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Graded mapping cone theorem, multisecants and syzygies

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ABSTRACT

Let X be a reduced closed subscheme in $\mathbb{P}^n$. As a slight generalization of property $\mathbf{N}_p$ due to Green–Lazarsfeld, one says that X satisfies property $\mathbf{N}_{2,p}$ scheme-theoretically if there is an ideal I generating the ideal sheaf $\mathcal{I}_{X/\mathbb{P}^n}$ such that I is generated by quadrics and there are only linear syzygies up to p-th step (cf. Eisenbud et al. (2005) [8], Vermeire (2001) [20]). Recently, many algebraic and geometric results have been proved for projective varieties satisfying property $\mathbf{N}_{2,p}$ (cf. Choi, Kwak, and Park (2008) [6], Eisenbud et al. (2005) [8], Kwak and Park (2005) [15]). In this case, the Castelnuovo regularity and normality can be obtained by the blowing-up method as $\operatorname{reg}(X) \leqslant e+1$ where e is the codimension of a smooth variety X (cf. Bertram, Ein, and Lazarsfeld (2003) [3]). On the other hand, projection methods have been very useful and powerful in bounding Castelnuovo regularity, normality and other classical invariants in geometry (cf. Beheshti and Eisenbud (2010) [2], Kwak (1998) [14], Kwak and Park (2005) [15], Lazarsfeld (1987) [16].

We first prove the graded mapping cone theorem on partial eliminations as a general algebraic tool to study syzygies of the non-complete embedding of X. For applications, we give an optimal bound on the length of zero-dimensional intersections of X and a linear space L in terms of graded Betti numbers. We also deduce several theorems about the relationship between X and its projections with respect to the geometry and syzygies for a projective scheme X satisfying property $\mathbf{N}_{2,p}$ scheme-theoretically. In addition, we give not only interesting information on the regularity of

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fibers of the projection for the case of $\mathbf{N}_{d,p}$, $d \geqslant 2$, but also geometric structures for projections according to moving the center.

Contents

1. Introduction

Let V be a vector space of dimension $n+1$ over an algebraically closed field k of characteristic zero with basis $x_0, \ldots, x_n$. If X is a non-degenerate reduced closed subscheme in $\mathbb{P}^n_k = \mathbb{P}(V)$ we write I_X for the saturated defining ideal of X in the coordinate ring $R = k[x_0, \ldots, x_n]$ of $\mathbb{P}(V)$.

Eisenbud et al. in [8] introduced the notion $\mathbf{N}_{d,p}$ for some $d \geqslant 2$. We say that X satisfies property $\mathbf{N}_{d,p}$ if $\mathrm{Tor}_i^R(R/I_X, k)$ is concentrated in degree less than $d+i$ for all $i \leqslant p$. This is the same as property $\mathbf{N}_p$ defined by Green and Lazarsfeld in [10] if X is projectively normal and $d = 2$.

As a slight generalization, one says that $X \subset \mathbb{P}^n$ *satisfies property* $\mathbf{N}_{2,p}$ *scheme-theoretically* if there is an ideal I generating the ideal sheaf $\mathcal{I}_{X/\mathbb{P}^n}$ such that I is generated by quadrics and there are only linear syzygies up to the p-th step (see [8,20]). This condition is weaker than property $\mathbf{N}_{2,p}$, but useful for local study via sheafification.

One of the main interesting problems is to understand geometric properties of projective schemes satisfying property $\mathbf{N}_{2,p}$. A great deal of research has been conducted with the aim of extracting geometric informations from the condition $\mathbf{N}_{2,p}$. The papers [1,11,8,2,3,14–16] give a small sample of the kinds of investigations that have been carried out in this direction.

This paper falls into that tradition of trying to understand the geometric consequences obtained from property $\mathbf{N}_{2,p}$. Our main goal is to present a method for bounding the Castelnuovo–Mumford regularity, higher normality and other classical invariants of the projection image of X satisfying property $\mathbf{N}_{2,p}$ *scheme-theoretically*. For this purpose, we establish the graded mapping cone associated to the projection from a point, and then we deduce the long exact sequence of Tor-modules coming from the partial eliminations (Theorem 3.2). Unlike the Koszul techniques of Green and Lazarsfeld used in [5,6,15], our construction turns out to be useful for dealing with non-complete embeddings of X satisfying property $\mathbf{N}_{2,p}$. (cf. Theorems 3.10, 4.2 and 4.8). A significant portion of our work is to generalize earlier results in [6] and [15] from complete to non-complete linear systems. We also provide some illuminating examples of our results via calculations done with Macaulay 2 (Examples 3.12, 4.4 and 4.13).

Organization of the paper. In Section 2 we recall some notations and definitions which will be used throughout the remaining part of the paper. In Section 3 we make use of the graded mapping cone theorem (Theorem 3.2) to bound the number of zero-dimensional intersections of X with a plane when it satisfies $\mathbf{N}_{2,p}$ and $\mathbf{N}_{3,p+1}$ scheme-theoretically (Theorem 3.10). We also give Example 3.12 showing that this result is sharp. Finally, Section 4 is devoted to investigate the influence of property $\mathbf{N}_{2,p}$ on syzygies, higher normality and other invariants of the projected images of X when moving the center in $\mathbb{P}^n$. Some remarks and examples are also provided.

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2. Notations and definitions

- We work over an algebraically closed field k of characteristic zero.
- For an $(n+1)$-dimensional k-vector space V with basis $x_0, \dots, x_n$, we form the symmetric algebra $R = \mathrm{Sym}(V) = k[x_0, \dots, x_n]$. If W is a subspace of V with a basis $x_t, \dots, x_n$ we write S_t for the symmetric algebra $\mathrm{Sym}(W) = k[x_t, x_{t+1}, \dots, x_n]$.
- Unless otherwise stated, X is a non-degenerate and reduced closed subscheme in $\mathbb{P}^n_k = \mathbb{P}(V)$, which is not necessarily a complete embedding.
- Let Λ be a linear subvariety in $\mathbb{P}^n_k = \mathbb{P}(V)$ with homogeneous coordinates $x_0, \dots, x_{t-1}$. Consider an outer projection of X from the center Λ

$$\pi_\Lambda : X \to \mathbb{P}^{n-t}_k = \mathbb{P}(W).$$

  We write $Y_t \subset \mathbb{P}^{n-t}_k$ for the image $\pi_\Lambda(X)$.
- For a finitely generated graded R-module $M = \bigoplus_{\ell \geqslant 0} M_\ell$, consider a minimal free resolution

$$\cdots \to \bigoplus_j R(-i-j)^{\beta_{i,j}(M)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}(M)} \to M \to 0$$

  of M as a graded R-modules. Thus $\beta^R_{i,j}(M) := \dim_k \mathrm{Tor}^R_i(M,k)_{i+j}$. We say that M is m-regular if $\beta_{i,j}(M) = 0$ for all $i \geqslant 0$ and $j \geqslant m$. The Castelnuovo–Mumford regularity of M is defined by

$$\mathrm{reg}(M) := \min\{m \mid M \text{ is } m\text{-regular}\}.$$

- For a coherent sheaf $\mathcal{M}$ on $\mathbb{P}(V)$, let $M = \bigoplus_{\ell \in \mathbb{Z}} H^0(\mathcal{M}(\ell))$ be its associated graded R-module. Then we write

$$\mathrm{reg}(\mathcal{M}) := \min\bigl\{m \bigm| H^i\bigl(\mathcal{M}(m-i)\bigr) = 0 \text{ for all } i \geqslant 1\bigr\}.$$

  In this case, it is well known that $\mathrm{reg}(M) = \mathrm{reg}(\mathcal{M})$.
- For a closed subscheme $X \subset \mathbb{P}^n$, we say that X is m-normal if $H^1(\mathcal{I}_{X/\mathbb{P}^n}(m)) = 0$. The normality index of X is defined by

$$\alpha_X := \min\{m \mid X \text{ is } d\text{-normal for all } d \geqslant m\}.$$

3. The graded mapping cone theorem and its applications

The graded mapping cone construction on partial eliminations and its related long exact sequence are our starting point to understand algebraic and geometric structures of projections.

Let us consider the following situation which is associated with a simple projection from one point: let $W = \bigoplus_{i=1}^n k \cdot x_i$ and $V = \bigoplus_{i=0}^n k \cdot x_i$ be vector spaces over k with symmetric algebras $S_1 = k[x_1, \dots, x_n]$ and $R = k[x_0, \dots, x_n]$ respectively. Let M be a graded R-module. Then we can also think of M as a graded S_1-module by the inclusion $S_1 \hookrightarrow R$. One can define the graded Koszul complex $K^{S_1}_\bullet(M)$ of M as follows:

$$0 \to \bigwedge^n W \otimes M \to \cdots \to \bigwedge^2 W \otimes M \to W \otimes M \to M \to 0$$

where $K^{S_1}_i(M)_{i+j} = \bigwedge^i W \otimes M_j$ for all i, j.

Now consider the map $\overline{\varphi}$ between graded complexes of S_1-modules

$$\overline{\varphi} : K^{S_1}_\bullet\bigl(M(-1)\bigr) \xrightarrow{\times x_0} K^{S_1}_\bullet(M),$$

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which is induced by the multiplicative map $\varphi : M(-1) \xrightarrow{\times x_0} M$. Then there is the mapping cone $(C_\bullet(\overline{\varphi}), \partial_{\overline{\varphi}})$ by the map $\overline{\varphi}$ such that we have the following long exact sequence (Proposition A3.19 in [7]):

$$\begin{aligned} \cdots \to \mathrm{Tor}_i^{S_1}(M,k)_{i+j} &\to H_i\big(C_\bullet(\overline{\varphi})\big)_{i+j} \\ &\to \mathrm{Tor}_{i-1}^{S_1}(M,k)_{i+j-1} \xrightarrow{\delta=\times x_0} \mathrm{Tor}_{i-1}^{S_1}(M,k)_{i+j} \to \cdots. \end{aligned} \tag{3.1}$$

In the following Lemma 3.1, we claim that $\mathrm{Tor}^R(M,k)$ can be obtained from the homology of the mapping cone.

Lemma 3.1. *Let M be a graded R-module. Then we have the following natural isomorphism*:

$$\mathrm{Tor}_i^R(M,k)_{i+j} \simeq H_i\big(C_\bullet(\overline{\varphi})\big)_{i+j}.$$

Proof. Let $K_\bullet^R(M)$ be the Koszul complex of a graded R-module M. Then the graded component in degree $i+j$ of $K_i^R(M)$ is $K_i^R(M)_{i+j} = \bigwedge^i V \otimes M_j$. Note that

$$\bigwedge^i V \cong \bigwedge^i W \oplus \Big[x_0 \wedge \Big(\bigwedge^{i-1} W\Big)\Big],$$
$$C_i(\overline{\varphi})_{i+j} = \Big(\bigwedge^i W \otimes M_j\Big) \oplus \Big(\bigwedge^{i-1} W \otimes M_j\Big).$$

Hence we see that the Koszul complex $K_i^R(M)$ has the following canonical decomposition in each graded component:

$$\begin{aligned} K_i^R(M)_{i+j} &\cong \Big(\bigwedge^i W \otimes M_j\Big) \oplus \Big[\Big(x_0 \wedge \Big(\bigwedge^{i-1} W\Big)\Big) \otimes M_j\Big] \\ &\cong C_i(\overline{\varphi})_{i+j}. \end{aligned} \tag{3.2}$$

Using the decomposition (3.2), one can verify that the following diagram is commutative:

$$\begin{array}{ccc} K_i^R(M)_{i+j} & \xrightarrow{\cong} & C_i(\overline{\varphi})_{i+j} \\ \downarrow \partial & & \downarrow \partial_{\overline{\varphi}} \\ K_{i-1}^R(M)_{i+j} & \xrightarrow{\cong} & C_{i-1}(\overline{\varphi})_{i+j} \end{array} \tag{3.3}$$

Therefore, we have a natural isomorphism $\mathrm{Tor}_i^R(M,k)_{i+j} \simeq H_i(C_\bullet(\overline{\varphi}))_{i+j}$. □

From the long exact sequence (3.1) and Lemma 3.1, we have the following theorem which will be crucial throughout the remaining part of the paper.

Theorem 3.2 *(Graded mapping cone theorem). For a graded R-module M which is also a graded S_1-module, we have the following long exact sequence*:

$$\begin{aligned} \cdots \to \mathrm{Tor}_i^{S_1}(M,k)_{i+j} &\to \mathrm{Tor}_i^R(M,k)_{i+j} \to \mathrm{Tor}_{i-1}^{S_1}(M,k)_{i+j-1} \\ \xrightarrow{\delta} \mathrm{Tor}_{i-1}^{S_1}(M,k)_{i+j} &\to \mathrm{Tor}_{i-1}^R(M,k)_{i+j} \to \mathrm{Tor}_{i-2}^{S_1}(M,k)_{i+j-1} \xrightarrow{\delta} \cdots \end{aligned}$$

whose connecting homomorphism δ is the multiplicative map $\times x_0$.

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Note that Theorem 3.2 gives us useful information about syzygies of outer projections (i.e. isomorphic or birational projections) of projective varieties. As a first step, we obtain this interesting information on the minimal free resolution of R/I as a graded S_1-module.

Corollary 3.3. *Let $I \subset R$ be a homogeneous ideal such that R/I is a finitely generated S_1-module. Assume that I admits d-linear resolution as an R-module up to the p-th step for some $p \geqslant 2$. Then, for $1 \leqslant i \leqslant p-1$,*

(a) *the minimal free resolution of R/I as a graded S_1-module is*

$$\to L_{p-1} \to \cdots \to L_1 \to S_1(-d+1) \oplus \cdots \oplus S_1(-1) \oplus S_1 \to R/I \to 0,$$

where $L_i = S_1(-d+1-i)^{\beta^{S_1}_{i,d-1}}$ for all $1 \leqslant i \leqslant p-1$;

(b) *in particular, $\beta^{S_1}_{i,d-1} = (-1)^i + \sum_{1 \leqslant j \leqslant i} (-1)^{j+i} \beta^R_{j,d-1}(R/I)$.*

Proof. (a) First, consider the exact sequence

$$\cdots \to \operatorname{Tor}^R_1(R/I,k)_j \to \operatorname{Tor}^{S_1}_0(R/I,k)_{j-1}$$
$$\xrightarrow{\delta} \operatorname{Tor}^{S_1}_0(R/I,k)_j \to \operatorname{Tor}^R_0(R/I,k)_j \to 0.$$

Since $\operatorname{Tor}^R_1(R/I,k)_j = 0$ for all $j \neq d$ and $\operatorname{Tor}^R_0(R/I,k)_j = 0$ for all $j \neq 0$, we see that $\beta^R_{0,0} = \beta^{S_1}_{0,j} = 1$ for all $0 \leqslant j \leqslant d-1$ and $\beta^{S_1}_{0,j} = 0$ for all $j \notin \{0,1,\ldots,d-1\}$ from the finiteness of R/I as an S_1-module.

Note that $\operatorname{Tor}^R_i(R/I,k)_{i+j} = 0$ for $1 \leqslant i \leqslant p$ and $j \neq d-1$ by assumption that I is d-linear up to the p-th step. Applying Theorem 3.2 for $M = R/I$, we have an isomorphism induced by $\delta = \times x_0$:

$$\operatorname{Tor}^{S_1}_{i-1}(R/I,k)_{(i-1)+j} \xrightarrow{\delta} \operatorname{Tor}^{S_1}_{i-1}(R/I,k)_{(i-1)+(j+1)}$$

for $1 \leqslant i \leqslant p$ and for all $j \notin \{d-2, d-1\}$. Hence we conclude that

$$\operatorname{Tor}^{S_1}_{i-1}(R/I,k)_{(i-1)+j} = 0 \quad \text{for } 2 \leqslant i \leqslant p \text{ and } j \neq d-1,$$

since R/I is finitely generated as an S_1-module, which means that

$$L_i = S_1(-d-i+1)^{\oplus \beta^{S_1}_{i,d-1}} \quad \text{for } 1 \leqslant i \leqslant p-1.$$

(b) Note that we have

$$0 \to \operatorname{Tor}^{S_1}_i(R/I,k)_{i+d-1} \to \operatorname{Tor}^R_i(R/I,k)_{i+d-1} \to \operatorname{Tor}^{S_1}_{i-1}(R/I,k)_{i+d-2} \to 0$$

for $1 \leqslant i \leqslant p-1$, consequently

$$\beta^{S_1}_{i,d-1}(R/I) = \beta^R_{i,d-1}(R/I) - \beta^{S_1}_{i-1,d-1}(R/I).$$

Then, by induction on p, we get the desired result. □

From now on, we consider an outer projection $\pi_\Lambda : X \to Y_t \subset \mathbb{P}(W)$ where $\dim \Lambda = t-1 \geqslant 0$. Then, the following basic sequence

$$0 \to R/I_X \to E \to H^1_*(\mathcal{J}_X) \to 0 \quad (\text{as } S_t\text{-modules})$$

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is also exact as finitely generated S_t-modules by Lemma 3.4. Furthermore, it would be very useful to compare their graded Betti tables by the graded mapping cone theorem.

Lemma 3.4. *Let I be a homogeneous ideal defining X scheme-theoretically in $\mathbb{P}^n$. Then R/I and $E = \bigoplus_{\ell\in\mathbb{Z}} H^0(X, \mathcal{O}_X(\ell))$ are finitely generated S_t-modules.*

Proof. For $0 \leqslant i \leqslant t-1$, let $p_i = [0, \ldots, 0, 1, 0, \ldots, 0] \in \Lambda$ be the point whose $(i+1)$-th coordinate is 1. Since $p_i \notin X$, for some $m_i > 0$, there is a homogeneous polynomial f_i in I which is of the form $f_i = x_i^{m_i} + g_i$ where $g_i \in k[x_0, x_1, \ldots, x_n]$ is a homogeneous polynomial of degree m_i with the power of x_i less than m_i. Hence R/I is generated by monomials of the form $x_0^{n_0} \cdots x_{t-1}^{n_{t-1}}$, $n_i < m_i$ for all $0 \leqslant i \leqslant t-1$ as an S_t-module. Next, from the exact sequence $0 \to R/I_X \to E \to H^1_*(\mathcal{I}_X) \to 0$ as S_t-modules, E is also a finitely generated S_t-module. □

Remark 3.5. For an inner projection of X from the center $q \in X$, let $Y_1 = \overline{\pi_q(X)}$ be the Zariski-closure of $\pi_q(X)$ in $\mathbb{P}^{n-1}$. Then R/I_X is *not finitely generated* as a graded S_1-module.

The following theorem is a generalization of Corollary 3.3, which is related to the existence of multisecant plane (cf. Theorem 3.10).

Theorem 3.6. *Suppose that X satisfies property $\mathbf{N}_{d,p}$ scheme-theoretically with an ideal I. Consider an outer projection $\pi_\Lambda : X \to \mathbb{P}(W)$ from the center $\Lambda = \mathbb{P}(U) = \mathbb{P}^{t-1}$. Then, $\overline{R} = R/I_{\geqslant d}$ has the simplest syzygies up to $(p-t)$-th step as S_t-module for $1 \leqslant t \leqslant p$,*

$$\to L_{p-t} \to \cdots \to L_1 \to \bigoplus_{i=0}^{d-1} \mathrm{Sym}^i(U) \otimes S_t(-i) \to \overline{R} \to 0 \tag{3.4}$$

where $L_i = S_t(-i-d+1)^{\beta^{S_t}_{i,d-1}}$ for $1 \leqslant i \leqslant p-t$ and $\mathrm{Sym}^i(U) = H^0(\mathcal{O}_\Lambda(i))$ is a vector space of homogeneous forms of degree i generated by U.

In particular, if $d = 2$ then the minimal free resolutions of R/I is

$$\to S_t(-p+t-1)^{\beta^{S_t}_{p-t,1}} \to \cdots \to S_t(-2)^{\beta^{S_t}_{1,1}} \to S_t \oplus S_t(-1)^t \to R/I \to 0.$$

Proof. Let $S_t = k[x_t, \ldots, x_n]$ be a polynomial ring for $0 \leqslant t \leqslant p$ and let

$$\to L_{p-t} \to \cdots \to L_1 \to L_0 \to \overline{R} \to 0$$

be the minimal free resolution of $\overline{R}$ as an S_t-module. We will give a proof by induction on $t \geqslant 1$. For $t = 1$, the result (3.4) follows directly from Corollary 3.3. For $t > 1$, by induction hypothesis, we can assume that for $1 \leqslant \alpha \leqslant p-(t-1)$,

$$\mathrm{Tor}^{S_{t-1}}_\alpha(\overline{R}, k)_j = 0 \quad \text{if } j \neq \alpha + d - 1.$$

Using an exact sequence by the mapping cone theorem for $\alpha \geqslant 1$,

$$\begin{aligned} \cdots &\to \mathrm{Tor}^{S_{t-1}}_{\alpha+1}(\overline{R}, k)_j \to \mathrm{Tor}^{S_t}_\alpha(\overline{R}, k)_{j-1} \\ &\xrightarrow{\delta} \mathrm{Tor}^{S_t}_\alpha(\overline{R}, k)_j \to \mathrm{Tor}^{S_{t-1}}_\alpha(\overline{R}, k)_j \to \cdots \end{aligned}$$

we can also show, by an argument similar to that of Corollary 3.3, that

$$\mathrm{Tor}_{\alpha}^{S_t}(\overline{R},k)_j=0 \quad \text{if } j\neq\alpha+d-1;$$

equivalently, $L_\alpha=S_t(-\alpha-d+1)^{\beta^{S_t}_{\alpha,d-1}}$ for $1\leqslant\alpha\leqslant p-t$.

It remains to show that $L_0=\bigoplus_{i=0}^{d-1}\mathrm{Sym}^i(U)\otimes S_t(-i)$. Note that the set $\{\mathrm{Sym}^i(U)\mid 0\leqslant i\leqslant d-1\}$ should be contained in any generating set of $\overline{R}$ as an S_t-module because there is no relation of degree at most $d-1$ in $\overline{R}$. So, we have to show that $\{\mathrm{Sym}^i(U)\mid 0\leqslant i\leqslant d-1\}$ is actually the set of all generators. This can be done by dimension counting. Let us prove this by induction on t. In the case of $t=1$, the result easily follows from Corollary 3.3(a). If $t>1$ then, by the induction hypothesis, we see that for all $i\leqslant d-1$,

$$\dim_k \mathrm{Tor}_0^{S_{t-1}}(\overline{R},k)_i=\binom{i+t-2}{t-2},\quad \mathrm{Tor}_1^{S_{t-1}}(\overline{R},k)_i=0$$

and we have the following sequence from the mapping cone construction

$$0\to\mathrm{Tor}_0^{S_t}(\overline{R},k)_{i-1}\to\mathrm{Tor}_0^{S_t}(\overline{R},k)_i\to\mathrm{Tor}_0^{S_{t-1}}(\overline{R},k)_i\to 0,$$

for each $0\leqslant i<d$. Hence, we obtain

$$\dim_k \mathrm{Tor}_0^{S_t}(\overline{R},k)_i=\sum_{m=0}^{i}\binom{m+t-2}{t-2}=\binom{i+t-1}{t-1},$$

as we wished. □

Definition 3.7. (See [15].) Consider three vector spaces $W\subset V\subset H^0(X,\mathcal{O}_X(1))$ and suppose that $t=\mathrm{codim}(W,V)$ and $\alpha=\mathrm{codim}(W,H^0(\mathcal{O}_X(1)))$. We say that R/I_X (resp. E) satisfies property $\mathbf{N}_p^S$ if R/I_X (resp. E) have the simplest minimal free resolutions up to the p-th step as graded S_t-modules:

$$\cdots\to E_p\to E_{p-1}\to\cdots\to E_1\to S_t\oplus S_t(-1)^{\oplus\alpha}\to E\to 0 \tag{3.5}$$

where $E_i=S_t(-i-1)^{\oplus\beta_{i,1}}$ for $1\leqslant i\leqslant p$, and

$$\cdots\to\tilde{L}_p\to\tilde{L}_{p-1}\to\cdots\to\tilde{L}_1\to S_t\oplus S_t(-1)^{\oplus t}\to R/I_X\to 0 \tag{3.6}$$

where $\tilde{L}_i=S_t(-i-1)^{\oplus\tilde{\beta}_{i,1}}$, $1\leqslant i\leqslant p$.

On the other hand, we have a similar result for $E=\bigoplus_{\ell\in\mathbb{Z}}H^0(X,\mathcal{O}_X(\ell))$ as the following proposition shows.

Proposition 3.8. *With the same hypotheses as Theorem* 3.6*, suppose E (or R/I_X) satisfies property $\mathbf{N}_p^S$ as an R-module for some $p\geqslant 2$. Then E (or R/I_X) also satisfies property $\mathbf{N}_{p-t}^S$ as an S_t-module under the projection morphism $\pi_\Lambda: X\to Y_t\subset\mathbb{P}^{n-t}=\mathbb{P}(W)$, $1\leqslant t\leqslant p$.*

Proof. When $t=1$, we can similarly show that E satisfies property $\mathbf{N}_{p-1}^S$ as an S_1 module by using Theorem 3.2 for $M=E$ and the vanishing $\beta_{i,j}^R(E)=0$ when $0\leqslant i\leqslant p$ and $j\geqslant 2$. As a consequence, E has the following simplest resolution:

$$\cdots\to E_{p-1}\to E_{p-2}\to\cdots\to E_1\to S_1\oplus S_1(-1)^{\oplus\alpha}\to E\to 0 \tag{3.7}$$

where $E_i = S_1(-i-1)^{\oplus \beta_{i,1}}$ for $1 \leqslant i \leqslant p-1$. For $t \geqslant 2$, letting $S_i = k[x_i, x_{i+1}, \ldots, x_n]$, we inductively check that if E satisfies property $\mathbf{N}^S_{p-i}$ as an S_i-module, then E also satisfies property $\mathbf{N}^S_{p-i-1}$ as an S_{i+1}-module by the same argument as in Theorem 3.6. For R/I_X, the proof is exactly the same. □

Remark 3.9. For isomorphic projections, the above Proposition 3.8 is in fact a simple algebraic restatement of Theorem 2 in [6], Theorem 1.2 in [15], and for birational projections, see a part of Theorem 3.1 in [19]. Indeed, for any regular projection $\pi_\Lambda : X \to Y_t \subset \mathbb{P}^{n-t} = \mathbb{P}(W)$, $1 \leqslant t \leqslant p$, there is an exact sequence:

$$0 \to \operatorname{Tor}_i^{S_t}(E,k)_{i+j} \to H^1\Big(\bigwedge^{i+1} \mathcal{M} \otimes \pi_{\Lambda *}\mathcal{O}_X(j-1)\Big) \xrightarrow{\alpha_{i,j}} \bigwedge^{i+1} W \otimes H^1\big(\pi_{\Lambda *}\mathcal{O}_X(j-1)\big) \to \cdots. \tag{3.8}$$

From the following commutative diagram:

$$\begin{array}{ccc} H^1(Y_t, \bigwedge^{i+1} \mathcal{M}_W \otimes \pi_{\Lambda *}\mathcal{O}_X(j-1)) & \xrightarrow{\alpha_{i,j}} & \bigwedge^{i+1} W \otimes H^1(\pi_{\Lambda *}\mathcal{O}_X(j-1)) \\ \| & & \| \\ H^1(X, \bigwedge^{i+1} \pi_\Lambda^* \mathcal{M}_W \otimes \mathcal{O}_X(j-1)) & \xrightarrow{\tilde{\alpha}_{i,j}} & \bigwedge^{i+1} W \otimes H^1(\mathcal{O}_X(j-1)) \end{array}$$

it was shown (cf. [6,15,19]) that $\tilde{\alpha}_{i,j}$ is injective for all $1 \leqslant i \leqslant p-t$ and $j \geqslant 2$. Thus, E satisfies property $\mathbf{N}^S_{p-t}$ as S_t-module. □

In the following theorem, we make use of the graded mapping cone theorem to bound the number of zero-dimensional intersection of X with a plane. As a special case, we recover Theorem 1.1 in [8] with a different method. This gives us a geometric meaning of property $\mathbf{N}_{d,p}$ with respect to multisecant planes.

Theorem 3.10. *Suppose that X satisfies property $\mathbf{N}_{d,p}$ scheme-theoretically in $\mathbb{P}^n$. Consider the projection $\pi_\Lambda : X \to Y_t \subset \mathbb{P}^{n-t}$ from the center $\Lambda = \mathbb{P}(U) = \mathbb{P}^{t-1}$, $t \leqslant p$. Then:*

(a) *Every fiber of π_Λ is $(d-1)$-normal, i.e. $\operatorname{reg}(\pi_\Lambda^{-1}(y)) \leqslant d$ for all $y \in Y_t$. So, $\operatorname{length}(\pi_\Lambda^{-1}(y)) \leqslant \binom{t+d-1}{t}$.*
(b) *$\operatorname{reg}(X \cap L) \leqslant d$ for any linear section $X \cap L$ as a finite scheme where $L = \mathbb{P}^m$, $1 \leqslant m \leqslant p$. In particular, for a projective variety satisfying property $\mathbf{N}_{2,p}$, there is no $(p+2)$-secant p-plane.*
(c) *Suppose X satisfies $\mathbf{N}_{2,p}$ and $\mathbf{N}_{3,p+1}$ scheme-theoretically with an ideal I for some $p \geqslant 1$. If there is an ℓ-secant $(p+1)$-plane to X then*

$$\ell \leqslant p + 2 + \min\big\{p+1, \beta^R_{p+1,2}(R/I)\big\}.$$

Proof. Choose an ideal I with property $\mathbf{N}_{d,p}$ defining X scheme-theoretically. For a proof of (a), consider the minimal free resolution of $R/(I)_{\geqslant d}$ as S_t-module given in Proposition 3.6, namely,

$$\cdots \to S_t(-d)^{\oplus \beta^{S_t}_{1,d-1}} \to \bigoplus_{i=0}^{d-1} \operatorname{Sym}^i(U) \otimes S_t(-i) \to R/(I)_{\geqslant d} \to 0$$

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where $\mathrm{Sym}^i(U) = H^0(\mathcal{O}_\Lambda(i))$ is a vector space of homogeneous forms of degree i generated by U. By sheafifying this exact sequence and tensoring $\bigotimes \mathcal{O}_{\mathbb{P}^{n-t}}(d-1)$, we have the surjective morphism of sheaves

$$\cdots \to \bigoplus_{i=0}^{d-1} \mathrm{Sym}^i(U) \otimes \mathcal{O}_{\mathbb{P}^{n-t}}(d-1-i) \to \pi_{\Lambda *}\mathcal{O}_X(d-1) \to 0.$$

For all $y \in Y_t$, we have the following surjective commutative diagram $(*)$ by Nakayama's lemma:

$$\begin{array}{ccc} \bigoplus_{i=0}^{d-1} \mathrm{Sym}^i(U) \otimes \mathcal{O}_{\mathbb{P}^{n-t}}(d-1-i) \otimes k(y) & \twoheadrightarrow & \pi_{\Lambda *}\mathcal{O}_X(d-1) \otimes k(y) \\ \| & (*) & \| \\ H^0(\langle \Lambda, y\rangle, \mathcal{O}_{\langle \Lambda, y\rangle}(d-1)) & \twoheadrightarrow & H^0(\mathcal{O}_{\pi_\Lambda{}^{-1}(y)}(d-1)) \end{array}$$

Therefore, as a finite scheme, $\pi_\Lambda{}^{-1}(y)$ is $(d-1)$-normal for all $y \in Y_t$.

For a proof of (b), suppose that $\mathrm{reg}(X \cap L) > d$ for some linear section $X \cap L$ as a finite scheme where $1 \leqslant m = \dim(L) \leqslant p$. Then we can take a linear subspace $\Lambda_1 \subset L$ of dimension $m-1$ disjoint from $X \cap L$. Then $X \cap L$ is the fiber of a projection $\pi_{\Lambda_1} : X \to \mathbb{P}^{n-m-1}$ at $\pi_{\Lambda_1}(L)$. However, this contradicts (a).

Let's proceed to prove (c). Suppose that I satisfies $\mathbf{N}_{2,p}$ and $\mathbf{N}_{3,p+1}$. If $\beta^R_{p+1,2} = 0$ then it is clear by (b). So let us assume that $\beta^R_{p+1,2}$ is nonzero and $\beta^R_{p+1,j} = 0$ for all $j \geqslant 3$. Suppose that L is an ℓ-secant $(p+1)$-plane to X, and then choose a linear subspace Λ of dimension p disjoint from X with homogeneous coordinates $x_0, \ldots, x_p$. Let $S_{p+1} = k[x_{p+1}, x_{p+2}, \ldots, x_n] \subset S_p = k[x_p, x_{p+1}, \ldots, x_n]$. Then it follows from Theorem 3.6 that the minimal free presentation of R/I as an S_p-module is of the form:

$$\cdots \to F_1 \to S_p \oplus S_p(-1)^p \to R/I \to 0.$$

Now consider the following long exact sequence for each $j = 0, 1, 2$,

$$\begin{aligned} \cdots \to \mathrm{Tor}_1^{S_p}(R/I, k)_j \to \mathrm{Tor}_0^{S_{p+1}}(R/I, k)_{j-1} \\ \xrightarrow{\delta = \times x_p} \mathrm{Tor}_0^{S_{p+1}}(R/I, k)_j \to \mathrm{Tor}_0^{S_p}(R/I, k)_j \to 0. \end{aligned} \tag{3.9}$$

By property $\mathbf{N}_{3,p+1}$ of R/I, we can easily verify that the minimal free resolution of R/I as an S_{p+1}-module is of the form

$$\cdots \to S_{p+1} \oplus S_{p+1}(-1)^{p+1} \oplus S_{p+1}(-2)^\alpha \to R/I \to 0 \tag{3.10}$$

for some α in $\mathbb{Z}_{\geqslant 0}$. Then it follows from 3.9 and $j = 2$ that

$$\dim_k \mathrm{Tor}_0^{S_{p+1}}(R/I, k)_1 = p + 1 \geqslant \alpha.$$

On the other hand, we have the following surjections from the fact that $\mathrm{Tor}^{S_i}_{p+1-i}(R/I, k)_{p+1-i+3} = 0$ for $1 \leqslant i \leqslant p+1$ (cf. Proposition 3.8):

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$$\mathrm{Tor}^R_{p+1}(R/I,k)_{p+1+2} \to \mathrm{Tor}^{S_1}_p(R/I,k)_{p+2} \xrightarrow{\times x_0} \mathrm{Tor}^{S_1}_p(R/I,k)_{p+3} = 0,$$
$$\mathrm{Tor}^{S_i}_{p+1-i}(R/I,k)_{p+1-i+2} \to \mathrm{Tor}^{S_{i+1}}_{p-i}(R/I,k)_{p-i+2} \xrightarrow{\times x_i} \mathrm{Tor}^{S_{i+1}}_{p-i}(R/I,k)_{p-i+3} = 0,$$
$$\mathrm{Tor}^{S_p}_1(R/I,k)_3 \to \mathrm{Tor}^{S_{p+1}}_0(R/I,k)_2 \xrightarrow{\times x_p} \mathrm{Tor}^{S_{p+1}}_0(R/I,k)_3 = 0$$

for all $0 \leqslant i \leqslant p+1$, which implies that

$$\dim_k \mathrm{Tor}^R_{p+1}(R/I,k)_{p+3} = \beta^R_{p+1,2}(R/I) \geqslant \alpha = \dim_k \mathrm{Tor}^{S_{p+1}}_0(R/I,k)_2.$$

So, we have the inequality

$$\alpha \leqslant \min\{p+1, \beta^R_{p+1,2}(R/I)\}.$$

By the sheafification of the sequence (3.10), the length of any fiber of

$$\pi_\Lambda : X \to \mathbb{P}^{n-p-1}$$

is at most $1+(p+1)+\alpha$. This completes the proof of (c). □

Remark 3.11. In the process of proving (a) in Theorem 3.10, we learn that the global property $\mathbf{N}_{d,p}$ scheme-theoretically gives local information on the length of fibers in any linear projection from the center Λ of dimension at most $p-1$. The commutative diagram $(*)$ in the proof of (a) can also be understood geometrically as follows:

$$\begin{array}{ccc} \mathrm{Bl}_\Lambda(\mathbb{P}^n) & = & \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-t}}(1) \oplus U \otimes \mathcal{O}_{\mathbb{P}^{n-t}}) \\ \sigma \downarrow & & \rho \downarrow \\ \mathbb{P}^n & \overset{\pi_\Lambda}{\dashrightarrow} & \mathbb{P}^{n-t} \end{array}$$

where $\sigma : \mathrm{Bl}_\Lambda \mathbb{P}^n \to \mathbb{P}^n$ is a blow-up of $\mathbb{P}^n$ along Λ.

Note that $\rho_*\sigma^*\mathcal{O}_{\mathbb{P}^n}(d-1) = \mathrm{Sym}^{d-1}(\mathcal{O}_{\mathbb{P}^{n-t}}(1) \oplus U \otimes \mathcal{O}_{\mathbb{P}^{n-t}})$. We actually showed that the natural morphism of sheaves

$$\rho_*\sigma^*\mathcal{O}_{\mathbb{P}^n}(d-1) \to \rho_*\sigma^*\mathcal{O}_X(d-1) = \pi_{\Lambda *}\mathcal{O}_X(d-1)$$

is surjective from property $N_{d,p}$ because the following morphism

$$\begin{array}{ccc} \mathrm{Sym}^{d-1}(\mathcal{O}_{\mathbb{P}^{n-t}}(1) \oplus U \otimes \mathcal{O}_{\mathbb{P}^{n-t}}) \otimes k(y) & \twoheadrightarrow & \pi_{\Lambda *}\mathcal{O}_X(d-1) \otimes k(y) \\ \| & & \| \\ \bigoplus_{i=0}^{d-1} \mathrm{Sym}^i(U) \otimes \mathcal{O}_{\mathbb{P}^{n-t}}(d-1-i) \otimes k(y) & \twoheadrightarrow & H^0(\mathcal{O}_{\pi_\Lambda^{-1}(y)}(d-1)) \end{array}$$

is surjective for all $y \in Y_t$. Similar constructions were used in bounding regularity of smooth surfaces and threefolds in [14] and [16].

The following examples show that the upper bounds of (a) and (c) in Theorem 3.10 are sharp.

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Table 1
Linear projection $\pi_q(C) \subset \mathbb{P}^9$ from $q \in S^5(C) \setminus S^4(C)$.

Total	0	1	2	3	4	5	6	7	8	9
	1	34	151	315	371	265	125	43	10	1
0	1	.	.	.	.	.	.	.	.	.
1	.	34	151	314	365	230	69	7	.	.
2	.	.	.	1	6	35	56	36	10	1

Example 3.12. (a) Let $S^\ell(C)$ be the ℓ-th higher secant variety of a rational normal curve C in $\mathbb{P}^n$. Then the defining ideal of $S^\ell(C)$ is generated by maximal minors of a 1-generic matrix of linear forms of size $\ell+1$ in $S = \mathbb{C}[x_0, \dots, x_n]$. It is known that $S^\ell(C)$ is arithmetically Cohen–Macaulay of degree $\binom{n-\ell+1}{\ell}$ having a $(\ell+1)$-linear resolution from the Eagon–Northcott complex. Now, consider a linear projection from a general linear space $\Lambda = \mathbb{P}^{n-2\ell}$. The length of a general fiber is the degree of $S^\ell(C)$ and thus the bound in Theorem 3.10(a) is sharp.

(b) Let C be an elliptic normal curve of degree d in $\mathbb{P}^{d-1}$ which satisfies property $\mathbf{N}_{d-3}$ but fails to satisfy property $\mathbf{N}_{d-2}$ with $\beta^R_{d-2,2} = 1$. Since $\deg(C) = d = d-1+\min\{d-2, \beta^R_{p+1,2}(R/I)\}$, the bound in Theorem 3.10(c) is also sharp.

(c) (Macaulay 2 in [13]) Let $C = \nu_{10}(\mathbb{P}^1)$ be a rational normal curve in $\mathbb{P}^{10}$. Let $S^\ell(C)$ be the ℓ-th higher secant variety of $\dim S^\ell(C) = \min\{2\ell-1, 10\}$. Then,

$$C \subsetneq \mathrm{Sec}(C) = S^2(C) \subsetneq S^3(C) \subsetneq \cdots \subsetneq S^6(C) = \mathbb{P}^{10}.$$

Then, for any point $q \in S^5(C) \setminus S^4(C)$, $\pi_q(C) \subset \mathbb{P}^9$ is a smooth rational curve with Betti Table 1.

Note that $\pi_q(C) \subset \mathbb{P}^9$ satisfies property $\mathbf{N}_{2,2}$ with $\beta^R_{3,2} = 1$. Since $\pi_q(C)$ has a 5-secant 3-plane in $\mathbb{P}^9$, the bound in Theorem 3.10(c) is also sharp. If $q \in \mathbb{P}^{10} \setminus S^5(C)$ then $\pi_q(C)$ satisfies $\mathbf{N}_{2,3}$ with $\beta_{4,2} = 6$.

(d) Note that two optimal examples (b) and (c) that we found in Example 3.12 all satisfy

$$\ell = p+2+\min\{p+1, \beta^R_{p+1,2}(R/I)\} = p+2+\beta^R_{p+1,2}(R/I).$$

It would be interesting to find other examples having optimal bound $\ell = 2p+3$ when $p+1 < \beta^R_{p+1,2}(R/I)$ for some $p \geqslant 2$.

4. Effects of property $\mathbf{N}_{2,p}$ on projections and moving the center

For a projective variety $X \subset \mathbb{P}^n$, property $\mathbf{N}_{2,p}$ is a natural generalization of property $\mathbf{N}_p$ to the case of non-complete linear systems. Note that a smooth variety $X \subset \mathbb{P}^n$ satisfying property $\mathbf{N}_{2,p}$, $p \geqslant 1$, scheme-theoretically has $\mathrm{reg}(X) \leqslant e+1$ where $e = \mathrm{codim}(X, \mathbb{P}^n)$ and so X is m-normal for all $m \geqslant e$ (cf. [3]). The main theorems in this section show that property $\mathbf{N}_{2,p}$ plays an important role to control the normality and defining equations of projected varieties under isomorphic and birational outer projections up to the $(p-1)$-th step.

Proposition 4.1. *Suppose that $X \subset \mathbb{P}^n$ satisfies property $\mathbf{N}_{2,p}$ scheme-theoretically for some $p \geqslant 2$. Consider an isomorphic projection $\pi_\Lambda : X \to Y_t \subset \mathbb{P}^{n-t}$ for some $1 \leqslant t \leqslant p-1$. Then we have*

$$H^1(\mathcal{J}_{X/\mathbb{P}^n}(m)) = H^1(\mathcal{J}_{Y_t/\mathbb{P}^{n-t}}(m))$$

for all $m \geqslant t+1$. Thus, if X is m-normal for all $m \geqslant \alpha_X$ then Y_t is m-normal for all $m \geqslant \max\{\alpha_X, t+1\}$ and $\mathrm{reg}(Y_t) \leqslant \max\{\mathrm{reg}(X), t+2\}$.

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Proof. Let $R = k[x_0, x_1, \ldots, x_n]$ and $S_t = k[x_t, x_{t+1}, \ldots, x_n]$ be the coordinate rings of $\mathbb{P}^n$ and $\mathbb{P}^{n-t}$ respectively. Choose an ideal I defining X with $\mathbf{N}_{2,p}$ scheme-theoretically. Then, by Theorem 3.6, we have the minimal free resolution of R/I as a graded S_t-module:

$$\to S_t(-p+t-1)^{\oplus \beta_{p-t,1}} \to \cdots \to S_t(-2)^{\oplus \beta_{1,1}} \to S_t \oplus S_t(-1)^{\oplus t} \xrightarrow{\varphi_0} R/I \to 0.$$

Note that $\pi_{\Lambda_*}(\mathcal{O}_X) \simeq \mathcal{O}_{Y_t}$ and $(R/I)_d = H^0(\mathcal{O}_{Y_t}(d))$ for all $d \gg 0$. Therefore, by sheafifying the resolution of R/I, we have the following familiar two diagrams by using Snake Lemma (see [12,15]):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{I}_{Y_t/\mathbb{P}^{n-t}} & \to & \mathcal{O}_{\mathbb{P}^{n-t}} & \to & \mathcal{O}_{Y_t} & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{L} & \to & \mathcal{O}_{\mathbb{P}^{n-t}}(-1)^{\oplus t} \oplus \mathcal{O}_{\mathbb{P}^{n-t}} & \xrightarrow{\widetilde{\varphi_0}} & \mathcal{O}_{Y_t} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}^{n-t}}(-1)^{\oplus t} & = & \mathcal{O}_{\mathbb{P}^{n-t}}(-1)^{\oplus t} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{4.1}$$

and in the first syzygies of R/I, we have the following diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \mathcal{I}_{Y_t/\mathbb{P}^{n-t}} & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & \mathcal{K} & \to & \mathcal{O}_{\mathbb{P}^{n-t}}(-2)^{\oplus \beta_{1,1}} & \to & \mathcal{L} & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & \mathcal{N} & \to & \mathcal{O}_{\mathbb{P}^{n-t}}(-2)^{\oplus \beta_{1,1}} & \to & \mathcal{O}_{\mathbb{P}^{n-t}}(-1)^{\oplus t} & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathcal{I}_{Y_t/\mathbb{P}^{n-t}} & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array} \tag{4.2}$$

Claim. *From the commutative diagrams* (4.1) *and* (4.2):

(a) $\mathrm{reg}(\mathcal{N}) \leqslant t+2$.
(b) *For all* $m \geqslant t+1$, *we have the isomorphisms on* m*-normality*: $H^1(\mathcal{I}_{Y_t/\mathbb{P}^{n-t}}(m)) \simeq H^2(\mathcal{K}(m)) \simeq H^1(\mathcal{L}(m)) \simeq H^1(\mathcal{I}_{X/\mathbb{P}^n}(m))$.

Proof of Claim. First of all, we can control the Castelnuovo-regularity of $\mathcal{N}$ (cf. [12,15,17]) by using the Eagon–Northcott complex associated to the exact sequence

$$0 \to \mathcal{N} \to \mathcal{O}_{\mathbb{P}^{n-t}}(-2)^{\oplus \beta_{1,1}} \to \mathcal{O}_{\mathbb{P}^{n-t}}(-1)^{\oplus t} \to 0.$$

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As a consequence, $\operatorname{reg}(\mathcal{N}) \leqslant t+2$. Thus, from the leftmost column and first row of (4.2), we have the following isomorphisms for all $m \geqslant t+1$:

$$H^1\big(\mathcal{I}_{Y_t/\mathbb{P}^{n-t}}(m)\big) \simeq H^2\big(\mathcal{K}(m)\big) \simeq H^1\big(\mathcal{L}(m)\big). \tag{4.3}$$

On the other hand, by taking global sections and using simple linear syzygies of R/I as an S_t-module, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & H^0_*(\mathcal{L}) & \to & S_t(-1)^{\oplus t} \oplus S_t & \xrightarrow{H^0_*(\widetilde{\varphi_0})} & \bigoplus_{\ell\in\mathbb{Z}} H^0(\mathcal{O}_{Y_t}(\ell)) & \twoheadrightarrow & H^1_*(\mathcal{L}) \\ & & \uparrow & & \| & & \uparrow & & \\ 0 & \to & K_0 & \to & S_t(-1)^{\oplus t} \oplus S_t & \xrightarrow{\varphi_0} & R/I & \to & 0 \end{array}$$

Since $\operatorname{im} H^0_*(\widetilde{\varphi_0}) = R/I_X$ and $\bigoplus_{\ell\in\mathbb{Z}} H^0(\mathcal{O}_{Y_t}(\ell)) = \bigoplus_{\ell\in\mathbb{Z}} H^0(\mathcal{O}_X(\ell))$, we have $H^1_*(\mathcal{L}) = H^1_*(\mathcal{I}_{X/\mathbb{P}^n})$. Combining this with (4.3), we obtain the proof of Claim (b). Hence we conclude that

$$H^1\big(\mathcal{I}_{X/\mathbb{P}^n}(m)\big) = H^1\big(\mathcal{I}_{Y_t/\mathbb{P}^{n-t}}(m)\big)$$

for all $m \geqslant t+1$ and thus $\operatorname{reg}(Y_t) \leqslant \max\{\operatorname{reg}(X), t+2\}$, as we wished. □

Theorem 4.2 *(Isomorphic projections for the case of $\mathbf{N}_{2,p}$). Suppose that $X \subset \mathbb{P}^n$ satisfy property $\mathbf{N}_{2,p}$ for some $p \geqslant 2$. Consider an isomorphic projection $\pi_\Lambda : X \to Y_t \subset \mathbb{P}^{n-t}$ for some $1 \leqslant t \leqslant p-1$. Then I_{Y_t} is also cut out by equations of degree at most $t+2$ and further satisfies property $\mathbf{N}_{t+2,p-t}$.*

Proof. Let us use the same notation as that in Proposition 4.1. Let $I = I_X$ be the defining ideal of X. From the leftmost column of (4.2), consider the following diagram for all $\ell \geqslant 1$:

$$\begin{array}{ccccc} H^0(\mathcal{O}_{\mathbb{P}^{n-t}}(\ell)) \otimes H^0(\mathcal{N}(t+2)) & \twoheadrightarrow & H^0(\mathcal{N}(t+2+\ell)) & \to & 0 \\ \downarrow & & \downarrow & & \\ H^0(\mathcal{O}_{\mathbb{P}^{n-t}}(\ell)) \otimes H^0(\mathcal{I}_{Y_t/\mathbb{P}^{n-t}}(t+2)) & \twoheadrightarrow & H^0(\mathcal{I}_{Y_t}(t+2+\ell)) & \to & 0 \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

Note that surjectivity of the first row is given by $\operatorname{reg}(\mathcal{N}) \leqslant t+2$ and surjectivity of two vertical columns are given by the fact that $H^1_*(\mathcal{K}) = 0$. Thus, the second row is also surjective and consequently Y_t is cut out by equations of degree at most $(t+2)$. For the syzygies of I_{Y_t}, consider the exact sequence by taking global sections

$$0 \to H^0_*(\mathcal{K}) \to H^0_*(\mathcal{N}) \to I_{Y_t} \to H^1_*(\mathcal{K}) = 0.$$

Since $H^0_*(\mathcal{K}) = K_1$ is the first syzygy module of R/I_X, $H^0_*(\mathcal{K})$ has the following resolution:

$$\to S_t(-p+t-1)^{\oplus\beta_{p-t,1}} \to \cdots \to S_t(-4)^{\oplus\beta_{3,1}} \to S_t(-3)^{\oplus\beta_{2,1}} \to H^0_*(\mathcal{K}) \to 0$$

and so, $\operatorname{Tor}_i^{S_t}(H^0_*(\mathcal{K}), k)_{i+j} = 0$ for $0 \leqslant i \leqslant p-t-2$ and $j \geqslant 4$. On the other hand, we have the following equivalence:

$$\operatorname{reg} H^0_*(\mathcal{N}) = \operatorname{reg}(\mathcal{N}) \leqslant t+2 \iff \operatorname{Tor}_i^{S_t}\big(H^0_*(\mathcal{N}), k\big)_{i+j} = 0 \quad \text{for } i \geqslant 0,\ j \geqslant t+3.$$

Table 2
Linear projection $C_3 = \pi_\Lambda(C) \subset \mathbb{P}^{10}$.

Total	0	1	2	3	4	5	6	7	8	9	10
	1	39	183	415	627	728	643	395	153	33	3
0	1	.	.	.	.	.	.	.	.	.	.
1	.	39	183	387	424	245	69	7	.	.	.
2	.	.	.	28	203	483	574	388	153	33	3

Table 3
Linear projection $C_4 = \pi_p(C_3) \subset \mathbb{P}^9$.

Total	0	1	2	3	4	5	6	7	8	9
	1	31	142	351	548	568	391	168	40	4
0	1	.	.	.	.	.	.	.	.	.
1	.	28	103	161	134	50	6	.	.	.
2	.	3	39	190	414	518	385	168	40	4

Thus, from the long exact sequence:

$$\mathrm{Tor}_i^{S_t}(H^0_*(\mathcal{K}),k)_{i+j} \to \mathrm{Tor}_i^{S_t}(H^0_*(\mathcal{N}),k)_{i+j} \to \mathrm{Tor}_i^{S_t}(I_{Y_t},k)_{i+j}$$
$$\xrightarrow{\delta} \mathrm{Tor}_{i-1}^{S_t}(H^0_*(\mathcal{K}),k)_{i+j} \to \mathrm{Tor}_{i-1}^{S_t}(H^0_*(\mathcal{N}),k)_{i+j} \to \mathrm{Tor}_{i-1}^{S_t}(I_{Y_t},k)_{i+j},$$

we get $\mathrm{Tor}_i^{S_t}(I_{Y_t},k)_{i+j} = 0$ for $0 \leqslant i \leqslant p-t-1$ and $j \geqslant t+3$, and Y_t satisfies property $N_{2+t,p-t}$. □

Remark 4.3. (a) As a special case of Proposition 4.1 and Theorem 4.2, the same results were proved in [6] and [15] for the complete linear embedding $X \subset \mathbb{P}(H^0(\mathcal{O}_X(1)))$.

(b) A. Alzati and F. Russo gave a necessary and sufficient condition for the isomorphic projection of an m-normal variety to be also m-normal. As an application, they showed that for a variety $X \subset \mathbb{P}^n$ satisfying property $\mathbf{N}_2$, one point isomorphic projection of X in $\mathbb{P}^{n-1}$ is k-normal for all $k \geqslant 2$ (Theorem 3.2 and Corollary 3.3 in [1]). Theorem 4.2 generalizes the results in [1] to non-linearly normal embeddings.

Example 4.4 *(Macaulay 2 in [13]).* Let C be a rational normal curve in $\mathbb{P}^{13}$ and $S^m(C)$ be the m-th higher secant variety of dimension $\min\{2m-1, 13\}$. Now choose three points q_1, q_2 in $S^6(C) \setminus S^5(C)$ and $q_3 \in \mathbb{P}^{13} \setminus S^6(C)$. If we let $\ell = \overline{q_1 q_2}$ and $\Lambda = \langle q_1, q_2, q_3 \rangle$ then it follows from computations with Macaulay 2 that we have:

(a) $C_1 = \pi_q(C) \subset \mathbb{P}^{12}$ is a smooth rational curve with property $\mathbf{N}_{2,3}$;
(b) $C_2 = \pi_\ell(C) \subset \mathbb{P}^{11}$ is a smooth rational curve with property $\mathbf{N}_{2,3}$;
(c) $C_3 = \pi_\Lambda(C) \subset \mathbb{P}^{10}$ is a smooth rational curve with property $\mathbf{N}_{2,2}$.

Moreover, if we consider a projection of $C_3 = \pi_\Lambda(C)$ from a point $p \in S^5(C) \setminus S^4(C)$ then $C_4 = \pi_p(C_3) \subset \mathbb{P}^9$ satisfies property $\mathbf{N}_{3,1}$, which fails to be property $\mathbf{N}_{2,1}$. By Proposition 4.1, we know that curves C_1, C_2, C_3 and C_4 are all m-normal for all $m \geqslant 2$ and 3-regular (see Tables 2 and 3).

On the other hand, for a point $q \in \mathrm{Sec}(X) \cup \mathrm{Tan}(X)$ we can also consider a birational projection and syzygies of the projected varieties. To begin with, let us explain the basic situation and information on the partial elimination ideals under outer projections. For $q = (1, 0, \ldots, 0, 0) \notin X$, consider an outer projection $\pi_q : X \to Y_1 \subset \mathbb{P}^{n-1} = \mathrm{Proj}(S_1)$, $S_1 = k[x_1, x_2, \ldots, x_n]$. Suppose the ideal I define X scheme-theoretically. For the degree lexicographic order, if $f \in I$ has leading term $\mathrm{in}(f) = x_0^{d_0} \cdots x_n^{d_n}$, we set $d_0(f) = d_0$, the leading power of x_0 in f. Then it is well known that

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$K_0(I) = \bigoplus_{m \geqslant 0}\{f \in I_m \mid d_0(f) = 0\} = I \cap S_1$ and defines Y_1 scheme-theoretically. More generally, let us give the definition and basic properties of partial elimination ideals, which was introduced by M. Green in [9].

Definition 4.5. (See [9].) Let $I \subset R$ be a homogeneous ideal and let

$$\tilde{K}_i(I) = \bigoplus_{m \geqslant 0}\bigl\{f \in I_m \bigm| d_0(f) \leqslant i\bigr\}.$$

If $f \in \tilde{K}_i(I)$, we may write f uniquely as $x_0^i \bar{f} + g$ where $d_0(g) < i$. Now we define $K_i(I)$ by the image of $\tilde{K}_i(I)$ in S_1 under the map $f \mapsto \bar{f}$ and we call $K_i(I)$ the i-th partial elimination ideal of I. Note that $\tilde{K}_i(I)$ and $K_i(I)$ are graded S_1-modules.

Proposition 4.6. *(See [9].) Set theoretically, the i-th partial elimination ideal $K_i(I)$ is the ideal of $Z_i = \{q \in Y_1 \mid \mathrm{mult}_q(Y_1) \geqslant i+1\}$ for every $i \geqslant 1$.*

Lemma 4.7. *Let $X \subset \mathbb{P}^n$ be a reduced non-degenerate projective variety satisfying property $\mathbf{N}_{2,p}$, $p \geqslant 2$, scheme-theoretically. Consider a projection $\pi_q : X \to Y_1 \subset \mathbb{P}^{n-1}$ where $q \notin X$. Let*

$$\Sigma_q(X) := \bigl\{x \in X \bigm| {\pi_q}^{-1}\bigl(\pi_q(x)\bigr) \text{ has length} \geqslant 2\bigr\}$$

be the nonempty secant locus of one-point projection. Then,

(a) *$\Sigma_q(X)$ is a quadric hypersurface in a linear subspace L and $q \in L$;*
(b) *$\pi_q(\Sigma_q(X)) = Z_1$ is a linear space which is the support of cokernel of $\mathcal{O}_{Y_1} \hookrightarrow \pi_{q*}(\mathcal{O}_X)$;*
(c) *for a point $q \in \mathrm{Sec}(X) \setminus \mathrm{Tan}(X) \cup X$, $\Sigma_q(X) = \{$two distinct points$\}$.*

Proof. Since X satisfies $\mathbf{N}_{2,p}$, $p \geqslant 2$, there is no 4-secant 2-plane to X by Theorem 3.10(b). Let $Z_1 := \{y \in Y_1 \mid {\pi_q}^{-1}(y)$ has length $\geqslant 2\}$ and choose two points y_1, y_2 in Z_1. Consider the line $\ell = \overline{y_1, y_2}$ in $\mathbb{P}^{n-1}$. If $\langle y_1, y_2\rangle \cap Y_1$ is finite, then we have 4-secant plane $\langle q, y_1, y_2\rangle$ which is a contradiction. So, $\mathrm{Sec}(Z_1) = Z_1$ and finally, we conclude that Z_1 is a linear space. Since $\pi_q : \Sigma_q(X) \twoheadrightarrow Z_1 \subset Y_1$ is a 2:1 morphism, $\Sigma_q(X)$ is a quadric hypersurface in $L = \langle Z_1, q\rangle$. For a proof of (c), if $\dim \Sigma_q(X)$ is positive, then clearly, $q \in \mathrm{Tan}\, \Sigma_q(X) \subset \mathrm{Tan}(X)$. □

As shown in Lemma 4.7, the fact that Z_1 is a linear space is crucial in the proof of the following theorem.

Theorem 4.8 *(Birational projections for the case of $\mathbf{N}_{2,p}$). Let $X \subset \mathbb{P}^n$ be a reduced non-degenerate projective variety satisfying property $\mathbf{N}_{2,p}$ scheme-theoretically for $p \geqslant 2$. Consider a projection $\pi_q : X \to Y_1 \subset \mathbb{P}^{n-1}$ where $q \in \mathrm{Sec}(X) \cup \mathrm{Tan}(X) \setminus X$. Then we have the following*:

(a) $H^1_*(\mathcal{I}_{X/\mathbb{P}^n}) = H^1_*(\mathcal{I}_{Y_1})$. *Thus, Y_1 is m-normal if and only if X is m-normal for all $m \geqslant 1$, and* $\mathrm{reg}(Y_1) \leqslant \max\{\mathrm{reg}(X), \mathrm{reg}(\mathcal{O}_{Y_1}) + 1\}$.
(b) *Y_1 is cut out by at most cubic hypersurfaces and satisfies property $N_{3,p-1}$.*

Proof. We may assume that $q = (1, 0, \dots, 0) \in \mathrm{Sec}(X) \cup \mathrm{Tan}(X) \setminus X$. Let $R = k[x_0, x_1, \dots, x_n]$ be a coordinate ring of $\mathbb{P}^n$, $S_1 = k[x_1, x_2, \dots, x_n]$ be a coordinate ring of $\mathbb{P}^{n-1}$. Let the ideal I define X with the condition $\mathbf{N}_{2,p}$ scheme-theoretically. Then, it is easily checked that $K_0(I)$ also defines Y_1 scheme-theoretically. By Theorem 3.6, we have the minimal free resolution of R/I as a graded S_1-module:

$$\cdots \to S_1(-p)^{\oplus \beta_{p-1,1}} \to \cdots \to S_1(-2)^{\oplus \beta_{1,1}} \xrightarrow{\varphi_1} S_1 \oplus S_1(-1) \xrightarrow{\varphi_0} R/I \to 0.$$

Furthermore, we have the following diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K_0(I) & \to & S_1 & \to & S_1/K_0(I) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow \alpha & & \\ 0 & \to & \tilde{K}_1(I) & \to & S_1 \oplus S_1(-1) & \xrightarrow{\varphi_0} & R/I & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K_1(I)(-1) & \to & S_1(-1) & \to & \operatorname{coker} \alpha & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Note that $\varphi_0(f, g) = f + g \cdot x_0$ and thus, $K_1(I)$ is the first partial elimination ideal of I associated to the projection π_q. Since $\tilde{K}_1(I)$ has the following minimal free resolution as a graded S_1-module:

$$\cdots \to S_1(-p)^{\oplus \beta_{p-1,1}} \xrightarrow{\varphi_{p-1}} \cdots \to S_1(-2)^{\oplus \beta_{1,1}} \xrightarrow{\varphi_1} \tilde{K}_1(I) \to 0,$$

we know that $K_1(I)$ is generated by linear forms and

$$\operatorname{reg}\bigl(K_1(I)(-1)\bigr) = 2, \qquad \operatorname{coker} \alpha = S_1/K_1(I)(-1).$$

Moreover, by usual Tor-computations, Y_1 satisfies property $N_{3,p-1}$.

On the other hand, consider the following exact sequence

$$0 \to \mathcal{O}_{Y_1} \xrightarrow{\tilde{\alpha}} \pi_{q_*}(\mathcal{O}_X) \to \operatorname{coker} \tilde{\alpha} \to 0. \tag{4.4}$$

By Lemma 4.7, since $\operatorname{coker} \tilde{\alpha}$ has the support Z_1 which is a linear space in $\mathbb{P}^{n-1}$ and $\pi_q : \Sigma_q(X) \twoheadrightarrow Z_1$ is 2:1, we have

$$\pi_{q_*}(\mathcal{O}_X)|_{Z_1} = \mathcal{O}_{Z_1} \oplus \mathcal{O}_{Z_1}(-1) \quad \text{and} \quad \operatorname{coker} \tilde{\alpha} = \mathcal{O}_{Z_1}(-1).$$

Therefore, $H^0_*(\operatorname{coker} \alpha) = S_1/I_{Z_1}(-1)$. Then, by taking global sections from the above sequence (4.4), we have the following commutative diagram as S_1-modules with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & S_1/I_{Y_1} & \to & \bigoplus_{m=0}^{\infty} H^0(\mathcal{O}_{Y_1}(m)) & \to & H^1_*(\mathcal{J}_{Y_1}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & R/I_X & \to & \bigoplus_{m=0}^{\infty} H^0(\mathcal{O}_X(m)) & \to & H^1_*(\mathcal{J}_{X/\mathbb{P}^n}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & S_1/I_{Z_1}(-1) & = & S_1/I_{Z_1}(-1) & & 0 & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

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The reason why the left column is exact is that $K_1(I) = I_{Z_1}$ for any ideal I defining X scheme-theoretically. Thus, $H^1_*(\mathcal{I}_{Y_1}) \simeq H^1_*(\mathcal{I}_{X/\mathbb{P}^n})$ and so, X is m-normal if and only if Y_1 is m-normal. So we complete the proof of (a) and (b). □

Remark 4.9. For a complete embedding of $X \subset \mathbb{P}(H^0(\mathcal{O}(1)))$ satisfying property $\mathbf{N}_p$, Lemma 4.7 and Theorem 4.8 was proved in [19] with different method. However, the point is that we can also deal with non-complete embeddings of X in $\mathbb{P}^n$ satisfying property $\mathbf{N}_{2,p}$ by virtue of the graded mapping cone theorem without using Green–Lazarsfeld's vector bundle technique on restricted Euler sequence on X.

So far, we proved the *uniform* properties of higher normality and syzygies of projections when a variety X satisfies property $\mathbf{N}_{2,p}$, $p \geqslant 2$, scheme-theoretically. On the other hand, by moving the center, we have a lot of interesting varieties with different structures in geometry and syzygies as the following example shows:

Example 4.10. Let $C = \nu_d(\mathbb{P}^1)$ in $\mathbb{P}^d$ be a rational normal curve. Consider the following filtration on the ℓ-th higher secant variety $S^\ell(C)$ of dimension $\min\{2\ell-1, d\}$:

$$C \subsetneq \mathrm{Sec}(C) = S^2(C) \subsetneq S^3(C) \subsetneq \cdots \subsetneq S^{\lfloor \frac{d}{2} \rfloor}(C) \subsetneq S^{\lfloor \frac{d}{2} \rfloor + 1}(C) = \mathbb{P}^d.$$

Then we have (see [6,18]):

(a) $\overline{\pi_q(C)} \subset \mathbb{P}^{d-1}$ satisfies property $\mathbf{N}_{2,d-2}$ for $q \in C$,
(b) $\pi_q(C) \subset \mathbb{P}^{d-1}$ is a rational curve with one node satisfying property $\mathbf{N}_{2,d-3}$ for $q \in \mathrm{Sec}(C) \setminus C$,
(c) $\pi_q(C) \subset \mathbb{P}^{d-1}$ satisfies property $\mathbf{N}_{2,\ell-3}$ for $q \in S^\ell(C) \setminus S^{\ell-1}(C)$.

Note that all projected curves are m-normal for all $m \geqslant 2$ and thus 3-regular.

Note that for varieties of next to minimal degree, the arithmetic properties of projected varieties by moving the center were investigated in [4] for the first time. The following proposition show that the number of quadrics and the depth of projected varieties depend on the position of the center. For a complete embedding $X \subset \mathbb{P}(H^0(\mathcal{L}))$, the same result is given in [19]. Let $s = \dim \Sigma_q(X)$ and if the secant locus $\Sigma_q(X) = \emptyset$, then $s = -1$.

Proposition 4.11 *(Moving the center: quadrics and depths). Let $X \subset \mathbb{P}^n$ be a reduced non-degenerate projective variety satisfying property $\mathbf{N}_{2,p}$, $p \geqslant 2$. Consider the projection $\pi_q : X \to Y_1 \subset \mathbb{P}^{n-1}$ where $q \notin X$. Let $\Sigma_q(X)$ be the secant locus of the projection π_q. Then the following hold*:

(a) $h^0(\mathbb{P}^{n-1}, \mathcal{I}_{Y_1}(2)) = h^0(\mathbb{P}^n, \mathcal{I}_{X/\mathbb{P}^n}(2)) - n + s$,
(b) $\mathrm{depth}(Y_1) = \min\{\mathrm{depth}(X), s+2\}$ *under the condition that*

$$H^i\big(\mathcal{O}_X(j)\big) = 0, \quad \forall j \leqslant -i,\ 1 \leqslant i \leqslant \dim(X).$$

Proof. (a) First, for the isomorphic projection case, we obtained the following fact from the commutative diagram (3.2):

$$\mathrm{reg}(\mathcal{N}) = 3, \qquad h^1\big(\mathcal{I}_{Y_1}(\ell)\big) = h^2\big(\mathcal{K}(\ell)\big) = h^1\big(\mathcal{L}(\ell)\big) = h^1\big(\mathcal{I}_{X/\mathbb{P}^n}(\ell)\big) \quad \text{for } \ell \geqslant 2.$$

From the basic equalities

$$\begin{cases} h^0(\mathcal{J}_{X/\mathbb{P}^n}(2)) + h^0(\mathcal{O}_X(2)) = \binom{n+2}{2} + h^1(\mathcal{J}_{X/\mathbb{P}^n}(2)) \quad \text{and} \\ h^0(\mathcal{J}_{Y_1}(2)) + h^0(\mathcal{O}_{Y_1}(2)) = \binom{n+1}{2} + h^1(\mathcal{J}_{Y_1}(2)), \end{cases}$$

we get $h^0(\mathcal{J}_{Y_1}(2)) = h^0(\mathcal{J}_{X/\mathbb{P}^n}(2)) - n - 1$. In the case of finite birational projections, the secant locus $\Sigma_q(X)$ is not empty and $\pi_q(\Sigma_q(X)) = Z_1 = \mathbb{P}^s$. In the proof of Theorem 4.8, we got the following fact:

$$H^1_*(\mathbb{P}^{n-1}, \mathcal{J}_{Y_1}) \simeq H^1_*(\mathbb{P}^n, \mathcal{J}_{X/\mathbb{P}^n}), \quad 0 \to S_1/I_{Y_1} \to R/I_X \to S_1/I_{Z_1}(-1) \to 0.$$

Therefore, by simple computation we have $h^0(\mathcal{J}_{Y_1}(2)) = h^0(\mathcal{J}_{X/\mathbb{P}^n}(2)) - n + s$. For a proof of (b), consider the following exact sequence

$$0 \to \mathcal{O}_{Y_1} \xrightarrow{\alpha} \pi_{q*}(\mathcal{O}_X) \to \mathcal{O}_{Z_1}(-1) \to 0. \tag{4.5}$$

If $s = -1$, then $Z_1 = \emptyset$ and $H^1(\mathcal{J}_{Y_1}(1)) \neq 0$. So, $\operatorname{depth}(Y_1) = 1$. Suppose $\operatorname{depth}(X) = 1$, $s \geqslant 0$. Then, by Theorem 4.8(a), $H^1_*(\mathcal{J}_{Y_1}) \simeq H^1_*(\mathcal{J}_{X/\mathbb{P}^n}) \neq 0$ and $\operatorname{depth}(Y_1) = 1$.

Now, suppose $\operatorname{depth}(X) \geqslant 2$, $s \geqslant 0$. When $s = 0$, then Z_1 is one point. Therefore we have

$$0 \to H^0(\mathcal{O}_{Z_1}(\ell - 1)) \to H^1(\mathcal{O}_{Y_1}(\ell)) \to H^1(\mathcal{O}_X(\ell)) \to 0$$

and $0 \neq H^0(\mathcal{O}_{Z_1}(\ell - 1)) \subset H^1(\mathcal{O}_{Y_1}(\ell))$ for all $\ell \leqslant 0$. So, $\operatorname{depth}(Y_1) = 2 = \min\{\operatorname{depth}(X), s + 2\}$. For $s \geqslant 1$ and $\operatorname{depth}(X) \geqslant s + 2$, we obtain the sequence

$$0 \to H^s_*(\mathcal{O}_{Z_1}(-1)) \to H^{s+1}_*(\mathcal{O}_{Y_1}) \to H^{s+1}_*(\mathcal{O}_X) \to 0,$$

$H^i_*(\mathcal{O}_{Y_1}) \simeq H^i_*(\mathcal{O}_X) = 0$ for all $1 \leqslant i \leqslant s$ and $H^s_*(\mathcal{O}_{Z_1}(-1)) \neq 0$. Thus, $\operatorname{depth}(Y_1) = s + 2 = \min\{\operatorname{depth}(X), s + 2\}$. Finally, in the case of $2 \leqslant \operatorname{depth}(X) \leqslant s + 1$, $s \geqslant 1$, under the assumption that $H^i(\mathcal{O}_X(j)) = 0$, $\forall j \leqslant -i$, we can easily check that $\operatorname{depth}(Y_1) = \operatorname{depth}(X) = \min\{\operatorname{depth}(X), s + 2\}$. □

Example 4.12. (A non-normal variety with non-vanishing cohomology.) We give some examples related to our proposition. For a projective normal variety X, we define

$$\delta(X) := \min\{\operatorname{depth} \mathcal{O}_{X,x} \mid x \text{ is a closed point}\}.$$

Then $H^i(\mathcal{O}_X(\ell)) = 0$ for all $\ell \ll 0$ and $i < \delta(X)$ by vanishing theorem of Enriques–Severi–Zariski–Serre. In the proof of Proposition 4.11, for $s = 0$ we have an interesting example Y_1 such that Y_1 has only one isolated non-normal singular point and in fact, $H^1(\mathcal{O}_{Y_1}(\ell)) \neq 0$ for all $\ell \leqslant 0$. As examples, suppose that a projective variety X has no lines and plane conics in $\mathbb{P}^n$ with the condition $\mathbf{N}_{2,p}$, $p \geqslant 2$ (e.g., the Veronese variety $\upsilon_d(\mathbb{P}^n)$, $d \geqslant 3$, or its isomorphic projections). Then, the singular locus of any simple projection is either empty or only one point because the secant locus is a quadric hypersurface in some linear subspace.

Example 4.13 *(Macaulay 2 in [13]).* Consider a rational normal 3-fold scroll $S_{1,1,4} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(4))$ in $\mathbb{P}^8$. From the Eagon–Northcott complex, we obtain the minimal free resolution of $S_{1,1,4}$ as follows:

$$0 \to R(-6)^5 \to R(-5)^{24} \to R(-4)^{45} \to R(-3)^{40} \to R(-2)^{15} \to I_{S_{1,1,4}} \to 0.$$

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As the center of projection $q \in \mathbb{P}^8$ moves toward $S_{1,1,4}$, we will see that the number of cubic generators decreases and the number of quadric generators increases in the following:

(a) Let $q \in \mathbb{P}^8 \setminus \mathrm{Sec}(S_{1,1,4})$. The isomorphic projection $Y \subset \mathbb{P}^7$ has the following resolution with $\mathrm{depth}(Y) = 1$

$$\cdots \to S(-4)^{40} \oplus S(-3)^8 \to S(-3)^{10} \oplus S(-2)^6 \to I_Y \to 0.$$

(b) Suppose $q \in \mathrm{Sec}(S_{1,1,4}) \setminus \mathrm{Tan}(S_{1,1,4})$. Then $s = 0$ and I_Y has the following resolution with $\mathrm{depth}(Y) = 2$:

$$\cdots \to S(-4)^{19} \oplus S(-3)^8 \to S(-3)^3 \oplus S(-2)^7 \to I_Y \to 0.$$

(c) For a point $q \in \mathrm{Tan}(S_{1,1,4}) \setminus S_{1,1,4}$, Y has different two types of resolutions: First, consider a linear span $\mathbb{P}^3 = \langle \ell_1, F\rangle$ where ℓ_1 is a line embedded by $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)) \hookrightarrow \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(4)) \subset \mathbb{P}^8$ and F be any fiber of the morphism $\varphi : S_{1,1,4} \to \mathbb{P}^1$. For a general point $q \in \mathbb{P}^3 = \langle \ell_1, F\rangle$, Y has a singular locus $\mathbb{P}^1$, only one cubic generator and the following minimal resolution of length 5:

$$\cdots \to S(-4)^4 \oplus S(-3)^{12} \to S(-3) \oplus S(-2)^8 \to I_Y \to 0.$$

Second, take a general point $q \in \mathbb{P}^3$ where the quadric hypersurface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \subset \mathbb{P}^3$ is a subvariety of $S_{1,1,4} \subset \mathbb{P}^8$. Then the projected variety Y clearly has the singular locus $\mathbb{P}^2$, $\mathrm{depth}(Y) = 4$ and the following resolution:

$$0 \to S(-6) \to S(-4)^9 \to S(-3)^{16} \to S(-2)^9 \to I_Y \to 0.$$

(d) For a general point $q \in S_{1,1,4}$, an inner projection Y is a smooth 3-fold scroll of type $S_{1,1,3}$ and has the following resolution:

$$0 \to S(-5)^4 \to S(-4)^{15} \to S(-3)^{20} \to S(-2)^{10} \to I_Y \to 0.$$

From [8], we know that if X satisfies the condition $\mathbf{N}_{2,p}$, $p = \mathrm{codim}(X, \mathbb{P}^n)$ then there are no outer projections $\pi_q(X)$ satisfying property $\mathbf{N}_{2,p-1}$. However, we proved that $\pi_q(X)$ satisfies at least property $\mathbf{N}_{3,p-1}$ by Theorems 4.2 and 4.8.

In contrast with the outer projections, it is known that for a non-degenerate smooth variety X in $\mathbb{P}(H^0(\mathcal{L}))$ with property $\mathbf{N}_p$, the inner projection $\overline{\pi_q(X)}$ satisfies $\mathbf{N}_{p-1}$ for a point $q \in X \setminus \mathrm{Trisec}(X)$ where $\mathrm{Trisec}(X)$ is the union of all proper trisecant lines or lines in X (see [5] for details).

We close the paper with the following question.

Question 4.14. Let X be a projective reduced scheme in $\mathbb{P}^n$ satisfying property $\mathbf{N}_{2,p}$, $p \geqslant 1$. Consider the inner projection $\overline{\pi_L(X \setminus L)}$ from a linear subvariety $L \subset X$ in $\mathbb{P}^{n-t-1}$ where $\dim L = t < p$. Is it true that $\overline{\pi_L(X \setminus L)}$ satisfies $\mathbf{N}_{2,p-t-1}$?

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