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Sharp bounds for higher linear syzygies and classifications of projective varieties

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Abstract In the present paper, we consider upper bounds of higher linear syzygies i.e. graded Betti numbers in the first linear strand of the minimal free resolutions of projective varieties in arbitrary characteristic. For this purpose, we first remind 'Partial Elimination Ideals (PEIs)' theory and introduce a new framework in which one can study the syzygies of embedded projective varieties well using PEIs theory and the reduction method via inner projections. Next we establish fundamental inequalities which govern the relations between the graded Betti numbers in the first linear strand of an algebraic set X and those of its inner projection X_q. Using these results, we obtain some natural sharp upper bounds for higher linear syzygies of any nondegenerate projective variety in terms of the codimension with respect to its own embedding and classify what the extremal case and the next-to-extremal case are. This is a generalization of Castelnuovo and Fano's results on the number of quadrics containing a given variety and another characterization of varieties of minimal degree and del Pezzo varieties from the viewpoint of 'syzygies'. Note that our method could also be applied

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to get similar results for more general categories (e.g. connected in codimension one algebraic sets).

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Contents

1 Introduction

Let $X \subset \mathbb{P}^N$ be any nondegenerate variety (i.e. irreducible and reduced closed subscheme) of dimension n and of $\operatorname{codim}(X, \mathbb{P}^N) = e$ over an algebraically closed field k of arbitrary characteristic. Let $R := k[x_0, \dots, x_N]$ be the coordinate ring of $\mathbb{P}^N$, I_X be the (saturated) defining ideal of X, and $R_X := R/I_X$ also be the coordinate ring of X. The *graded Betti numbers* of X is defined by

$$\beta_{p,q}(X) := \dim_k \operatorname{Tor}_p^R(R_X, k)_{p+q} \tag{1.1}$$

and the *Betti table* of X, $\mathbb{B}(X)$ consists of these graded Betti numbers of X. This table is usually considered to represent the type of the minimal free resolution of R_X. For instance, $\beta_{1,1}$ corresponds to the number of (independent) *quadrics* containing X and so does $\beta_{2,1}$ to the *linear* syzygies on them. We may present $\mathbb{B}(X)$ typically as follows (Fig. 1):

		0	1	$\cdots$	$a(X)$	$a+1$	$\cdots$	$b-1$	$b(X)$	$\cdots$	p	$\cdots$
	0	1	–	$\cdots$	–	–	$\cdots$	–	–	$\cdots$	–	$\cdots$
	1	–	$\beta_{1,1}$	$\cdots$	$\beta_{a,1}$	$\beta_{a+1,1}$	$\cdots$	$\beta_{b-1,1}$	–	$\cdots$	–	$\cdots$
$\mathbb{B}(X)$	2	–	–	$\cdots$	–	$\beta_{a+1,2}$	$\cdots$	$\beta_{b-1,2}$	$\beta_{b,2}$	$\cdots$	$\beta_{p,2}$	$\cdots$
	$\vdots$	–	–	$\cdots$	–	$\ddots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\ddots$	$\cdots$
	q	–	–	$\cdots$	–	$\beta_{a+1,q}$	$\cdots$	$\beta_{b-1,q}$	$\beta_{b,q}$	$\cdots$	$\beta_{p,q}$	$\cdots$
	$\vdots$	–	–	$\cdots$	–	$\ddots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\ddots$	$\cdots$

Fig. 1 Betti table of a nondegenerate variety X in $\mathbb{P}^N$. We denote *zero* by –. By two pivotal places, determined by $a = a(X)$, $b = b(X) \geq 0$, we could characterize the *first* linear strand of this resolution

Since Green [11] showed through his foundational paper several results which imply some of strong connections between geometry of projective varieties and their syzygies, there have been many problems and conjectures concerning shapes of $\mathbb{B}(X)$ and structures on some or all of $\{\beta_{p,q}$'s$\}$. In this paper we will consider some interesting problems based on the first linear strand of Betti tables of projective varieties (or schemes) particularly.

By convention, we call the subcomplex (or the corresponding part of the table) represented by Betti numbers $\beta_{1,1}, \ldots, \beta_{b-1,1}$ in the second row of $\mathbb{B}(X)$ *the (first) linear strand of* $\mathbb{B}(X)$. Following the notations in [8], we also denote the (homological) index to which the resolution admits *only* linear syzygies by $a(X)$ and the first index from which there exists *no more* linear syzygy by $b(X)$. Then, the linear strand of the minimal free resolution of R_X can be characterized by these invariants $a(X)$ and $b(X)$.

Classically, there have been well-known results on the number of quadratic equations containing X, i.e. $\beta_{1,1}(X)$ (see [3,9] and also [15,17,22] for modern references). Before stating them, let us make our terminology clear. Say $d = \deg(X)$, *degree* of X. One can say that X is a *variety of minimal degree* (abbr. VMD) if $d = e + 1$. Here we call X *of next-to-minimal degree* when $d = e + 2$. Furthermore, throughout this paper, we call X a *del Pezzo* variety if X is *arithmetically Cohen–Macaulay* (abbr. ACM) and of next-to-minimal degree. Then, the theorems say

(a) [3] *Let* $X^n \subset \mathbb{P}^{n+e}$ *be a nondegenerate variety of codimension* e,

$$\beta_{1,1}(X) \leq \binom{e+1}{2}$$

and the "=" *holds if and only if* X *is a variety of minimal degree.*

(b) [9] *Unless* X *is a variety of minimal degree,*

$$\beta_{1,1}(X) \leq \binom{e+1}{2} - 1$$

and the "=" *holds if and only if* X *is a del Pezzo variety.*

But when we move on higher p's, it is not so feasible to handle *higher linear syzygies* (i.e. $\beta_{p,1}(X)'s$) directly as to manipulate them in case of p being very low (e.g. considering generators, their relations, and so on). In this paper we introduce a useful way to treat higher linear syzygies in a quite effective manner, that is

Projecting higher linear syzygies of X to those of its projected image X_{q}.

Especially, we will focus on *inner* projection process (i.e. a projection taking its center from inside of X) here (see Remark 2.12 for details). We denote the Zariski closure of the image of $\pi_{\mathrm{q}} : X \backslash \{\mathrm{q}\} \to \mathbb{P}^{N-1}$ by X_{q}. Note that this inner projection process often transplants much of favorable structures on syzygies and Betti table into its projected image, in contrast with outer projection (e.g. see [15]).

Main results Now we present our main results. First, we are giving a very useful inequality through which we can explain the relations between the Betti numbers in the first linear strand of X and X_{q} essentially.

Theorem 1.1 (Fundamental Inequality in the 1st linear strand) *Let $X^n \subset \mathbb{P}^{n+e}$ be a nondegenerate variety of codimention e, $\mathrm{q} \in X$ be any closed point of X. For any $p \geq 1$, the following holds*

$$\beta_{p,1}(X) \leq \beta_{p,1}(X_{\mathrm{q}}) + \beta_{p-1,1}(X_{\mathrm{q}}) + \binom{e}{p} \tag{1.2}$$

and equality holds if $1 \leq p \leq a(X)$ and q is a smooth point of X.

Here the fundamental inequality (1.2) is stated in a simplified form. We will present and prove a more strengthened version of Theorem 1.1 in Sect. 3 (see Theorem 3.1) for the sake of future use.

As a direct consequence of Theorem 1.1, we can obtain optimal upper bounds on $\beta_{p,1}$ of *every* variety for more higher p in the linear strand (note that Castelnuovo's bound could be easily recovered by the inequality of $p = 1$).

Theorem 1.2 *Let $X^n \subset \mathbb{P}^{n+e}$ be any nondegenerate variety of codim $e \geq 1$. Then,*

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1} \quad for\ all\ \ p \geq 0 \tag{1.3}$$

Note that $p\binom{e+1}{p+1}$ is the p-th Betti number of varieties of minimal degree (VMD) of codimension e. We also remark that it could be possible to have $\beta_{p,1}$'s larger than these numbers in case of more general algebraic set (see Example 5.4).

We can also add new characterizations to classical ones of VMD's as follows:

Theorem 1.3 *Let $X^n \subset \mathbb{P}^{n+e}$ be a nondegenerate variety with $e \geq 1$. Then, the following are all equivalent:*

(a) *X^n is a variety of minimal degree in $\mathbb{P}^{n+e}$;*
(b) *$\mathcal{I}_X$ is 2-regular;*
(c) *$a(X) \geq e$;*
(d) *$h^0(\mathbb{P}^{n+e}, \mathcal{I}_X(2)) = \binom{e+1}{2}$;*
(e) *one of $\beta_{p,1}(X)$'s achieves the maximum for some $1 \leq p \leq e$;*
(f) *all the $\beta_{p,1}(X)$'s achieve the maxima.*

And we continue to give the next-to-extremal bounds on $\beta_{p,1}$'s as below:

Theorem 1.4 *Let $X^n \subset \mathbb{P}^{n+e}$ be any nondegenerate variety of codim $e \geq 1$. Unless X is a variety of minimal degree, then we have*

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1} - \binom{e}{p-1} \quad (1 \leq p \leq e). \tag{1.4}$$

Note that those equalities hold if and only if X is del Pezzo and this fact newly characterizes del Pezzo varieties in the viewpoint of higher linear syzygies (see Theorems 4.1 and 4.3) generalizing Fano's classical result. In larger categories, there are some cases which have their Betti numbers between these extremal and next-to-extremal bounds (see Example 5.5 and Question 5.6).

Organization of the paper For this purpose, we remind partial elimination ideals (PEIs) theory, give account for its relevance to the theory of projections of projective schemes briefly, and introduce a new framework in which one can study syzygies of embedded projective subschemes using PEIs theory and reduction method via inner projections in Sect. 2. In Sect. 3 we give proofs of our main results and add a remark which give some inspiration on how to carry out the computations of Betti numbers of projective varieties using projections. In Sect. 4 we treat *next-to-extremal case* which is a natural generalization of Fano's classical theorem as our previous theorems did for Castelnuovo's. Finally, we give examples and questions to improve our results into more general categories and more refined bounds in Sect. 5.

Notations and conventions We are working on the following conventions:

- (Betti numbers) For any commutative ring A and a graded A-module M, we also define *graded Betti numbers of* M, $\beta^A_{p,q}(M)$ by $\dim_k \mathrm{Tor}^A_p(M,k)_{p+q}$. For a polynomial ring R and its homogeneous ideal I, we remind an easy fact

$$\mathrm{Tor}^R_p(R/I,k)_{p+q} = \mathrm{Tor}^R_{p-1}(I,k)_{p-1+q+1} \quad \text{for any } p \geq 1,\ q \geq 0$$

  so that $\beta^R_{p,q}(R/I) = \beta^R_{p-1,q+1}(I)$. We'll write $\beta_{p,q}(M)$ or $\beta_{p,q}$ instead of $\beta^R_{p,q}(M)$ where it leads no confusion and denote $\beta_{p,q}(R_X)$ simply by $\beta_{p,q}(X)$.
- (Property $\mathbf{N}_{d,p}$) For a homogeneous ideal $I \subset R$, we say that I *satisfies property* $\mathbf{N}_{d,p}$ if every $\beta_{i,j}(I) = 0$ for any $0 \leq i < p$ and any $j > d$ (see also [5, 15]). When $d = 2$ and $I = I_X$, the saturated defining ideal of a projectively normal embedding $X \subset \mathbb{P}^N$, this property $\mathbf{N}_{2,p}$ coincides with Green–Lazarsfeld's property $\mathbf{N}_p$.
- (Tor modules) From now on, we often abbreviate $\mathrm{Tor}^A_p(M,k)_{p+q}$ as $\mathrm{T}^A_{p,q}(M)$ for any commutative ring A and a graded A-module M.
- (Arithmetic depth) When we refer the *depth of* X, denoted by $\mathrm{depth}_R(X)$, we mean the arithmetic depth of X, i.e. $\mathrm{depth}_R(R_X)$.
- (Nondegeneracy) Throughout the paper, the *nondegenerate* condition on a scheme X defined by I just means that I has *no linear forms*.

2 Partial elimination ideals and its application

Green [12] introduced the notion of PEIs in his lecture note to study lexicographic generic initial ideals (gins) and subsequent works concerning lex-gins have been done by some authors (e.g. see [1,2,4]). In this section we will briefly review PEIs theory, develop further and try to investigate another application of it. We will also recall some basic facts which are essential for the remaining part of the paper throughout this section.

2.1 A brief review of partial elimination ideals

Let $S = k[x_1, \dots, x_N] \subset R = k[x_0, x_1 \dots, x_N]$ be two polynomial rings and I be a homogeneous ideal of R. For the degree lexicographic order, if $f \in I_m$ has leading term $\mathrm{in}(f) = x_0^{d_0} \dots x_N^{d_N}$, we set $d_0(f) = d_0$, the leading power of x_0 in f. Then we can give the definition of PEIs of I as follows:

Definition 2.1 Let $I \subset R$ be a homogeneous ideal and let us define

$$\widetilde{K}_i(I) := \left(\bigoplus_{a=0}^{i} S \cdot x_0^a \right) \cap I = \bigoplus_{m \geq 0} \left\{ f \in I_m \mid d_0(f) \leq i \right\}.$$

If $f \in \widetilde{K}_i(I)$, we may write uniquely $f = x_0^i \bar{f} + g$ where $d_0(g) < i$. Now we consider the ideal $K_i(I)$ in S generated by the image of $\widetilde{K}_i(I)$ under the map $f \mapsto \bar{f}$ and we call $K_i(I)$ the *i-th PEI of I with respect to x_0*. We define $\widetilde{K}_i(I)$ (so, also $K_i(I)$) as zero for any $i < 0$ by convention.

Observation 2.2 We could observe some properties of these ideals.

(a) (Finiteness) $\widetilde{K}_i(I)$ is always a finitely generated graded S-module (even though I and R/I might not be) and $K_i(I)$ is a homogeneous ideal of S. We also define $\widetilde{K}_\infty(I) := I$ as S-module and $\widetilde{K}_i(I)$ could be regarded as a finite S-module approximation of I.

(b) 0-th PEI $K_0(I)$ of I is equal to

$$\widetilde{K}_0(I) = S \cap I = \bigoplus_{m \geq 0} \left\{ f \in I_m \mid d_0(f) = 0 \right\},$$

the *complete* elimination ideal of I with respect to x_0.

(c) (Stabilization) Since $\widetilde{K}_i(I)$'s form a natural filtration of I with respect to x_0, they induce an ascending chain of $K_i(I)$'s such as:

$$(0) = \widetilde{K}_{-1}(I) \subset \widetilde{K}_0(I) \subset \widetilde{K}_1(I) \subset \cdots \subset \widetilde{K}_s(I) \subset \widetilde{K}_{s+1}(I) \subset \cdots \subset R$$

$$(0) = K_{-1}(I) \subset K_0(I) \subsetneq K_1(I) \subsetneq \cdots \subsetneq K_s(I) = K_{s+1}(I) = \cdots \subset S,$$

where the ascending chain of $K_i(I)$'s is always stabilized in finite steps. Let's define *the stabilization number* $s(I)$, and *the stabilized PEI* $K_\infty(I)$ as below:

$$s(I) := \min\{i \in \mathbb{N} | \; K_i(I) = K_{i+1}(I) = \cdots\}, \quad K_\infty(I) := K_s(I).$$

(d) (Exact sequences) For any $i \in \mathbb{Z}$, there are two short exact sequences of graded S-modules such as

$$0 \to \frac{\widetilde{K}_{i-1}(I)}{\widetilde{K}_h(I)} \xrightarrow{incl.} \frac{\widetilde{K}_i(I)}{\widetilde{K}_h(I)} \xrightarrow{f} K_i(I)(-i) \to 0 \tag{2.1}$$

for every $h \leq i-1$ and

$$0 \to \frac{\widetilde{K}_{i-1}(I)}{\widetilde{K}_{i-2}(I)}(-1) \xrightarrow{\times x_0} \frac{\widetilde{K}_i(I)}{\widetilde{K}_{i-1}(I)} \xrightarrow{g} \frac{K_i(I)}{K_{i-1}(I)}(-i) \to 0. \tag{2.2}$$

For convenience, we can similarly define a finitely generated graded S-module $\widetilde{Q}_i(I)$ and $Q_i(I)$ as follows:

$$\widetilde{Q}_i(I) := \left(\bigoplus_{a=0}^{i} S \cdot x_0^a\right) / \widetilde{K}_i(I) \quad \text{and} \quad Q_i(I) := S/K_i(I). \tag{2.3}$$

Note that $\widetilde{Q}_i(I)$ can be considered as a sort of finite approximation for $\widetilde{Q}_\infty(I) := R/I$ as S-module as $\widetilde{K}_i(I)$ does for I and $\widetilde{Q}_0(I) = Q_0(I)$ defines the image scheme by the elimination. We introduce a diagram in which all these notions fit into well.

Using the syzygies of $\widetilde{K}_i(I)$ (resp. of $\widetilde{Q}_i(I)$), we can approximate S-module syzygy structures of I or, more generally, $I/\widetilde{K}_h(I)$ (resp. of R/I).

Proposition 2.3 (Approximation of syzygies) *Let $I \subset R$ be any homogeneous ideal. For given any $p, q \geq 0$ and $h \in \mathbb{Z}$, we have*

(a) *For any $d \geq q$,* $\mathrm{Tor}_p^S(R/I, k)_{p+q} \simeq \mathrm{Tor}_p^S(\widetilde{Q}_d(I), k)_{p+q}$.
(b) *For any $d \geq q$,* $\mathrm{Tor}_p^S(I, k)_{p+q} \simeq \mathrm{Tor}_p^S(\widetilde{K}_d(I), k)_{p+q}$. *In general, for any $d \geq q$ and $d \geq h$ it holds that*

$$\mathrm{Tor}_p^S(I/\widetilde{K}_h(I), k)_{p+q} \simeq \mathrm{Tor}_p^S(\widetilde{K}_d(I)/\widetilde{K}_h(I), k)_{p+q}.$$

Moreover, if $K_\infty(I) \neq (1)$, these are also true for any $d \geq q-1$.

Proof We could prove (a) and (b) by almost same arguments. Let's try to prove (b). Note that $I = \widetilde{K}_\infty(I)$ as S-module. So, it is enough to show that for any $d \in \mathbb{Z}$ such that $d \geq q-1$,

$$\mathrm{Tor}_p^S(\widetilde{K}_{d'}(I), k)_{p+q} \simeq \mathrm{Tor}_p^S(\widetilde{K}_d(I), k)_{p+q} \quad \text{for any } d' \geq d \quad (*).$$

But, this directly comes from the iterated use of exact sequence (2.1), the first row in Fig. 2. Because, from the induced long exact sequence of (2.1) we have

$$\mathrm{T}^S_{p+1,q-d-2}(K_{d+1}(I)) \to \mathrm{T}^S_{p,q}(\widetilde{K}_d(I)) \simeq \mathrm{T}^S_{p,q}(\widetilde{K}_{d+1}(I)) \to \mathrm{T}^S_{p,q-d-1}(K_{d+1}(I)),$$

where the first and fourth terms vanish for any $d \geq q$ and furthermore for any $d \geq q-1$ in case of $K_{d+1}(I) \subset K_\infty(I) \neq (1)$ so that we obtain the middle isomorphism (recall our notation for $\mathrm{Tor}_p^A(M, k)_{p+q}$ as $\mathrm{T}^A_{p,q}(M)$ for any commutative ring A and a graded A-module M). Repeating this, we can prove $(*)$. For (a), we could prove in a similar way (use the third row exact sequence in Fig. 2). □

As a consequence, we get a simple, but frequently used lemma.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{K}_i(I)/\widetilde{K}_h(I) & \longrightarrow & \widetilde{K}_{i+1}(I)/\widetilde{K}_h(I) & \longrightarrow & K_{i+1}(I)(-i-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \left(\bigoplus_{a=0}^{i} S\cdot x_0^a\right)/\widetilde{K}_h(I) & \longrightarrow & \left(\bigoplus_{a=0}^{i+1} S\cdot x_0^a\right)/\widetilde{K}_h(I) & \longrightarrow & S(-i-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{Q}_i(I) & \longrightarrow & \widetilde{Q}_{i+1}(I) & \longrightarrow & Q_{i+1}(I)(-i-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Fig. 2 PEIs diagram. Here $i \geq 0, h$ are integers and $i \geq h$. The $K_i(I)$'s (resp. their quotient $Q_i(I)$'s) measures the growth of $\widetilde{K}_i(I)$ (resp. of $\widetilde{Q}_i(I)$) as i getting large

Lemma 2.4 *Let $I \subset R$ be any homogeneous ideal such that $K_\infty(I) \neq (1)$.*

$$\mathrm{Tor}_p^S(I/\widetilde{K}_h(I), k)_{p+q} = 0 \quad \textit{for every } p \geq 0 \textit{ and any } q \leq h+1.$$

Proof It is straightforward from Proposition 2.3 (b). □

2.2 Applications to projection mappings

Geometrically, PEIs are closely related to projection mappings of schemes by nature. Consider our scheme $X \subset \mathbb{P}^N = \mathrm{Proj}(R)$ defined by a homogeneous ideal $I \subset R$, take a closed point $\mathfrak{q}$ of $\mathbb{P}^N$ as centre of our projection. Let $X_\mathfrak{q}$ be its image of the projection map $\pi_\mathfrak{q} : X\backslash\{\mathfrak{q}\} \to \mathbb{P}^{N-1} = \mathrm{Proj}(S)$ if $\mathfrak{q} \notin X$ and be the Zariski closure of the image if $\mathfrak{q} \in X$.

We define the *PEIs of I with respect to* $\mathfrak{q}$ (denoted by $K_i(\mathfrak{q}, I)$) by the PEI $K_i(I)$'s of I with respect to x_0 assuming $\mathfrak{q} = (1 : 0 : \cdots : 0)$ by a suitable linear change of coordinates. This definition makes sense, because we may define *coordinate-free* version of PEIs with no much difficulty (e.g. [16]) and could show that taking these PEIs commutes with coordinate transformations. We often denote $K_i(I)$ and $s(\mathfrak{q})$ (or just s) simply instead of $K_i(\mathfrak{q}, I)$ and $s(\mathfrak{q}, I)$ where no confusion occurs.

Now, let's regard the PEIs of I with respect to $\mathfrak{q}$. First of all, the 0-th PEI $K_0(I) = I \cap S$ gives a natural scheme structure on $X_\mathfrak{q}$ itself. Further, from higher partial elimination ideals we could extract more information on the given projection $\pi_\mathfrak{q}$. For *outer* projection case (i.e. $\mathfrak{q} \notin X$), they turned out to be related *multiple loci* of $\pi_\mathfrak{q}$ (see [4,12,13]). Here, we introduce an extended version including inner projection case also (see [14]).

Proposition 2.5 *Let I be a homogeneous ideal of R defining $X \subset \mathbb{P}^N$ as a scheme, $\mathrm{q} \in \mathbb{P}^N$ be any closed point and let $\mathscr{M}_{i+1}(\pi_{\mathrm{q}})$ be the multiple loci in $\mathbb{P}^{N-1}$ where each fiber of π_{q} is a finite scheme of length at least $i+1$. Set-theoretically, we have*

$$Z(K_\infty(I)) \cup \mathscr{M}_{i+1}(\pi_{\mathrm{q}}) = Z(K_i(I)).$$

Thus it is important to see *when the $K_i(I)$'s are stabilized* (i.e. the stabilization number $s(\mathrm{q})$) and *what they do look like* (i.e. the stabilized ideal $K_\infty(I)$) for studying of projections. In general, we can give bounds for $s(\mathrm{q})$ in terms of degrees of generators and the $K_\infty(I)$ matches an interesting geometric notion in inner projection case as the following proposition says.

Proposition 2.6 *Let $X \subset \mathbb{P}^N$ be a projective subscheme with a defining ideal I and $\mathrm{q} = (1, 0, \ldots, 0) \in \mathbb{P}^N$. Suppose that I is generated by homogeneous polynomials of degree at most d.*

(a) *Outer case (i.e. $\mathrm{q} \notin X$): $s = s(\mathrm{q}) \leq d$ and $K_\infty(I) = (1)$.*
(b) *Inner case (i.e. $\mathrm{q} \in X$): $s = s(\mathrm{q}) \leq d-1$ and $K_\infty(I) = I_{TC_{\mathrm{q}}X}$, where $I_{TC_{\mathrm{q}}}X$ is the ideal of projective tangent cone of X at q. In particular, if q is smooth, $K_{d-1}(I)$ consists of linear forms which defines the projective tangent space, $T_{\mathrm{q}}X$.*

Proof (a) Comes from a fact, i.e. there always exists a homogeneous $f \in I$ with its leading term $\mathrm{in}(f) = x_0^\nu$ and $\nu \leq d$. For (b), see proposition 2.5 in [15]. □

Remark 2.7 (Computations of Betti numbers using PEIs) Using Proposition 2.6, we could compute some pieces of syzygies of an infinitely generated S-module I (or more generally, of $I/\widetilde{K}_h(I)$). Let $\mathrm{q} \in X = V(I)$ be any smooth point and consider the PEIs of I with respect to q. For any $p \geq 0$, $\beta^S_{p,q}(I/\widetilde{K}_h(I))$ is zero for every $q \leq h+1$ (Lemma 2.4). When $q = h+2$, by Proposition 2.3 (b) and a short exact sequence (2.1) this is equal to

$$\beta^S_{p,q}(\widetilde{K}_{h+1}(I)/\widetilde{K}_h(I)) = \beta^S_{p,q}(K_{h+1}(I)(-h-1)) = \beta^S_{p,1}(K_{h+1}(I)) \tag{2.4}$$

and it can be computed by the Koszul resolution of the (independent) linear forms in $K_{h+1}(I)$.

In particular, when I is generated in degree d, $h = d-2$, by Proposition 2.6,

$$K_i(I) = (\ell_1, \ldots, \ell_e) =: I_L \quad \text{for every } i \geq d-1, \tag{2.5}$$

where $e = N - \dim_k T_{\mathrm{q}}X$ and I_L defines the projective tangent space $T_{\mathrm{q}}X$. Hence, an infinitely generated S-module $I/\widetilde{K}_{d-2}(I)$ has a rather simple minimal free S-resolution such as:

$$0 \to \bigoplus_{q=0}^{\infty} S(-d-e+1-q)^{b_{e-1}} \to \cdots \to \bigoplus_{q=0}^{\infty} S(-d-1-q)^{b_1}$$
$$\to \bigoplus_{q=0}^{\infty} S(-d-q)^{b_0} \to I/\widetilde{K}_{d-2}(I),$$

where $\beta^S_{p,q}(I/\widetilde{K}_{d-2}(I)) = b_p = \binom{e}{p+1}$.

2.3 Syzygies of inner projections

In this subsection, we explain how we can compare the graded Betti numbers of X with those of $X_{\mathfrak{q}}$ and give some general rules for behaviors of Betti tables under inner projections. First, we recall a mapping cone construction as follows (see e.g. [15]):

Proposition 2.8 (Elimination mapping cone sequence) *Let $S = k[x_1, \dots, x_N]$, $R = k[x_0, x_1 \dots, x_N]$ be two polynomial rings. Let M be any graded R-module which is not necessarily finitely generated. Then, we have a natural long exact sequence:*

$$\begin{tikzcd} \cdots \arrow[r, "\bar{\mu}"] & \mathrm{Tor}^S_p(M,k)_{p+q} \arrow[r] & \mathrm{Tor}^R_p(M,k)_{p+q} \arrow[r] & \mathrm{Tor}^S_{p-1}(M,k)_{p-1+q} \arrow[dll, "\bar{\mu}"'] \\ & \mathrm{Tor}^S_{p-1}(M,k)_{p-1+q+1} \arrow[r] & \mathrm{Tor}^R_{p-1}(M,k)_{p-1+q+1} \arrow[r] & \mathrm{Tor}^S_{p-2}(M,k)_{p-2+q+1} \arrow[r, "\bar{\mu}"] & \cdots \end{tikzcd}$$

whose connecting homomorphism $\bar{\mu}$ is induced by an S-module homomorphism μ $M(-1) \xrightarrow{\times x_0} M$, the multiplicative map.

Using the elimination mapping cone sequence (EMCS) and Betti number calculations of PEIs, we could put Betti numbers of X and those of $X_{\mathfrak{q}}$ together and relate them each other.

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \widetilde{K}_0(I_X) \arrow[r] \arrow[d] & \widetilde{K}_q(I_X) \arrow[r] \arrow[d] & \widetilde{K}_q(I_X)/\widetilde{K}_0(I_X) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & S \arrow[r] \arrow[d] & \left(\bigoplus_{a=0}^{q} S \cdot x_0^a\right) \arrow[r] \arrow[d] & \left(\bigoplus_{a=1}^{q} S \cdot x_0^a\right) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \widetilde{Q}_0(I_X) \arrow[r, "f_q"] \arrow[d] & \widetilde{Q}_q(I_X) \arrow[r] \arrow[d] & \operatorname{coker} f_q \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & , \end{tikzcd}$$

Here we are able to obtain the Tor-modules $\mathrm{Tor}^S_p(R_{X_{\mathfrak{q}}}, k)_{p+q}$ and $\mathrm{Tor}^S_p(R_X, k)_{p+q}$ from $\mathrm{Tor}^S_p(\widetilde{Q}_0(I_X), k)_{p+q}$ and $\mathrm{Tor}^S_p(\widetilde{Q}_q(I_X), k)_{p+q}$ by the approximation of syzygies (Proposition 2.3). Additionally, syzygy structures of $K_i(I_X)$'s give essential information on syzygies of $\widetilde{K}_q(I_X)/\widetilde{K}_0(I_X)$ (eventually, on those of a S-module $\operatorname{coker} f_q$). Combined with EMCS which connects between S-module and R-module syzygies of R_X, we could draw the principle on which this study stands as below (see Fig. 3).

$$\begin{tikzcd}
\mathrm{Tor}_p^S(X_{\mathrm{q}})_{p+q} \arrow[r, "f_q"] \arrow[dr, dashed, leftrightarrow] & \mathrm{Tor}_p^S(X)_{p+q} \arrow[r] \arrow[d, "\mathsf{EMCS}"] & \mathrm{Tor}_p^S(\mathrm{coker} f_q, k)_{p+q} \\
 & \mathrm{Tor}_p^R(X)_{p+q} \arrow[d, "\mathsf{EMCS}"] \arrow[dl, dashed, leftrightarrow] & {[\text{from Betti } \#\text{'s } \beta_{i,j}^S(\mathsf{PEIs})]} \arrow[u, squiggly] \arrow[d, squiggly] \\
\mathrm{Tor}_{p-1}^S(X_{\mathrm{q}})_{p-1+q} \arrow[r, "f_q"] & \mathrm{Tor}_{p-1}^S(X)_{p-1+q} \arrow[r] & \mathrm{Tor}_p^S(\mathrm{coker} f_q, k)_{p-1+q}
\end{tikzcd}$$

Fig. 3 How to connect $\beta^R_{p,q}(X)$ with $\beta^S_{p,q}(X_{\mathrm{q}})$? Here $\mathrm{Tor}^{\square}_*(V)_\star$ means $\mathrm{Tor}^{\square}_*(R_V, k)_\star$ where R_V is the coordinate ring of V

Now, we state some general theorems for syzygies of inner projections, which are a generalization of main results in [15]. This will be used for the proof of Theorem 1.3 in Sect. 3.

Theorem 2.9 *Let $X \subset \mathbb{P}^N$ be a nondegenerate subscheme defined by an ideal I, $\mathrm{q} \in X$ be any smooth point. Set $J = \widetilde{K}_0(I)$, the elimination ideal defining the image scheme X_{q}. Suppose that the stabilization number $s = s(\mathrm{q}, I) = 1$.*

(a) *Suppose that I satisfies property $\mathbf{N}_{d,p_0}$ as R-module for some $d \geq 1$ and $p_0 \geq 1$. Then, J satisfies at least property $\mathbf{N}_{d,p_0-1}$ as S-module.*
(b) *Suppose that for some $d \geq 2$ and $p_0 \geq 1$, J satisfies property $\mathbf{N}_{d,p_0}$ as S-module. Then, I satisfies at least property $\mathbf{N}_{d,p_0}$ as R-module.*
(c) $\mathrm{reg}_R(I) = \max\{\mathrm{reg}_S(J), 2\}$.

Proof (a) is a natural generalization of corollary 3.4 in [15] and (b) can be also obtained by similar arguments. For (c), let M be the $\max\{\mathrm{reg}_S(J), 2\}$. First, we see that $2 \leq \mathrm{reg}_R(I)$ by Proposition 2.6 (b). Since I satisfies $\mathbf{N}_{\mathrm{reg}_R(I),\infty}$, we also see that $\mathrm{reg}_S(J) \leq \mathrm{reg}_R(I)$ by (a). Thus, $\mathsf{M} \leq \mathrm{reg}_R(I)$. Conversely, the fact that $\beta^S_{p,q}(J) = 0$ for any $p \geq 0$ and $q > \mathsf{M}$ implies that I satisfies $\mathbf{N}_{\mathsf{M},\infty}$ by (b) so that $\mathsf{M} \geq \mathrm{reg}_R(I)$. □

Theorem 2.10 (Depth of inner projection) *Let $X \subset \mathbb{P}^N$ be a nondegenerate equidimensional subscheme defined by the saturated ideal I_X and q be any smooth point of X. Suppose that the stabilization number $s(\mathrm{q}, I_X) = 1$. Then,*

$$\mathrm{depth}_R(X) = \mathrm{depth}_S(X_{\mathrm{q}}). \tag{2.6}$$

Proof Almost same as the proof of theorem 4.1 in [15]. □

Remark 2.11 (*A condition for $s(\mathrm{q}) = 1$*) As we have seen in Theorems 2.9 and 2.10, one of the most favorable cases is $s(\mathrm{q}) = 1$ (in this case, $\widetilde{K}_{s-1}(I)$ coincides with a defining ideal of the image scheme X_{q}). First of all, $s(\mathrm{q}) = 1$ if I is *quadratic* by Proposition 2.6 (b).

Let us consider a little more refined condition. Then, we know in general

$$t := \dim_k[K_1(I)_1] \leq \mathrm{codim}_{\mathrm{q}}(X, \mathbb{P}^N) =: e, \tag{2.7}$$

Inner projection reduction	Hyperplane section reduction
For a general closed point $\mathsf{q} \in X$, $$\begin{array}{c} X \subset \mathbb{P}^N \\ \Big\downarrow{\scriptstyle \pi_{\mathsf{q}}} \\ X_{\mathsf{q}} := \overline{\pi_q(X \setminus \{q\})} \subset \mathbb{P}^{N-1} \end{array}$$	For a general hyperplane $H \subset \mathbb{P}^N$, $$\begin{array}{c} X \subset \mathbb{P}^N \\ \rightsquigarrow \\ X_H := X \cap H \subset \mathbb{P}^{N-1} \end{array}$$
$\operatorname{codim}(X_{\mathsf{q}}) = \operatorname{codim}(X) - 1$	$\operatorname{codim}(X_H) = \operatorname{codim}(X)$
$\deg(X_{\mathsf{q}}) = \deg(X) - 1$	$\deg(X_H) = \deg(X)$
$\Delta_{X_{\mathsf{q}}} = \Delta_X$	$\Delta_{X_H} = \Delta_X$

Fig. 4 Two different reduction methods in projective algebraic geometry. Inner projection reduction versus Hyperplane section reduction. Here, $\Delta(X) := \deg(X) - \operatorname{codim}(X) - 1$ for an embedded variety $X \subset \mathbb{P}^N$ (originally, due to Fujita for a polarized pair $(X, \mathcal{L})$)

where $X = V(I)$ is a subscheme of $\mathbb{P}^N$ and $\operatorname{codim}_{\mathrm{q}}(X, \mathbb{P}^N)$ denotes the codimension of the component containing q in $\mathbb{P}^N$. If we assume the case of taking q as a *general* (so, smooth) point of X, then

$$\text{the condition } t = e \text{ is equivalent to } s(\mathrm{q}) = 1,$$

because $K_1(I) = I_{T_{\mathrm{q}}X} = I_{TC_{\mathrm{q}}X}$ in both assumptions, so it is by Proposition 2.6 (b).

Remark 2.12 (*Reduction via inner projections*) In general, this inner projection method sometimes gives us a useful way to reduce many given problems into the situation of some small invariants (such as degree, codimension, etc.) in which one might often solve them with the help of many nice properties of *small world* in the same way as hyperplane section method did in classical algebraic geometry (see Fig. 4). In case of taking a general hyperplane section of a variety $X \subset \mathbb{P}^N$ the geometry goes into a relatively easier/well-known situation, while the complexity of defining equations/syzygies is almost the same. But, in case of taking a general inner projection, the syzygies seem to go into a much simpler stage as compensating for a big payment of the complexity of the geometry. Furthermore, in contrast with *outer* projection, note that the reduction via inner projection also preserves Δ-*genus* in the sense of Fujita (see e.g. [10]) as same as the hyperplane reduction does. See also [21] for a typical example using both hyperplane section and inner projection reductions in a clever way.

3 Proofs of main results

3.1 Proof of fundamental inequality (1.2)

In this subsection, we prove the fundamental inequality (1.2) in Theorem 1.1. In fact, we give a proof of the more strengthened form of the theorem as follows:

Theorem 3.1 (In the first linear strand) *Let $X \subset \mathbb{P}^N$ be a nondegenerate subscheme, I_X be the defining ideal of X, $\mathrm{q} \in X$ be a closed point and $K_i(I_X)$'s be the PEIs of I_X with respect to q. Set $t = \dim_k(K_1(I_X))_1$ and $e = \operatorname{codim}_{\mathrm{q}}(X, \mathbb{P}^N)$ the codimension of the component containing q in $\mathbb{P}^N$. Then,*

(a) *For any $p \geq 1$, the following holds*

$$\beta_{p,1}(X) \leq \beta_{p,1}(X_{\mathrm{q}}) + \beta_{p-1,1}(X_{\mathrm{q}}) + \binom{t}{p} \leq \beta_{p,1}(X_{\mathrm{q}}) + \beta_{p-1,1}(X_{\mathrm{q}}) + \binom{e}{p} \tag{3.1}$$

$$\beta_{p,1}(X) \geq \beta_{p,1}(X_{\mathrm{q}}) + \beta_{p-1,1}(X_{\mathrm{q}}) + \binom{t}{p} - \beta_{p-1,2}(X_{\mathrm{q}}) - \beta^S_{p-2,2}(K_1(I)). \tag{3.2}$$

When $p = 1$, in particular, we have

$$\beta_{1,1}(X) = \beta_{1,1}(X_{\mathrm{q}}) + \binom{t}{1} \leq \beta_{1,1}(X_{\mathrm{q}}) + \binom{e}{1}. \tag{3.3}$$

(b) *Furthermore, if $a = a(X) \geq 1$, then for any smooth point of $\mathrm{q} \in X$*

$$\beta_{p,1}(X) \ = \ \beta_{p,1}(X_{\mathrm{q}}) + \beta_{p-1,1}(X_{\mathrm{q}}) + \binom{e}{p} \quad for\ any\ \ p \leq a \tag{3.4}$$

holds and for the case of $p = a + 1$ it holds that

$$\beta_{a+1,1}(X) \ = \ \beta_{a+1,1}(X_{\mathrm{q}}) + \beta_{a,1}(X_{\mathrm{q}}) - \beta_{a,2}(X_{\mathrm{q}}) + \binom{e}{a+1}. \tag{3.5}$$

Proof We prove the theorem by treating $\beta^R_{p,q}(I_X)$ instead of $\beta^R_{p,q}(R_X)$ (also for $\beta^S_{p,q}(S_{X_{\mathrm{q}}})$). Because $\beta^R_{p,q}(R_X) = \beta^R_{p-1,q+1}(I_X)$ for all $p > 0$ and any $q \in \mathbb{Z}$, keep in mind that from now on,

every p in the proof is by one less than the p in the statement.

For simplicity, let I be the defining ideal I_X and J be the ideal $K_0(I_X) = \widetilde{K}_0(I_X)$ defining X_{q} scheme-theoretically. Being a nondegenerate subscheme, I has no linear forms. Consider the commutative diagram such as:

$$\begin{tikzcd} 0 \arrow[r] & J(-1) \arrow[r] \arrow[d] & I(-1) \arrow[r] \arrow[d] & I/J(-1) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \widetilde{K}_1(I) \arrow[r] & I \arrow[r] & I/\widetilde{K}_1(I) \arrow[r] & 0, \end{tikzcd} \tag{3.6}$$

where the vertical maps are induced by x_0-multiplications.

Then, from the above diagram and EMCS (Proposition 2.8), we have an induced commutative diagram as follows:

$$\begin{tikzcd}
 & & \vdots \arrow[d, phantom] & & \mathrm{T}^S_{p,2}\left(\frac{K_2(I)}{K_1(I)}(-2)\right) \arrow[d] \\
 & & \mathrm{T}^R_{p,2}(I) \arrow[d] & & \\
0 \arrow[r] & \mathrm{T}^S_{p-1,2}(J) \arrow[r] & \mathrm{T}^S_{p-1,2}(I) \arrow[r, "\upsilon"] \arrow[d, "\times x_0"', "\mu"] & \mathrm{T}^S_{p-1,2}\left(I/J\right) \arrow[r, "\sim"] \arrow[d, "\phi"] & \mathrm{T}^S_{p-1,2}\left(\widetilde{K}_1(I)/J\right) \arrow[d, "\widetilde{\phi}"] \\
0 \arrow[r] & \mathrm{T}^S_{p-1,3}(\widetilde{K}_1(I)) \arrow[r] & \mathrm{T}^S_{p-1,3}(I) \arrow[r, "\nu"] \arrow[d] & \mathrm{T}^S_{p-1,3}\left(I/\widetilde{K}_1(I)\right) \arrow[r, "\sim"] & \mathrm{T}^S_{p-1,3}\left(\widetilde{K}_2(I)/\widetilde{K}_1(I)\right) \arrow[d] \\
 & & \mathrm{T}^R_{p-1,3}(I) & & \mathrm{T}^S_{p-1,3}\left(\frac{K_2(I)}{K_1(I)}(-2)\right),
\end{tikzcd}$$

where the vertical sequence in the rightmost comes from the short exact sequence (2.2), $\mathrm{T}^S_{p,1}(I/J) = 0$ (since I has no linear forms) and $\mathrm{T}^S_{p,2}(I/\widetilde{K}_1(I)) = 0$ by Lemma 2.4. We could also identify ϕ with $\widetilde{\phi}$ in above diagram via the isomorphisms given by the approximation of syzygies (Proposition 2.3 (b)). Furthermore, since $\mathfrak{q} \in X$ so that $K_i(I)$ contains no units, we have $\mathrm{T}^S_{p,2}\big(\frac{K_2(I)}{K_1(I)}(-2)\big) = 0$ so that $\widetilde{\phi}$ (therefore ϕ also) is a monomorphism. This implies that $\ker \phi \circ \upsilon = \ker\ \upsilon$.

For (a), let us compare dimensions of kernels of morphisms in the commuting diagram above. For $\ker\ \mu$ is a subspace of $\ker \nu \circ \mu = \ker\ \phi \circ \upsilon$, we have

$$\beta^R_{p,2}(I) - \beta^S_{p,2}(\widetilde{K}_1(I)) = \dim\ker\ \mu \le \dim\ker\ \phi \circ \upsilon = \dim\ker\ \upsilon\ =\ \beta^S_{p-1,2}(J)$$

so that

$$\begin{aligned}\beta^R_{p,2}(I) &\le \beta^S_{p-1,2}(J) + \beta^S_{p,2}(\widetilde{K}_1(I))\\ &\le \beta^S_{p-1,2}(J) + \beta^S_{p,2}(J) + \beta^S_{p,2}(K_1(I)(-1)),\end{aligned}$$

because of a short exact sequence from (2.1)

$$0 \to J \to \widetilde{K}_1(I) \to K_1(I)(-1) \to 0. \tag{3.7}$$

Further, since the k-vector space $K_1(I)_1$ consists of t independent linear forms, we can compute via a linear Koszul resolution

$$\beta^S_{p,2}(K_1(I)(-1)) = \beta^S_{p,1}(K_1(I)) = \binom{t}{p+1} \le \binom{e}{p+1}$$

(note that always $t \le e$; see Remark 2.11) and obtain the inequality (3.1).

The inequality (3.2) comes from the following:

$$\begin{aligned}\dim\ker\ \upsilon &= \dim\ker\ \nu \circ \mu = \dim\ker\ \mu + \dim(\operatorname{im} \mu \cap \ker\ \nu)\\ &\le \dim\ker\ \mu + \dim\ker\ \nu\end{aligned} \tag{3.8}$$

so that

$$\beta^S_{p-1,2}(J) \le \beta^R_{p,2}(I) - \beta^S_{p,2}(\widetilde{K}_1(I)) + \beta^S_{p-1,3}(\widetilde{K}_1(I))$$

or

$$\beta^S_{p-1,2}(J) + \beta^S_{p,2}(\widetilde{K}_1(I)) - \beta^S_{p-1,3}(\widetilde{K}_1(I)) \le \beta^R_{p,2}(I).$$

Once again, using the induced long exact sequence from (3.7), we also have the desired inequality

$$\beta^S_{p,2}(J) + \beta^S_{p-1,2}(J) - \beta^S_{p-1,3}(J) + \binom{t}{p+1} - \beta^S_{p-1,2}(K_1(I)) \le \beta^R_{p,2}(I).$$

When $p = 0$, both inequalities (3.1) and (3.2) coincide and lead to the formula (3.3).

For (b), above all, note that $a = a(X) \ge 1$ means I is quadratic and has property $\mathbf{N}_{2,a}$. To prove the first part (3.4), it is enough to show that $t = e$, $\beta^S_{p-1,2}(K_1(I)) = 0$, and $\beta^S_{p-1,3}(J) = 0$ for any $p \le a-1$. Now that I is quadratic, $s(\mathrm{q}) = 1$. Thus, $t = e$ (see Remark 2.11) so that $\beta^S_{p-1,2}(K_1(I)) = 0$. Moreover, by Fact 5.10 we know that X_{q} has at least property $\mathbf{N}_{2,a-1}$ i.e. $\beta^S_{p-1,3}(J) = 0$ for any $p \le a-1$. So, the equality (3.4) is immediate from both (3.1) and (3.2). Furthermore, since $\mathrm{T}^R_{a-1,3}(I) = \mathrm{T}^R_{a,2}(R_X) = 0$ by property $\mathbf{N}_{2,a}$, μ becomes surjective in case of $p = a$ and in this case the inequality of (3.8) becomes equal so that this gives the equality (3.5). □

Remark 3.2 (*Case of non-saturated ideals*) Note that Theorem 3.1 can be easily generalized for any scheme-theoretic defining ideal (not necessarily saturated) I of X. Besides this theorem, most of results in this paper could drop the saturatedness.

As a test case, we could give a following corollary using Theorem 3.1 (this was introduced as a part of so-called $K_{p,1}$*-theorem* by Green for complex projective manifolds in [11] and also by [20] for a bit more general case).

Corollary 3.3 (Generalized $K_{p,1}$-theorem (a)) *Let* $X^n \subset \mathbb{P}^{n+e}$ *be any nondegenerate (possibly singular) variety of* codim e*. Then, we have*

$$\beta_{p,1}(X) = 0 \quad for\ any \quad p > e. \tag{3.9}$$

Proof Use induction on e. When $e = 1$ (i.e. hypersurface), it is obvious. Suppose that (3.9) holds if $e \le m$ for some $m \ge 1$. If $\operatorname{codim}(X, \mathbb{P}^{n+e}) = m+1$, then take an inner projection of X from any general point q of X. By Theorem 3.1 (a), for any $p > m+1$ we have

$$\begin{aligned}\beta_{p,1}(X) &\le \beta_{p-1}(X_{\mathrm{q}}) + \beta_{p,1}(X_{\mathrm{q}}) + \binom{m+1}{p}\\ &= 0 \quad (\because\ p-1 > \operatorname{codim}(X_{\mathrm{q}}, \mathbb{P}^{n+e-1}))\end{aligned}$$

so that $\beta_{p,1}(X) = 0$ and the proof is done. □

3.2 Proofs of Theorems 1.2 and 1.3

Now, we are ready to prove Theorem 1.2 and other results.

Proof of Theorem 1.2 We use induction on the homological index p. Set $X := X^{(e)}$ to respect its own codimension and consider iterated inner projections from a general (so, non-singular) point and denote the Zariski closure of the image of i-th inner projection π_i by $X^{(e-i)}$. Then, we have a chain of (birational) maps $\{\pi_k\}$ from X to some lower codimensional variety (for example, a hypersurface $X^{(1)}$) and the associated sequence of varieties $\{X^{(e)}, X^{(e-1)}, \dots, X^{(2)}, X^{(1)}\}$ such as

$$X = X^{(e)} \overset{\pi_1}{\dashrightarrow} X^{(e-1)} \overset{\pi_2}{\dashrightarrow} \cdots \overset{\pi_i}{\dashrightarrow} X^{(e-i)} \overset{\pi_{i+1}}{\dashrightarrow} \cdots \overset{\pi_{e-2}}{\dashrightarrow} X^{(2)} \overset{\pi_{e-1}}{\dashrightarrow} X^{(1)}. \tag{3.10}$$

$p = 1$ case: Here, we reprove the classical result (known by Castelnuovo and independently by Zak) using our own reduction method via inner projections. We start by recalling some binomial identity, a variant of Vandermonde identity, which will be used frequently in the remaining part of our paper:

$$\sum_{i=0}^{s} \binom{r+i}{i} = \sum_{i=0}^{s} \binom{r+i}{r} = \binom{r+s+1}{r+1}. \tag{3.11}$$

By the inequality (3.3) of Theorem 3.1 (a), for any $e \geq 1$ we know

$$\begin{aligned}
\beta_{1,1}(X^{(e)}) &\leq \beta_{1,1}(X^{(e-1)}) + \binom{e}{1} \\
&\leq \beta_{1,1}(X^{(e-2)}) + \binom{e-1}{1} + \binom{e}{1} \\
&\quad \vdots \\
&\leq \beta_{1,1}(X^{(1)}) + \binom{2}{1} + \cdots + \binom{e-1}{1} + \binom{e}{1} \\
&\leq \binom{e+1}{2} \quad \text{by binomial identity (3.11),}
\end{aligned} \tag{3.12}$$

because $X^{(1)}$ is a hypersurface so that $\beta_{1,1}(X^{(1)}) \leq 1$.

Now, for some $m \geq 1$ suppose the induction hypothesis as follows:

$$\text{“our desired upper bound (1.3) holds for every nondegenerate variety of all } p \leq m \text{ and of any codimension } e \geq 1\text{”.} \tag{3.13}$$

$p = m + 1$ case: Using the inequality (3.1), we have

$$\begin{aligned}
\beta_{m+1,1}(X^{(e)}) &\leq \beta_{m+1,1}(X^{(e-1)}) + \beta_{m,1}(X^{(e-1)}) + \binom{e}{m+1} \\
&\leq \beta_{m+1,1}(X^{(e-2)}) + \beta_{m,1}(X^{(e-2)}) + \beta_{m,1}(X^{(e-1)}) + \binom{e-1}{m+1} \\
&\quad + \binom{e}{m+1} \\
&\quad \vdots \\
&\leq \beta_{m+1,1}(X^{(m)}) + \sum_{i=m}^{e-1} \beta_{m,1}(X^{(i)}) + \sum_{i=m+1}^{e} \binom{i}{m+1} \\
&\leq \beta_{m+1,1}(X^{(m)}) + m\binom{e+1}{m+2} + \binom{e+1}{m+2} \\
&\qquad \text{by hypothesis (3.13) and (3.11)} \\
&\leq (m+1)\binom{e+1}{m+2},
\end{aligned} \tag{3.14}$$

because $\beta_{m+1,1}(X^{(m)}) \leq 0$ by Corollary 3.3. This completes our proof. □

Remark 3.4 Here the *irreducibility* assumption on X is necessary for the upper bound (1.3) in Theorem 1.2. Otherwise, we need some condition on the *connectedness* among components to pursue the same upper bound as (1.3) (see Example 5.4 for details).

As one of by-products of Theorem 1.2, we have the following new characterizations of varieties of minimal degree which generalize Castelnuovo's bound on quadrics to higher linear syzygy level.

Theorem 3.5 (Theorem 1.3) *Let $X^n \subset \mathbb{P}^{n+e}$ be a nondegenerate variety with $e \geq 1$. Then, the following are all equivalent:*

(a) *X^n is a variety of minimal degree (abbr. VMD) in $\mathbb{P}^{n+e}$;*
(b) *$\mathcal{I}_X$ is 2-regular;*
(c) *$a(X) \geq e$;*
(d) *$h^0(\mathbb{P}^{n+e}, \mathcal{I}_X(2)) = \binom{e+1}{2}$;*
(e) *one of $\beta_{p,1}(X)$'s achieves the maximal upper bound (1.3) for some $1 \leq p \leq e$;*
(f) *all the $\beta_{p,1}(X)$'s achieve the maximal upper bound (1.3).*

Proof of Theorem 1.3 First, note that $(a) \Leftrightarrow (b) \Leftrightarrow (c)$. $(a) \Leftrightarrow (b)$ is well-known fact (e.g. see [7]) and $(b) \Leftrightarrow (c)$ also comes from so-called *rigidity* of property $\mathbf{N}_{2,p}$ (see [5,15]). For the remaining part, we take an order such as $(f) \Rightarrow (e) \Rightarrow (d) \Rightarrow (b) \Rightarrow (f)$.

$(f) \Rightarrow (e)$ is trivial. To see $(e) \Rightarrow (d)$, use induction on p. For $p = 1$, this implication is tautological. Assume that this is true for when $p \leq m$ for some $m \geq 1$. If $\beta_{p,1}(X)$ meets its own maximum at $p = m + 1$, then for any sequence of *iterated* *general inner projections* $\{X = X^{(e)}, X^{(e-1)}, \ldots, X^{(m)}\}$ as (3.10) it means that all the inequalities in (3.14) should be equalities and $\beta_{m,1}(X^{(m)}) = m\binom{m+1}{m+1} = m$. This shows that we have the stabilization number $s = 1$ at every reduction step from X to $X^{(m)}$, because for all $i \geq m+1$, the inequality in (3.1)

$$\begin{aligned}\beta_{m+1,1}(X^{(i)}) &\leq \beta_{m+1,1}(X^{(i-1)}) + \beta_{m,1}(X^{(i-1)}) + \binom{\dim_k[K_1(I^{(i)})_1]}{m+1} \\ &\leq \beta_{m+1,1}(X^{(i-1)}) + \beta_{m,1}(X^{(i-1)}) + \binom{\operatorname{codim}(X^{(i)})}{m+1}\end{aligned}$$

goes to be equal and in particular

$$\dim_k[K_1(I^{(i)})_1] = \operatorname{codim}(X^{(i)})\text{for every } m+1 \leq i \leq e,$$

which implies $s = 1$ (see Remark 2.11). Here $I^{(e)} := I_X$ and the elimination ideal of $I^{(i+1)}$ is $I^{(i)}$ which defines $X^{(i)}$ scheme-theoretically.

Then, similarly as in (3.12), using the formula (3.3) we obtain

$$\begin{aligned}\beta_{1,1}(X) &= \beta_{1,1}(X^{(e-1)}) + \binom{e}{1} \\ &= \beta_{1,1}(X^{(e-2)}) + \binom{e-1}{1} + \binom{e}{1} \\ &\ \ \vdots \\ &= \beta_{1,1}(X^{(m)}) + \binom{m+1}{1} + \cdots + \binom{e-1}{1} + \binom{e}{1} = \binom{e+1}{2},\end{aligned} \tag{3.15}$$

because $\beta_{m,1}(X^{(m)}) = m$ implies $\beta_{1,1}(X^{(m)}) = \binom{m+1}{2}$ by induction hypothesis.

To get $(d) \Rightarrow (b)$, take any sequence of iterated general inner projections from X to a hypersurface $X^{(1)}$, $\{X = X^{(e)}, X^{(e-1)}, \ldots, X^{(1)}\}$. By the same argument we did for $(e) \Rightarrow (d)$, every reduction step from X to a hypersurface $X^{(1)}$ has the stabilization number $s = 1$ and $\beta_{1,1}(X^{(1)}) = 1$ which means $X^{(1)}$ is a hyperquadric (in particular 2-regular). Now we can lift the regularity of $X^{(1)}$ up to the regularity of X through Theorem 2.9 (c). Hence, our X is 2-regular.

Finally, the part $(b) \Rightarrow (f)$ is also a fairly known fact (e.g. [7, 19]) and this completes the proof. □

Remark 3.6 (*Geometric description of VMDs*) Classically, the geometric classification of VMD has been known as *del Pezzo-Bertini* classification. It says that every VMD which is not a linear space is either a hyperquadric, a rational normal scroll, or a cone over the Veronese surface in $\mathbb{P}^5$. For a modern treatment, see [7].

3.3 Remarks for the proofs

It seems to be worthwhile to write down the calculations in the proof of Theorem 1.2 rather than to do it over through proof-by-induction. It makes one to see *how one could obtain such an upper bound* (1.3) more clearly and gives some inspiration for the *next-to-extremal case*.

Let us begin by meditating the formula (3.1) a bit more. For any associated sequence of iterated general inner projections $\{X = X^{(e_0)}, \dots, X^{(e)}, \dots, X^{(1)}\}$, this formula (3.1) tells us that

$$\beta_{p,1}(X^{(e)}) \le \beta_{p-1,1}(X^{(e-1)}) + \beta_{p,1}(X^{(e-1)}) + \binom{e}{p} \tag{3.16}$$

for every pair (e, p). Figuratively speaking, *one father* (i.e. $\beta_{p,1}(X^{(e)})$) has *two sons* (i.e. $\beta_{p-1,1}(X^{(e-1)})$ and $\beta_{p,1}(X^{(e-1)})$) and leaves an *inheritance* (i.e. $\binom{e}{p}$) to them.

For instance, if we keep on doing this from the forefather $\beta_{p_0,1}(X^{(e_0)})$ (for simplicity, denote it by $\beta^{(e_0)}_{p_0,1}$) to fourth generation, they become such a family and have the inheritance as appeared in Fig. 5. Here, the forefather's *worth* (i.e. the value of Betti number) can be counted as the worth of all his descendants in *last* (so, fourth) generation and *all* the inheritances they left up to that time.

Since $\beta^{(e)}_{p,1} = 0$ for any pair (e, p) such that $p \le 0$ or $p > e$ (see Corollary 3.3), let us continue this *Birth-Inheritance Game* (see Fig. 6 in page 18) till all the $\beta^{(p_0)}_{p_0,1}, \beta^{(p_0-1)}_{p_0-1,1}, \dots, \beta^{(1)}_{1,1}$ on the diagonal appear. Then, we can bound $\beta^{(e)}_{p_0,1}$ as follows:

$$\beta^{(e_0)}_{p_0,1} \le \underbrace{\sum_{i=0}^{p_0-1} \binom{e_0 - p_0 - 1 + i}{i} \cdot \beta^{(p_0-i)}_{p_0-i,1}}_{(A)} + \underbrace{\sum_{i=0}^{p_0-1} \left\{ \sum_{j=0}^{e_0-p_0-1} \binom{i+j}{i}\binom{e_0 - i - j}{p_0 - i} \right\}}_{(B)},$$

where (A) corresponds to the sum of diagonal Betti numbers in Fig. 6 and (B) corresponds to the lower parallelogram of all the inheritance there (i.e. the sum of *bold-faced*

$$\begin{array}{ccccccc} & & & \beta^{(e_0)}_{p_0,1} & & & \\ & & \beta^{(e_0-1)}_{p_0-1,1} & & \beta^{(e_0-1)}_{p_0,1} & & \\ & \beta^{(e_0-2)}_{p_0-2,1} & & 2\beta^{(e_0-2)}_{p_0-1,1} & & \beta^{(e_0-2)}_{p_0,1} & \\ \beta^{(e_0-3)}_{p_0-3,1} & & 3\beta^{(e_0-3)}_{p_0-2,1} & & 3\beta^{(e_0-3)}_{p_0-1,1} & & \beta^{(e_0-3)}_{p_0,1} \end{array} \qquad \begin{array}{ccccc} & & \binom{e_0}{p_0} & & \\ & \binom{e_0-1}{p_0-1} & & \binom{e_0-1}{p_0} & \\ \binom{e_0-2}{p_0-2} & & 2\binom{e_0-2}{p_0-1} & & \binom{e_0-2}{p_0} \end{array}$$

Fig. 5 Four generations of Betti numbers (on the *left side*) and their inheritances (on the *right side*). Note that both of them form Pascal's triangle

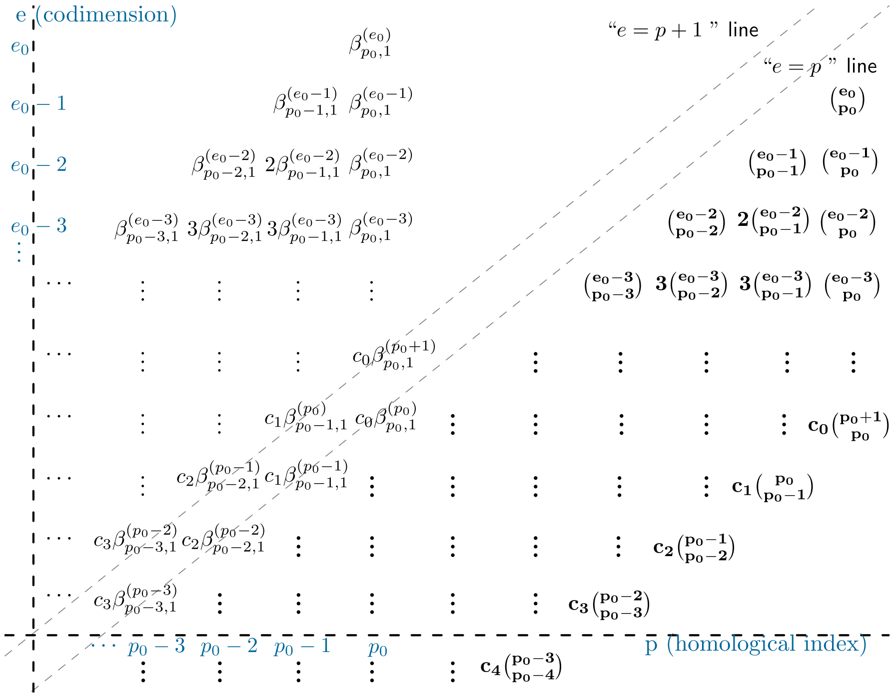

Fig. 6 Birth-Inheritance Game People (i.e. *Betti number* $\beta^{(e)}_{p,1}$'s) are located, according to the pair (e, p), in the upper triangular area and all their inheritance (i.e. *bold-faced binomial numbers*) are stacked up in the lower triangular area. Each person gives birth to two sons and leaves the inheritance until one reaches the diagonal (i.e. $e = p$ line). Note that all the inheritances form a *parallelogram* during this game and the coefficient $c_i = \binom{e_0-p_0-1+i}{i}$

binomial numbers in Fig. 6). Direct summand in (B) is the $(j+1)$-th binomial number from the top in the $(i+1)$-th column from the right of the parallelogram.

Say $c_i = \binom{e_0-p_0-1+i}{i}$. Now, if we do this game once more from $c_0\beta^{(p_0)}_{p_0,1}$ to $c_{p_0-1}\beta^{(1)}_{1,1}$ (i.e. on all the Betti numbers on (A)), then it follows that

$$\begin{aligned}(A) &= \sum_{i=0}^{p_0-1} c_i\beta^{(p_0-i)}_{p_0-i,1} \le \sum_{i=0}^{p_0-1}\left\{\left(\sum_{j=0}^{i} c_j\right)\binom{p_0-i}{p_0-i}\right\} \quad \left(\because \text{ all } \beta^{(p_0-i-1)}_{p_0-i,1} \text{vanish}\right)\\ &= \sum_{i=0}^{p_0-1}\binom{e_0-p_0+i}{i}\binom{p_0-i}{p_0-i} = \binom{e_0}{e_0-p_0+1} =: (A)' \end{aligned} \tag{3.17}$$

by the binomial identities (3.11) and that

$$\beta^{(e_0)}_{p_0,1} \le (A) + (B) \le (A)' + (B) = \sum_{i=0}^{p_0-1}\left\{\sum_{j=0}^{e_0-p_0}\binom{i+j}{i}\binom{e_0-i-j}{p_0-i}\right\}$$

$$= \sum_{i=0}^{p_0-1} \binom{e_0+1}{p_0+1} = p_0 \binom{e_0+1}{p_0+1} \tag{3.18}$$

by another variant of Vandermonde identity

$$\sum_{i=0}^{s} \binom{r+i}{r} \binom{s-i}{t} = \binom{r+s+1}{r+t+1} \quad \text{for integers } s \geq t \geq 0, \tag{3.19}$$

and the binomial identities (3.11). Hence, we obtain the desired upper bounds, which represent the Betti numbers of VMD.

4 Next-to-extremal case

Theorem 4.1 (Theorem 1.4) *Let $X^n \subset \mathbb{P}^{n+e}$ be any nondegenerate variety of codim $e \geq 1$. Unless X is a variety of minimal degree, then we have*

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1} - \binom{e}{p-1} \quad for\ all\ 1 \leq p \leq e. \tag{4.1}$$

Note that $p\binom{e+1}{p+1} - \binom{e}{p-1}$ is also the p-th Betti number of del Pezzo varieties of codimension e (e.g. [19]). Before proving Theorem 4.1, we introduce another relevant lemma as a direct consequence of theorem 3.5 in [20].

Lemma 4.2 (Generalized $K_{p,1}$-theorem (b)) *Let $X^n \subset \mathbb{P}^{n+e}$ be any nondegenerate variety of codim e. Unless X is a variety of minimal degree, then we have*

$$\beta_{e,1}(X) = 0.$$

Now, let's prove next-to-extremal upper bounds on $\beta_{p,1}$'s.

Proof of Theorem 4.1 First, we note that a general inner projection X_q is not of minimal degree, unless X is of minimal degree (due to so-called *Trisecant lemma*). Similarly as in the proof of extremal bounds, take a sequence of iterated general inner projections $\{X = X^{(e)}, X^{(e-1)}, \ldots, X^{(1)}\}$. As discussed in Sect. 3.3, we could bound

$$\beta_{p,1}(X) \leq \underbrace{\sum_{i=0}^{p-1} \binom{e-p-1+i}{i} \cdot \beta_{p-i,1}(X^{(p-i)})}_{(A)} + \underbrace{\sum_{i=0}^{p-1} \left[\sum_{j=0}^{e-p-1} \binom{i+j}{i} \binom{e-i-j}{p-i} \right]}_{(B)}$$
$$= (B) = p\binom{e+1}{p} - (A)' = p\binom{e+1}{p} - \binom{e-1}{p} \quad \text{(see (3.17) and (3.18))},$$

because $\beta_{p-i,1}(X^{(p-i)}) = 0$ for every $0 \leq i \leq p-1$ by Lemma 4.2. □

As an application, we can also add new characterizations of del Pezzo varieties which generalize Fano's classical bound on quadrics to higher linear syzygy level.

Theorem 4.3 *Let $X^n \subset \mathbb{P}^{n+e}$ be a nondegenerate variety with $e \geq 2$. Then, the following are all equivalent:*

(a) *X is a del Pezzo variety;*
(b) *$a(X) = e - 1$;*
(c) *$h^0(\mathbb{P}^{n+e}, \mathcal{I}_X(2)) = \binom{e+1}{2} - 1$;*
(d) *one of $\beta_{p,1}(X)$'s achieves the upper bound (4.1) for some $1 \leq p \leq e - 1$;*
(e) *all the $\beta_{p,1}(X)$'s achieve the upper bound (4.1).*

Proof $(a) \Leftrightarrow (b)$ is known by theorem 4.3 (b) in [15] and we prove by taking an order such as $(b) \Rightarrow (e) \Rightarrow (d) \Rightarrow (c) \Rightarrow (a)$.

$(b) \Rightarrow (e)$ comes from the known Betti numbers of del Pezzo varieties (e.g. [19]) and $(e) \Rightarrow (d)$ is trivial. Now let us see $(d) \Rightarrow (c)$. As seen in the proof of Theorems 1.3 and 4.1, the equality of next-to-extremal bound on some $\beta_{p,1}(X)$ means that every reduction step from $X = X^{(e)}$ to $X^{(1)}$ for any sequence of iterated general inner projections $\{X = X^{(e)}, X^{(e-1)}, \ldots, X^{(1)}\}$ should have the stabilization $s = 1$ and $\beta_{1,1}(X^{(1)}) = 0$. Thus, using the formula (3.3) repeatedly, we obtain $\beta_{1,1}(X) = \binom{e+1}{2} - 1$. Finally, to show $(c) \Rightarrow (a)$ note that the delta genus is preserved (see Fig. 4) under each reduction (i.e. $\Delta(X^{(i+1)}) = \Delta(X^{(i)})$ for every $i \geq 1$) and that $X^{(2)}$ is a complete intersection of two quadrics. Since $X^{(2)}$ is a variety of next-to-minimal degree (i.e. $\Delta = 1$) and ACM, we conclude that our original X is also of next-to-minimal degree and ACM (*depth* can be lifted by Theorem 2.10 whenever $s = 1$), in other words a del Pezzo variety. □

Remark 4.4 (*Geometric characterization of del Pezzo varieties*) Some works on the geometric characterization/classification of del Pezzo varieties have been done by Fujita for mainly normal singularities and recently by Brodmann and Park for non-normal cases (see Remark 4.4 (b) in [15] for references).

5 Examples and questions

More general categories As we explored through Theorems 1.2 and 1.3, in the category of k-varieties $\mathsf{Var}(k)$ all the notions *minimal degree, 2-regularity,* and *maximal Betti numbers* are equivalent. How about more general categories?

In [6] they appointed '2-regularity' as a generalization of the notion of 'minimal degree', clarified its geometric meaning (so-called *smallness*), and classified them completely in the category of algebraic sets $\mathsf{AlgSet}(k)$. We could also attempt to extend the notion of 'maximal Betti numbers' and to generalize similar characterizations on them into more general categories (even though not into the whole $\mathsf{AlgSet}(k)$).

For instance, let us consider the following category. One says that any algebraic set $X = \cup X_i$ is *connected in codimension 1* if X is equidimensional and all the irreducible component X_i's can be ordered in such a way that every $X_i \cap X_{i+1}$ is of codimension 1 in X. Denote the category of connected in codimension 1 algebraic sets by $\mathsf{CC}_1(k)$.

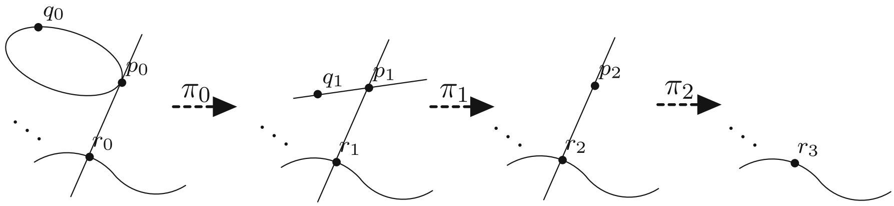

Fig. 7 How to reduce components following (i) and (ii) in $\mathsf{CC}_1(k)$. The *dashed arrows* represent inner projections π_i's from q_0, q_1, and p_2 respectively. Note that every reduction step diminishes codimension exactly by one

Note that a key ingredient for proofs of most of results in this paper is the reduction method via inner projections where the notion of *codimension* has an important role. $\mathsf{CC}_1(k)$ is the very case in which codimension is well-defined (*degree* is given by the sum of degrees of all the components. It is always at least codimension + 1.) and reduction process are well-behaved as following steps (see also Fig. 7):

(i) choose one component and take iterated general inner projections within the component until the component disappear (into the intersection with other components);
(ii) do these reductions component by component.

Therefore, our extremal bounds and characterizations for the maximal Betti numbers in $\mathsf{Var}(k)$ can be naturally generalized to this category $\mathsf{CC}_1(k)$.

Theorem 5.1 *Let $X^n \subset \mathbb{P}^{n+e}$ be any nondegenerate algebraic set of codim $e \geq 1$ in $\mathsf{CC}_1(k)$. Then,*

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1} \quad for\ all\ \ p \geq 0. \tag{5.1}$$

Further, the following are all equivalent:

(a) *X is of minimal degree in $\mathbb{P}^{n+e}$.*
(b) *$\mathcal{I}_X$ is 2-regular.*
(c) *$a(X) \geq e$.*
(d) *$h^0(\mathbb{P}^{n+e}, \mathcal{I}_X(2)) = \binom{e+1}{2}$.*
(e) *one of $\beta_{p,1}(X)$'s achieves the maximum for some $1 \leq p \leq e$.*
(f) *all the $\beta_{p,1}(X)$'s achieve the maxima.*

Remark 5.2 In $\mathsf{CC}_1(k)$, we can also see which algebraic set does attain the maximal Betti numbers geometrically. First, we recall that a sequence $\{X_1, X_2, \ldots, X_n\}$ of the components of an algebraic set $X = \cup X_i$ is *linearly joined* if we have

$$(X_1 \cup \cdots \cup X_i) \cap X_{i+1} = \langle X_1 \cup \cdots \cup X_i \rangle \cap \langle X_{i+1} \rangle$$

for every $i = 1, 2, \ldots, n-1$, where $\langle X_i \rangle$ means its span (so this definition may depend on the ordering). Being of minimal degree, we could easily obtain that they are just all the linearly joined union of VMDs. This also coincides with the classification of [6], because of 2-regularity.

Now, we look some interesting examples up. Since the theory is closely related to the geometry of codimension, the examples have been chosen among the curve cases.

Example 5.3 (*Reducible linearly joined unions of VMDs*) Let $X_1 \subset \mathbb{P}^4$ be a union of a line ℓ and a twisted cubic C such that $\ell \cap C = \langle \ell \rangle \cap \langle C \rangle = \mathit{one\ point}$. Let X_2 be a union of two plane conics Q_1, Q_2 meeting at *one point* (their spans also) in $\mathbb{P}^4$. Both X_1 and X_2 in $\mathsf{CC}_1(k)$ are linearly joined unions of VMDs and codim $e = 3$. Using `Macaulay 2` (see [18]), we can verify that they give the same Betti table having maximal Betti numbers as expected in Theorem 5.1.

$\mathbb{B}(X_1) = \mathbb{B}(X_2)$

	0	1	2	3
0	1	–	–	–
1	–	6	8	3

But, we can not drop the condition 'connected in codimension 1' in Theorem 5.1 even though 'linearly joined' condition holds as the following example says.

Example 5.4 (*Two unions of lines: three lines in* $\mathbb{P}^3$ *and skew lines in* $\mathbb{P}^3$) Let $I_{X_1} = (x_0x_3, x_1x_2, x_2x_3)$ and $I_{X_2} = (x_0x_2, x_0x_3, x_1x_2, x_1x_3)$ be two saturated ideals in $k[x_0, x_1, x_2, x_3]$. I_{X_1} defines a union of three lines $X_1 = \ell_1 \cup \ell_2 \cup \ell_3$ such that ℓ_1 and ℓ_2 meet at one point and so do ℓ_2 and ℓ_3. I_{X_2} defines $X_2 = \ell_1 \cup \ell_2$ be skew lines in $\mathbb{P}^3$. Both of X_i's are nondegenerate, linearly joined set of codim $e = 2$. But the skew lines X_2 is not connected in codimension 1 (by convention, consider dim $\emptyset = -1$), while X_1 satisfies to be connected in codimension 1. By `Macaulay 2`, we present the Betti table of X as below. Note that all $\beta_{p,1}(X_2)$'s *exceed* the maximal Betti numbers in (5.1) of codimension 2, in contrast with $\mathbb{B}(X_1)$ achieving the bound.

$\mathbb{B}(X_1)$

	0	1	2	3
0	1	–	–	–
1	–	3	2	–

$\mathbb{B}(X_2)$

	0	1	2	3
0	1	–	–	–
1	–	4	4	1

We also have examples which show that the bounds (4.1) may not serve as next-to-extremal bounds in $\mathsf{CC}_1(k)$. In other words, from the consideration of next-to-extremal case it might be possible to occur many interesting Betti tables according to the configurations of unions of small degree varieties even though in the category $\mathsf{CC}_1(k)$.

Example 5.5 (*On next-to-extremal bound*) Let $X_1 \subset \mathbb{P}^4$ be a union of a plane conic Q and a twisted cubic C meeting at *one double point* with $\langle Q \rangle \cap \langle C \rangle = \mathbb{P}^1$ ($e = 3$). This is nondegenerate, connected in codimension 1, but not linearly joined. X_1 is also of next-to-minimal degree and has the same Betti table as a del Pezzo variety does in $\mathsf{Var}(k)$ (see Fig. 8). On the other hand, if X_2 is a nondegenerate union of a plane nodal cubic C and a smooth conic Q in $\mathbb{P}^4$ ($e = 3$), then X_2 has a different Betti table with the one of X_1, although X_2 is of next-to-minimal degree, connected in codimension 1, and even *linearly joined* (see also Fig. 8). We see that $\beta_{2,1}(X_2)$ and $\beta_{3,1}(X_2)$ exceed

$\mathbb{B}(X_1)$

	0	1	2	3
0	1	–	–	–
1	–	5	5	–
2	–	–	–	1

$\mathbb{B}(X_2)$

	0	1	2	3
0	1	–	–	–
1	–	5	6	2
2	–	1	2	1

Fig. 8 Two Betti tables of X_1 and X_2, algebraic sets of next-to-minimal degree in $\mathsf{CC}_1(k)$ (by `Macaulay 2`). Note that two tables be the same after a *diagonal cancellation*

next-to-extremal bounds (4.1) though $\beta_{1,1}(X_2)$ *achieves* the maximum of (4.1). Note that two Betti tables get the same after taking a *diagonal cancellation*.

Question 5.6 Here are our questions.

(a) Is it possible to generalize upper bounds (5.1) and (4.1) into more general categories such as $\mathsf{AlgSet}(k)$ (possibly in terms of codimensions of components and other invariants, if needed)?
(b) Can we explain the reason of the difference of two Betti tables in Fig. 8 *geometrically*? Is it possible to heal the next-to-extremal case in $\mathsf{CC}_1(k)$ (see Example 5.5) by figuring out this *diagonal cancellation phenomena*?
(c) Classify or characterize those who have *next-to-simple* Betti tables (the *simplest* are the tables of 2-regular schemes) geometrically in $\mathsf{CC}_1(k)$ or more general categories (see also question 5.6 in [15]).

More improved bounds Concerning on linear syzygies of X at least, one could say in general

More quadrics X has, Nicer syzygies X has.

Here, *what 'niceness' does mean* could be spoken in many different ways, but in view of Theorems 1.3 and 4.3 we can say it means getting closer to maximal Betti numbers in the linear strand and higher $a(X)$ (or $b(X)$) our X has.

On this point there is an interesting fact such as (coming directly from corollary 3.8 in [15]):

Fact 5.7 Let $X^n \subset \mathbb{P}^{n+e}$ be a nondegenerate subscheme in $\mathsf{Var}(k)$ (or $\mathsf{CC}_1(k)$) of codim e. Then, we have

$$\binom{e+1}{2} - \binom{e+1-a(X)}{2} \leq \beta_{1,1}(X). \tag{5.2}$$

In other words, it means that $a(X)$ has some necessary conditions on $\beta_{1,1}$. Therefore, we suspect that the following question might be true:

$$\text{Is it possible to give an upper bound on } \beta_{1,1}(X) \text{ in terms of } b(X)?, \tag{5.3}$$

which is the question about whether $\beta_{1,1}$ does impose some sufficient condition for $b(X)$ or not. For a large $b(X)$ (to be precise, for $b(X) \geq e$ in $\mathsf{Var}(k)$), (5.3) is true. It is also considered as a kind of *converse* of the idea, say

High $b(X)$ guarantees many quadrics on X so that X can inherits interesting geometric structures from the embedding quadrics,

on which many problems (e.g. Green's conjectures on algebraic curves in [11]) are essentially based. As one of the ways to answer the question (5.3), we raise the following question:

Question 5.8 Let $X^n \subset \mathbb{P}^{n+e}$ be a nondegenerate reduced subscheme of codim e and X_{q} be its inner projected image.

$$\text{Does it hold that } b(X_{\mathrm{q}}) \leq b(X) - 1 \text{ for a general point } \mathrm{q} \in X?$$

Remark 5.9 We complete this section by making some relevant remarks.

(a) For $a(X)$, we have an interesting result from corollary 3.4 in [15]:

Fact 5.10. Let $X^n \subset \mathbb{P}^{n+e}$ be a nondegenerate reduced subscheme of codim e and X_{q} be its inner projected image. Then, we have

$$a(X_{\mathrm{q}}) \geq a(X) - 1 \text{ for a general (in fact, any smooth) point } \mathrm{q} \in X.$$

(b) We know that $b(X_{\mathrm{q}}) \leq b(X)$ for a general $\mathrm{q} \in X$ always holds. To the best of author's knowledge, there hasn't been a counterexample for Question 5.8 except the case of q being singular. If Question 5.8 is true, then through similar arguments in Sect. 3.3, we can answer the question (5.3) as follows:

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1} + \left\{\binom{e+1}{p+1} - \binom{b}{p+1}\right\} - (e-b+1)\binom{e+1}{p}, \tag{5.4}$$

which are more improved upper bounds in terms of e, p, $b := b(X)$ generalizing the bounds (1.3) and (1.4). [To be precise, it is enough to run *Birth-Inheritance game* till our Betti numbers arrive at the line "$e = p + (e(X) - b(X) + 1)$" in Fig. 6 as though they did at the line "$e = p$" in the *extremal* bound (i.e. $b(X) = e(X) + 1$) and at the line "$e = p + 1$" in the *next-to-extremal* case (i.e. $b(X) = e(X)$).]

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