mathcal{M}_W \otimes \sigma^*\mathcal{L}(-E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & \wedge^{i+1}\sigma^*\mathcal{M}_V & \longrightarrow & \wedge^{i+1}V \otimes \mathcal{O}_{\tilde{X}} & \longrightarrow & \wedge^{i}\sigma^*\mathcal{M}_V \otimes \sigma^*\mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \wedge^{i}\mathcal{M}_W \otimes \mathcal{O}_{\tilde{X}}(-E) & \longrightarrow & \wedge^{i}W \otimes \mathcal{O}_{\tilde{X}} & \longrightarrow & \operatorname{coker}\alpha & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{4.2}$$

Twisting through by $\sigma^*\mathcal{L}^{j-1}$ in diagram (4.2) and taking cohomology, one gets the following diagram:

$$\begin{array}{ccccccc}
\wedge^{i+1}W \otimes H^0(\sigma^*\mathcal{L}^{j-1}) & \longrightarrow & H^0(\wedge^{i}\mathcal{M}_W \otimes \sigma^*\mathcal{L}^{j}(-E)) & \longrightarrow & H^1(\wedge^{i+1}\mathcal{M}_W \otimes \sigma^*\mathcal{L}^{j-1}) & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
\wedge^{i+1}V \otimes H^0(\sigma^*\mathcal{L}^{j-1}) & \xrightarrow{\alpha_{i,j}} & H^0(\wedge^{i}\sigma^*\mathcal{M}_V \otimes \sigma^*\mathcal{L}^{j}) & \longrightarrow & H^1(\wedge^{i+1}\sigma^*\mathcal{M}_V \otimes \sigma^*\mathcal{L}^{j-1}) & \xrightarrow{\beta_{i,j}} & \\
\downarrow{\scriptstyle \mu_{i,j}} & & \downarrow{\scriptstyle \nu_{i,j}} & & \downarrow{\scriptstyle \xi_{i,j}} & & \\
\wedge^{i}W \otimes H^0(\sigma^*\mathcal{L}^{j-1}) & \xrightarrow{\gamma_{i,j}} & H^0(\operatorname{coker}\alpha \otimes \sigma^*\mathcal{L}^{j-1}) & \longrightarrow & H^1(\wedge^{i}\mathcal{M}_W \otimes \sigma^*\mathcal{L}^{j-1}(-E)) & \xrightarrow{\delta_{i,j}} & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & H^1(\wedge^{i}\mathcal{M}_W \otimes \sigma^*\mathcal{L}^{j}(-E)) & & H^2(\wedge^{i+1}\mathcal{M}_W \otimes \sigma^*\mathcal{L}^{j-1}) & & \\
 & & \downarrow{\scriptstyle \omega_{i,j}} & & & & \\
 & & H^1(\wedge^{i}\sigma^*\mathcal{M}_V \otimes \sigma^*\mathcal{L}^{j}) & & & &
\end{array} \tag{4.3}$$

Observe the following from the above diagram:

(1) $\mu_{i,j}$ is always surjective.
(2) By hypothesis of property N_p of $(X, \mathcal{L})$ (see Lemma 2.1) and Lemma 2.2, $\alpha_{i,j}$ is surjective for $0 \leqslant i \leqslant p$ and $j \geqslant 2$.
(3) $\mathrm{Bl}_q(X) \subset \mathbb{P}^{r-1}$ satisfies property N_{p-1} for $\sigma^*\mathcal{L}(-E)$ if

$$\delta_{i,j} \text{ is injective} \quad \text{for all } 1 \leqslant i \leqslant p \text{ and } j \geqslant 2$$

by Lemma 4.4 and the following diagram:

$$\begin{array}{ccc} H^1\big(\wedge^i \mathcal{M}_W \otimes (\sigma^*\mathcal{L} - E)^{j-1}\big) & \longrightarrow & \wedge^i W \otimes H^1\big((\sigma^*\mathcal{L} - E)^{j-1}\big) \\ \downarrow & & \downarrow \wr \text{ (by Corollary 2.4)} \\ H^1\big(\wedge^i \mathcal{M}_W \otimes \sigma^*\mathcal{L}^{j-1}(-E)\big) & \xrightarrow{\delta_{i,j}} & \wedge^i W \otimes H^1\big(\sigma^*\mathcal{L}^{j-1}\big) \end{array}$$

(4) For $1 \leqslant i \leqslant p$ and $j \geqslant 2$, the following holds since $\mu_{i,j}$ and $\alpha_{i,j}$ are surjective:

$$\begin{aligned} \delta_{i,j} \text{ is injective} &\iff \gamma_{i,j} \text{ is surjective} \\ &\iff \nu_{i,j} \text{ is surjective} \\ &\iff \omega_{i,j} \text{ is injective.} \end{aligned}$$

(5) We also have the following commutative diagram of cohomology groups:

$$\begin{array}{ccc} H^1\big(\tilde{X}, \wedge^i \mathcal{M}_W \otimes \sigma^*\mathcal{L}^j(-E)\big) & \xrightarrow{\delta_{i,j+1}} & \wedge^i W \otimes H^1\big(\tilde{X}, \sigma^*\mathcal{L}^j\big) \\ \downarrow{\scriptstyle \omega_{i,j}} & & \downarrow{\scriptstyle \rho_{i,j}} \\ H^1\big(\tilde{X}, \wedge^i \sigma^*\mathcal{M}_V \otimes \sigma^*\mathcal{L}^j\big) & \xrightarrow{\beta_{i-1,j+1}} & \wedge^i V \otimes H^1\big(\tilde{X}, \sigma^*\mathcal{L}^j\big) \end{array}$$

From this diagram, observe the following:
- $\rho_{i,j}$ is always injective.
- $\beta_{i-1,j+1}$ is injective for $1 \leqslant i \leqslant p+1$ and $j \geqslant 1$ by property N_p of $(X, \mathcal{L})$.

Thus for $1 \leqslant i \leqslant p+1$ and $j \geqslant 1$,

$$\delta_{i,j+1} \text{ is injective} \quad \text{if and only if} \quad \omega_{i,j} \text{ is injective.}$$

Note that $\delta_{i,j+1}$ is injective for a sufficient large j since $H^1(\tilde{X}, \wedge^i \mathcal{M}_W \otimes \sigma^*\mathcal{L}^j(-E)) = H^1(X, \sigma_*(\wedge^i \mathcal{M}_W \otimes \mathcal{O}_{\tilde{X}}(-E)) \otimes \mathcal{L}^j) = 0$, $j \gg 0$. Hence the following holds for $1 \leqslant i \leqslant p$:

$$\begin{aligned}
\delta_{i,j+1} \text{ is injective} \quad \text{for all } j \gg 0 \quad &\Longrightarrow \quad \omega_{i,j} \text{ and } \delta_{i,j} \text{ are injective} \\
&\Longrightarrow \quad \omega_{i,j-1} \text{ and } \delta_{i,j-1} \text{ are injective} \\
&\cdots \\
&\Longrightarrow \quad \omega_{i,2} \text{ and } \delta_{i,2} \text{ are injective.}
\end{aligned}$$

Therefore it is proved that $\delta_{i,j}$ is injective for all $1 \leqslant i \leqslant p$ and $j \geqslant 2$. □

Let X be a smooth projective variety of dimension n and $\mathcal{L}$ a very ample line bundle on X. Let $\sigma : \tilde{X} \to X$ be the blowing-up at distinct s points $q_1, q_2, \ldots, q_s$ in X with the exceptional divisors $E_1, \ldots, E_s$. Suppose $\mathcal{L}' := \sigma^*\mathcal{L}(-E_1 - \cdots - E_s)$ on $\tilde{X}$ is very ample. Then we have the following corollary from Theorem 4.6 immediately.

Corollary 4.7. *If $\mathcal{L}$ satisfies N_p for $p \geqslant 1$, then $\mathcal{L}'$ satisfies N_{p-s} for $p \geqslant s$. In particular, $\sigma^*\mathcal{L}(-E_1 - \cdots - E_p)$ is very ample and normally generated.*

We also get the generalization of Corollary 3.4 with extra conditions.

Corollary 4.8. *Let $X \subset \mathbb{P}^N$ be a smooth linearly normal variety satisfying $H^i(X, \mathcal{O}(j)) = 0$ for all $1 \leqslant i < \dim X$ and $j \geqslant 0$. If X has a $(p+2)$-secant p-plane Λ such that $X \cap \Lambda$ is a finite curvilinear scheme, then property N_p fails for X.*

Proof. From the given cohomological and curvilinear conditions, we can possibly get the linearly normal smooth curve section containing $(p+2)$-secant p-plane Λ and its graded Betti numbers are same as those of X by Green theorem [11, 3.b.7]. Thus we are done by Corollary 3.4. □

Corollary 4.9. *Let $X \subset \mathbb{P}(W)$ be a smooth variety with property N_p^S and $t = \operatorname{codim}(W, H^0(\mathcal{L}))$. Then, for any $q \in X \setminus \operatorname{Trisec}(X)$ we have*:

(1) *An inner projection $\overline{\pi_q(X \setminus \{q\})} \subset \mathbb{P}(W')$ satisfies property N_{p-1}^S.*
(2) *The defining equations of $\overline{\pi_q(X \setminus \{q\})}$ in $\mathbb{P}(W')$ have degree at most $t+2$ for $p \geqslant 2$.*

Proof. We can prove this by the exactly same methods as those shown in Theorem 4.6 except taking subsystems of the complete linear system of global sections. For a proof of (2), note that $t = \operatorname{codim}(W, H^0(\mathcal{L})) = \operatorname{codim}(W', H^0(\sigma^*\mathcal{L} - E))$. Thus, it follows from Theorem 1.1 in [13] because an inner projection of X satisfies N_1^S property for $p \geqslant 2$. □

Remark 4.10. We would like to mention that Corollary 4.8 can be proved with no extra cohomological conditions due to Eisenbud, Green, Hulek and Popescu [6, Theorem 1.1].

5. Applications to some examples and questions

The main theorems can be applied to the following examples.

• ***Veronese embedding of*** $\mathbb{P}^n$**.** Let $\nu_{2,d}:\mathbb{P}^2 \hookrightarrow \mathbb{P}^r$, $r=\binom{d+2}{2}-1$, be the Veronese embedding associated to the complete linear system $|\mathcal{O}_{\mathbb{P}^2}(d)|$. The image of the embedding $\nu_{2,d}$ with $d \geqslant 3$ satisfies property N_p if and only if $p \leqslant 3d-3$ (see [17]). Let $q_1,\ldots,q_m$ be distinct m points of $\mathbb{P}^2$ for $m \leqslant 3d-3$, and let $\tilde{X}$ be the blowing-up at $q_1,\ldots,q_m$ with the exceptional divisors $E_1,\ldots,E_m$. A. Noma proved in [16] that the line bundle $\mathcal{L}:=\sigma^*\mathcal{O}_{\mathbb{P}^2}(d)\otimes\mathcal{O}_{\tilde{X}}(-E_1-\cdots-E_m)$ is very ample if and only if, for all l with $1 \leqslant l \leqslant m$, any l-points of $\{q_1,\ldots,q_m\}$ do not lie on any rational normal curve of degree l on $\nu(\mathbb{P}^2)\subset\mathbb{P}^r$. From Corollary 4.7, $\mathcal{L}$ satisfies property N_{3d-3-m} if it is very ample.

For arbitrary n, let $\nu_{n,d}:\mathbb{P}^n \hookrightarrow \mathbb{P}^r$, $r=\binom{d+n}{n}-1$, be the Veronese embedding associated to the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$. It is well known that $\nu_{n,d}$ with $d \geqslant 2$ satisfies N_p for $p \leqslant d$ and the line bundle $\mathcal{L}:=\sigma^*\mathcal{O}_{\mathbb{P}^n}(d)\otimes\mathcal{O}_{\tilde{X}}(-E_1-\cdots-E_d)$ is very ample [3, Theorem 1], where $\tilde{X}$ is the blowing-up at general d points $x_1,\ldots,x_d$ with the exceptional divisors $E_1,\ldots,E_d$. Thus, Corollary 4.7 implies that $\mathcal{L}$ is normally generated.

On the other hand, it can be shown by using Corollary 4.8 that the image $\nu_{n,d}(\mathbb{P}^n)$ fails to hold property N_{3d-2}. This was first proved by Ottaviani and Paoletti [17], and then by Eisenbud, Green, Hulek and Popescu [6, Proposition 3.2].

Proposition 5.1. *The image* $\nu_{n,d}(\mathbb{P}^n)$ *for* $d \geqslant 3$ *fails to hold property* N_{3d-2}.

Proof. Since the image $\nu_{n,d}(\mathbb{P}^n)$ is projectively Cohen–Macaulay, the hypotheses of Corollary 4.8 are satisfied. We can also show that the image $\nu_{n,d}(\mathbb{P}^n)$ has $3d$-secant $(3d-2)$-plane (see [6, Proposition 3.2] for details). Therefore, $\nu_{n,d}(\mathbb{P}^n)$ for $d \geqslant 3$ fails to hold property N_{3d-2}. □

• ***Calabi–Yau manifolds.*** Let X be an n-dimensional smooth projective variety with $K_X=\mathcal{O}_X$ and let $\mathcal{L}$ be a very ample line bundle on X. Then $\mathcal{L}^{\otimes n+p+1}$ satisfies to hold N_p (see [7]). Assume that $p \geqslant 2n$. In [3], for general p points $x_1,\ldots,x_p$ with $p \leqslant N-2n-1$ where $N=\dim H^0(X,\mathcal{L}^{\otimes n+p+1})$, $\mathcal{L}':=\sigma^*\mathcal{L}^{n+p+1}(-E_1-\cdots-E_p)$ is very ample. So, we conclude that $\mathcal{L}'$ is normally generated.

• ***Rational surfaces.*** Gallego and Purnaprajna give a nice sufficient condition for a line bundle $\mathcal{L}$ on a rational surface X to hold property N_p.

Theorem 5.2. [10] *Let X be a rational surface, and let $\mathcal{L}$ be an ample line bundle on X. If $\mathcal{L}$ is base-point-free and $-K_X\cdot\mathcal{L} \geqslant p+3$, then $\mathcal{L}$ satisfies property N_p.*

If $-K_X\cdot\mathcal{L} \geqslant p+3$, then $-K_{\tilde{X}}\cdot(\sigma^*\mathcal{L}-E) \geqslant p+2$. So, this theorem is consistent with Theorem 4.6. Note also that Del Pezzo surfaces as the blow-ups of $\mathbb{P}^2$ are rational surfaces.

Remark 5.3. Many works have been done to show that some very ample line bundles on smooth projective varieties satisfy property N_p. For example:

- (Green [11]) Let X be a smooth projective curve of genus g and let $\mathcal{L}$ be a line bundle on X of degree d. If $d \geqslant 2g+1+p$, then $(X,\mathcal{L})$ satisfies property N_p.
- (Butler [2], Park [18]) Let C be a smooth projective curve of genus g. For a vector bundle $\mathbb{E}$ of rank n over C, let $X=\mathbb{P}(\mathbb{E})$ be the associated projective bundle with tautological line bundle H with projection map $\pi:X\to C$. For a line bundle $\mathcal{L}=aH+\pi^*B$ on X with $a \geqslant 1$ and $B\in \operatorname{Pic} C$, assume that $\mu^-(\pi_*\mathcal{L}) \geqslant 2g+2p$ for some $1 \leqslant p \leqslant a$. Then Butler

proved that $(X, \mathcal{L})$ satisfies property N_p. For the cases of $a = 1, n = 2$ and $n = 3, a = 2$, Park extended Butler's result to show property N_p for $\mu^-(\pi_*\mathcal{L}) \geqslant 2g + 2p$ with no assumption $1 \leqslant p \leqslant a$.

- (Ein–Lazarsfeld [7]) Let X be a smooth complex projective variety of dimension n with the canonical sheaf K_X. For $A, B \in \operatorname{Pic} X$, assume that A is very ample and B is numerically effective. Then, for $p \geqslant 0$, $K_X + (n + p)A + B$ satisfies property N_p except the case $(X, A, B) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n})$ and $p = 0$.

Note in these examples that $H^i(X, \mathcal{O}(j)) = 0$ for all $1 \leqslant i < \dim X$ and $j \geqslant 0$ if $\dim(X) \geqslant 2$. On the other hand, it has been very important to show very ampleness of the form of line bundles $\sigma^*\mathcal{L}(-m_1E_1 - \cdots - m_sE_s)$. However, we are interested in the syzygies of the embedding defined by the line bundle $\sigma^*\mathcal{L}(-m_1E_1 - \cdots - m_sE_s)$.

Finally, it seems to us that the following questions related to our paper are interesting. Note that if the line bundle $\sigma^*\mathcal{L}(-2E)$ is very ample then the embedding of $\tilde{X}$ by $\sigma^*\mathcal{L}(-2E)$ is the Zariski closure of the tangential projection of X from the projective tangent space of X at q. It would be very interesting to get any structural results on these linear systems with multiple base points.

• **Question 1.** What can we say about syzygies of the line bundle $\sigma^*\mathcal{L}(-m_1E_1 - \cdots - m_sE_s)$, i.e., of the linear system of multiple base points if it is very ample?

Another question is about the converse of Theorems 3.1 and 4.6.

• **Question 2.** We ask whether the converse of Theorems 3.1 and 4.6 hold or not. More specifically, assume that $\mathcal{L}$ is a very ample line bundle on a projective variety X. For an embedded projective variety $X \subset \mathbb{P}(H^0(\mathcal{L})) \simeq \mathbb{P}^r$ by the complete linear system of $\mathcal{L}$, property N_p holds for X if and only if property N_{p-1} holds for the projection $\phi_q(X)$ of X for any point $q \notin \operatorname{Trisec}(X)$.

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