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Generic initial ideals of singular curves in graded lexicographic order

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ABSTRACT

In this paper, we are interested in the generic initial ideals of *singular* projective curves with respect to the graded lexicographic order. Let C be a *singular* irreducible projective curve of degree $d \geqslant 5$ with the arithmetic genus $\rho_a(C)$ in $\mathbb{P}^r$ where $r \geqslant 3$. If $M(I_C)$ is the regularity of the lexicographic generic initial ideal of I_C in a polynomial ring $k[x_0, \dots, x_r]$ then we prove that $M(I_C)$ is $1 + \binom{d-1}{2} - \rho_a(C)$ which is obtained from the monomial

$$x_{r-3} x_{r-1}^{\binom{d-1}{2} - \rho_a(C)},$$

provided that $\dim \operatorname{Tan}_p(C) = 2$ for every singular point $p \in C$. This number is equal to one plus the number of secant lines through the center of general projection into $\mathbb{P}^2$. Our result generalizes the work of J. Ahn (2008) [1] for *smooth* projective curves and that of A. Conca and J. Sidman (2005) [9] for *smooth* complete intersection curves in $\mathbb{P}^3$. The case of singular curves was motivated by A. Conca and J. Sidman (2005) [9, Example 4.3]. We also provide some illuminating examples of our results via calculations done with *Macaulay* 2 and `Singular` (Decker et al., 2011 [10], Grayson and Stillman [16]).

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Contents

1. Introduction

Let $R = k[x_0, \ldots, x_r]$ be a polynomial ring over an algebraically closed field k of characteristic zero and I be a homogeneous ideal of R. If X is a non-degenerate reduced closed subscheme in $\mathbb{P}^r$ we write I_X for the saturated defining ideal of X in the polynomial ring R.

Bayer and Mumford in [4] introduced the *regularity* of the initial ideal of I with respect to a term order τ as a measure of the complexity of computing Gröbner bases. Even though this depends on the choice of coordinates, it is constant in generic coordinates by the result of Galligo [13]. He has proved that the initial ideals of I in generic coordinates are invariant, which is the so-called generic initial ideal of I with respect to τ, denoted by $\mathrm{Gin}_\tau(I)$. In characteristic zero, it was shown in [6] that the regularity of $\mathrm{Gin}_\tau(I)$ is exactly the maximum of the degrees of its minimal generators.

One of the important problems is to bound the regularity of the generic initial ideal of I for a given term order τ on monomials. Many people have studied generic initial ideals with respect to the reverse lexicographic term ordering, as these ideals have essentially best-case complexity due to a result of Bayer and Stillman (for examples, [4,6–8,14,15,18–24]). However, much less is known about the generic initial ideals with respect to the graded lexicographic term ordering. One expects them to require many more generators than the reverse lexicographic initial ideals, but little is known about their precise behavior [1,2,9].

In this paper, we continue the study of the lexicographical generic initial ideals of *singular* projective curves. Our main result gives a relationship between the complexity of algebraic computations with the ideal of a singular curve and the geometry of its generic projection to the plane. More precisely, let C be a *singular* irreducible curve of degree $d \geqslant 5$ with arithmetic genus $\rho_a(C)$ in $\mathbb{P}^r$ where $r \geqslant 3$. If $\dim \mathrm{Tan}_p(C) = 2$ for every singular point $p \in C$ then the regularity of the lexicographic generic initial ideal of C is exactly $1 + \binom{d-1}{2} - \rho_a(C)$, which is one plus the number of secant lines through the center of general projection into $\mathbb{P}^2$. Moreover it turns out that the regularity is obtained from the monomial generator $x_{r-3}x_{r-1}^{\binom{d-1}{2}-\rho_a(C)}$ of $\mathrm{Gin}(I_C)$.

We use M. Green's partial elimination ideals and careful work with their Hilbert functions to achieve the result, which previously has been used in [1]. Main ideas employed in this paper are to reduce the problem to the case of singular curves in $\mathbb{P}^3$ and to show that the first partial elimination ideal of $I_C \subset K[x_0, x_1, x_2, x_3]$ is a radical ideal in generic coordinates, under the assumption that $\dim \mathrm{Tan}_p(C) = 2$ for every singular point $p \in C$. In process of the proof, this ideal turns out to be the defining ideal of the set of non-isomorphic points under a generic projection of C into $\mathbb{P}^2$.

Our result generalizes the works of J. Ahn [1] and A. Conca and J. Sidman [9] who proved the same formula for the case of *smooth* projective curves and for *smooth* complete intersection curves in $\mathbb{P}^3$, respectively.

Finally, we remark that our result is not true if $\dim \mathrm{Tan}_p(C) > 2$. The example of A. Conca and J. Sidman [9, Example 4.3] is a complete intersection curve C defined by $x^3 - yz^2$ and $y^3 - z^2t$ with one singular point $p = [0, 0, 0, 1]$. One can compute $\dim \mathrm{Tan}_p(C) = 3$ and $\delta_p = 10$ with `Singular` [10]. In this case the regularity of the lexicographic generic initial ideal of I_C is 16, which is not $1 + \binom{9-1}{2} - \rho_a(C) = 19$ (see Example 3.6 for the details).

2. Notations and known facts

(a) We work over an algebraically closed field k of characteristic zero.

(b) For a homogeneous ideal I, the Hilbert function of R/I is defined by $H(R/I,m) := \dim_k(R/I)_m$ for any non-negative integer m. We denote its corresponding Hilbert polynomial by $P_{R/I}(z) \in \mathbb{Q}[z]$. If $I = I_X$ then we simply write $P_X(z)$ instead of $P_{R/I_X}(z)$.
(c) Given a homogeneous ideal $I \subset R$ and a term order τ, there is a Zariski open subset $U \subset GL_{r+1}(k)$ such that $\mathrm{in}_\tau(g(I))$ for $g \in U$ is constant. We will call $\mathrm{in}_\tau(g(I))$ the generic initial ideal of I for $g \in U$ and denote it by $\mathrm{Gin}_\tau(I)$. One can say that I is in generic coordinates if $\mathrm{in}_\tau(I) = \mathrm{Gin}_\tau(I)$.
(d) The generic initial ideal $\mathrm{Gin}_\tau(I)$ of I has Borel fixed property, which is a nice combinatorial property. In characteristic 0, we say that a monomial ideal J has Borel fixed property if $x_i m \in J$ for a monomial m, then $x_j m \in J$ for all $j \leqslant i$.
(e) For a homogeneous ideal $I \subset R$, let $M(I)$ denote the maximum of the degrees of minimal generators of $\mathrm{Gin}_{\mathrm{GLex}}(I)$.
(f) For a homogeneous ideal $I \subset R$, consider a minimal free resolution

$$\cdots \to \bigoplus_j R(-i-j)^{\beta_{i,j}(I)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}(I)} \to I \to 0$$

of I as a graded R-modules. We say that I is m-regular if $\beta_{i,j}(I) = 0$ for all $i \geqslant 0$ and $j > m$. The Castelnuovo–Mumford regularity of I is defined by

$$\mathrm{reg}(I) := \min\{m \mid I \text{ is } m\text{-regular}\}.$$

(g) If I is a Borel fixed monomial ideal then $\mathrm{reg}(I)$ is exactly the maximal degree of minimal generators of I (see [6,12]). This implies that $M(I) = \mathrm{reg}(\mathrm{Gin}_{\mathrm{GLex}}(I))$.
(h) Let C be an integral projective scheme of dimension 1 over k, and $f : \widetilde{C} \to C$ be its normalization. We write δ_p for the length of $(f_*\mathcal{O}_{\widetilde{C}})_p/\mathcal{O}_{C,p}$ as an $\mathcal{O}_{C,p}$-module for each $p \in C$. Note that if a singular point p is a node or an ordinary cusp then $\delta_p = 1$ [17, Exercise IV 1.8(c)].

We recall some definitions and known facts which will be used throughout the remaining parts of the paper. Unless otherwise stated, we always assume the graded lexicographic term ordering. Furthermore, for an irreducible reduced closed subscheme X, we also assume that I_X is in generic coordinates such that $\mathrm{in}(I_X) = \mathrm{Gin}(I_X)$.

Theorem 2.1. *(See [1, Theorem 1.2].) Let X be an integral scheme in $\mathbb{P}^r$ and let π be a generic projection of X to $\mathbb{P}^{r-1}$. Suppose that π is an isomorphism. Then $M(I_X) = M(I_{\pi(X)})$.*

Definition 2.2. (See [9,12].) Let I be a homogeneous ideal in $R = k[x_0, \ldots, x_r]$. If $f \in I_d$ has leading term $\mathrm{in}(f) = x_0^{d_0} \cdots x_r^{d_r}$, we will set $d_0(f) = d_0$, the leading power of x_0 in f. We let

$$\widetilde{K}_i(I) = \bigoplus_{d \geqslant 0} \{ f \in I_d \mid d_0(f) \leqslant i \}.$$

If $f \in \widetilde{K}_i(I)$, we may write uniquely $f = x_0^i \bar{f} + g$, where $d_0(g) < i$. Now we define $K_i(I)$ as the image of $\widetilde{K}_i(I)$ in $\bar{R} = k[x_1 \ldots x_r]$ under the map $f \to \bar{f}$ and we call $K_i(I)$ the i-th partial elimination ideal of I.

Remark 2.3. We have an inclusion of the partial elimination ideals of I:

$$I \cap \bar{R} = K_0(I) \subset K_1(I) \subset \cdots \subset K_i(I) \subset K_{i+1}(I) \subset \cdots \subset \bar{R} = k[x_1 \ldots x_r].$$

Note that if I is in generic coordinates and $i_0 = \min\{i \mid I_i \neq 0\}$ then $K_i(I) = \bar{R}$ for all $i \geqslant i_0$.

The following result gives a useful relationship between partial elimination ideals and the geometry of the projection map from $\mathbb{P}^r$ to $\mathbb{P}^{r-1}$. For a proof of this proposition, see [12, Proposition 6.2].

Proposition 2.4. *Let $X \subset \mathbb{P}^r$ be a reduced closed subscheme and let I_X be the defining ideal of X. Suppose $p = [1, 0, \ldots, 0] \in \mathbb{P}^r \setminus X$ and that $\pi : X \to \mathbb{P}^{r-1}$ is the projection from the point $p \in \mathbb{P}^r$ to $x_0 = 0$. In generic coordinates, the radical ideal $\sqrt{K_i(I_X)}$ defines the algebraic set $\{q \in \pi(X) \mid \mathrm{mult}_q(\pi(X)) > i\}$ set-theoretically.*

Thus, we can define the following two projective schemes associated with the partial elimination ideals:

$$Y_i(X) := \mathrm{Proj}\big(\bar{R}/\sqrt{K_i(I_X)}\big) \subset Z_i(X) := \mathrm{Proj}\big(\bar{R}/K_i(I_X)\big).$$

It is clear that $Z_i(X)_{\mathrm{red}} = Y_i(X)$ and if $K_i(I_X)$ is radical, then $Y_i(X) = Z_i(X)$.

It is natural to ask what is a Gröbner basis of $K_i(I)$? Recall that any non-zero polynomial f in R can be uniquely written as $f = x^t \bar{f} + g$ where $d_0(g) < t$. A. Conca and J. Sidman [9] proved that if G is a Gröbner basis for an ideal I then the set

$$G_i = \big\{\bar{f} \bigm| f \in G \text{ with } d_0(f) \leqslant i\big\}$$

is a Gröbner basis for $K_i(I)$. The following proposition shows that if I is in generic coordinates then there is a more refined Gröbner basis for $K_i(I)$, which plays an important role in this paper.

Proposition 2.5. *(See [2, Proposition 3.4].) Let I be a homogeneous ideal* in generic coordinates *and G be a Gröbner basis for I with respect to the graded lexicographic order. Then, for each $i \geqslant 0$,*

(a) *the i-th partial elimination ideal $K_i(I)$ is in generic coordinates*;
(b) *$G_i = \{\bar{f} \mid f \in G$ with $d_0(f) = i\}$ is a Gröbner basis for $K_i(I)$.*

We have the following immediate corollary from Proposition 2.5.

Corollary 2.6. *For a homogeneous ideal $I \subset R = k[x_0, \ldots, x_r]$* in generic coordinates, *we have*

$$M(I) = \max\big\{M\big(K_i(I)\big) + i \bigm| 0 \leqslant i \leqslant \beta\big\},$$

where $\beta = \min\{j \mid I_j \neq 0\}$.

3. Generic initial ideals of singular curves

As mentioned in the introduction, $M(I_C)$ can be computed precisely in terms of degree and genus of a *smooth* integral curve C in $\mathbb{P}^r$, $r \geqslant 3$. In this section, we generalize the results for smooth curves in [1] to non-degenerate *singular* curves in $\mathbb{P}^r$, $r \geqslant 3$. We are motivated by [9, Example 4.3] due to A. Conca and J. Sidman.

Remark 3.1. We will use the following well-known facts to prove our main results.

(a) (Trisecant Lemma) Let C be a reduced, irreducible curve in $\mathbb{P}^r$ where $r \geqslant 3$. The family of trisecant lines to C has dimension at most 1. This is equivalent to the assertion that not every pair of points of C lie on a trisecant line (see [3]).
(b) Let C be an integral curve in $\mathbb{P}^r$, $r \geqslant 3$, and $\dim \mathrm{Tan}_p(C) = 2$ for any $p \in \mathrm{Sing}(C)$. Then we can choose a generic point $q \notin \mathrm{Tan}_p(C)$ such that $\pi_q : C \to \mathbb{P}^{r-1}$ is an isomorphic projection. Furthermore, $M(I_C) = M(I_{\pi_q(C)})$ (see [1, Theorem 1.2]).

From now on, we consider the Hilbert functions of two subschemes $Y_i(C) \subset Z_i(C) \subset \mathbb{P}^2$ associated to the partial elimination ideals $K_i(I_C)$, $i = 0, 1$, for a singular projective curve C.

Lemma 3.2. *Let $I_C \subset k[x_0, \dots, x_3]$ be the defining ideal of an integral, possibly singular, curve C in $\mathbb{P}^3$. Then* $\deg(\bar{R}/K_1(I_C)) = \binom{d-1}{2} - \rho_a(C)$.

Proof. The Hilbert function of I_C is decomposed by the partial elimination ideals $K_i(I_C)$ as follows

$$H(R/I_C, m) = \sum_{i=0}^{\infty} H\big(\bar{R}/K_i(C), m-i\big). \tag{1}$$

This comes from the following combinatorial identity

$$\binom{m+d}{d} = \sum_{i=0}^{m} \binom{m-i+d-1}{d-1}.$$

By Remark 3.1(a), we know that there is no trisecant line to C passing through a general point of $\mathbb{P}^3$. This means that the zero locus of $K_i(I_C)$ is empty for $i \geqslant 2$ by Proposition 2.4. So, $H(\bar{R}/K_i(C), m) = 0$ for $m \gg 0$ and $i \geqslant 2$. Thus, the equality (1) can be reformulated by

$$P_C(m) = P_{\pi(C)}(m) + P_{Z_1(C)}(m-1) \quad \text{for } m \gg 0. \tag{2}$$

Since $\pi(C)$ is a plane curve of degree $d = \deg(C)$ and arithmetic genus $\rho_a(\pi(C)) = \binom{d-1}{2}$, we know that $P_C(m) = dm + 1 - \rho_a(C)$, and $P_{\pi(C)}(m) = dm + 1 - \binom{d-1}{2}$. Consequently,

$$\deg\big(\bar{R}/K_1(I_C)\big) = P_C(m) - P_{\pi(C)}(m) = \binom{d-1}{2} - \rho_a(C). \qquad \square$$

Theorem 3.3. *Let C be a non-degenerate integral curve of degree d and arithmetic genus $\rho_a(C)$ in $\mathbb{P}^3$. Assume that* $\dim \operatorname{Tan}_p(C) = 2$ *for every singular point $p \in C$. Then $K_1(I_C)$ is a radical ideal defining a set of reduced points $Y_1(C)$ of degree $\binom{d-1}{2} - p_a(C)$, which is the number of secant lines through the center of general projection into $\mathbb{P}^2$.*

Proof. Let $\varphi : \widetilde{C} \to C$ be the normalization of C. Then we have the following exact sequence

$$0 \to \mathcal{O}_C \to \varphi_* \mathcal{O}_{\widetilde{C}} \to \sum_{p \in C} (\varphi_* \mathcal{O}_{\widetilde{C}})_p / \mathcal{O}_{C,p} \to 0$$

where $(\varphi_* \mathcal{O}_{\widetilde{C}})_p$ is the integral closure of $\mathcal{O}_{C,p}$. Thus we have the equation

$$\begin{aligned} \sum_{p \in C} \delta_p &= \chi(\varphi_* \mathcal{O}_{\widetilde{C}}) - \chi(\mathcal{O}_C) \\ &= \big(1 - \rho_a(\widetilde{C})\big) - \big(1 - \rho_a(C)\big) \\ &= \rho_a(C) - \rho_a(\widetilde{C}) \end{aligned} \tag{3}$$

where $\delta_p = \text{length}((\varphi_* \mathcal{O}_{\widetilde{C}})_p / \mathcal{O}_{C,p})$. Now consider the following commutative diagram:

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{\varphi} & C \subset \mathbb{P}^3 \\ & \searrow^{\pi'} & \downarrow \pi \\ & & \pi(C) \subset \mathbb{P}^2 \end{array}$$

where $\pi' = \pi \circ \varphi : \widetilde{C} \to \mathbb{P}^2$. The assumption that $\dim \operatorname{Tan}_p(C) = 2$ for every singular point $p \in C$ implies that the generic projection $\pi : C \to \mathbb{P}^2$ gives a local isomorphism around every singular point $p \in C$ and thus we have

$$\begin{aligned} \delta_p &= \operatorname{length}\big((\varphi_* \mathcal{O}_{\widetilde{C}})_p / \mathcal{O}_{C,p}\big) \\ &= \operatorname{length}\big(\big(\pi'_* \mathcal{O}_{\widetilde{C}}\big)_q / \mathcal{O}_{\pi(C),q}\big) = \delta_q \end{aligned}$$

where $q = \pi(p)$. By virtue of Remark 3.1, we see that the fiber of a generic projection of the curve C contains at most a 2-points scheme and thus non-isomorphic points in $\pi(C)$ under a generic projection of C into $\mathbb{P}^2$ are only nodes, whose set is defined by $\sqrt{K_1(I_C)}$. If $q' = \pi(p')$ is such a node then one knows $\delta_{p'} = 0$ and $\delta_{q'} = 1$ since $p' \in C$ is a smooth point and $q' \in \pi(C)$ is a nodal point. Hence we have

$$\deg\big(\bar{R}/\sqrt{K_1(I_C)}\big) = \sum_{q \in \pi(C)} \delta_q - \sum_{p \in C} \delta_p. \tag{4}$$

On the other hand, consider the short exact sequence:

$$0 \to \mathcal{O}_{\pi(C)} \to \pi'_* \mathcal{O}_{\widetilde{C}} \to \sum_{q \in \pi(C)} \big(\pi'_* \mathcal{O}_{\widetilde{C}}\big)_q / \mathcal{O}_{\pi(C),q} \to 0.$$

Then we also obtain the following equation

$$\chi(\mathcal{O}_{\pi(C)}) - \chi\big(\pi'_*(\mathcal{O}_{\widetilde{C}})\big) + \sum_{q \in \pi(C)} \delta_q = 0.$$

Since $\pi(C)$ is a plane curve, it is clear that $\rho_a(\pi(C)) = \binom{d-1}{2}$. Hence,

$$\begin{aligned} \sum_{q \in \pi(C)} \delta_q &= \chi\big(\pi'_*(\mathcal{O}_{\widetilde{C}})\big) - \chi(\mathcal{O}_{\pi(C)}) \\ &= \big(1 - \rho_a(\widetilde{C})\big) - \big(1 - \rho_a\big(\pi(C)\big)\big) \\ &= \binom{d-1}{2} - \rho_a(\widetilde{C}). \end{aligned} \tag{5}$$

So, we have

$$\begin{aligned} \deg\big(\bar{R}/\sqrt{K_1(I_C)}\big) &= \sum_{q \in \pi(C)} \delta_q - \sum_{p \in C} \delta_p \quad \big(\text{by Eq. (4)}\big) \\ &= \binom{d-1}{2} - \rho_a(\widetilde{C}) - \big(\rho_a(C) - \rho_a(\widetilde{C})\big) \quad \big(\text{by Eqs. (3) and (5)}\big) \\ &= \binom{d-1}{2} - \rho_a(C). \end{aligned}$$

We know that $\deg(\bar{R}/K_1(I_C)) = \binom{d-1}{2} - \rho_a(C)$ by Lemma 3.2. Thus we have

$$\deg\big(\bar{R}/\sqrt{K_1(I_C)}\big) = \deg\big(\bar{R}/K_1(I_C)\big).$$

Since $K_1(I_C)$ defines a zero-dimensional scheme, we have $\sqrt{K_1(I_C)} = K_1(I_C)^{\mathrm{sat}}$. Then we conclude that $K_1(I_C)$ is a radical ideal defining a set of points with degree $\binom{d-1}{2} - \rho_a(C)$ since $K_1(I_C)$ is already saturated (see [1, Theorem 4.1]). □

Corollary 3.4. *Let C be a non-degenerate integral curve of degree d and arithmetic genus $\rho_a(C)$ in $\mathbb{P}^3$. Assume that $\delta_p = 1$ for every singular point $p \in C$. Then $K_1(I_C)$ is a radical ideal defining a set $Y_1(C)$ which consists of distinct $\binom{d-1}{2} - p_a(C)$ points.*

Proof. Note that it suffices to show that the condition $\delta_p = 1$ implies $\dim \mathrm{Tan}_p(C) = 2$. Let $\varphi : \widetilde{C} \to C$ be the normalization of C. If $p \in C$ is a singular point then the assumption that $\delta_p = \mathrm{length}(\varphi_* \mathcal{O}_{\widetilde{C}})_p / \mathcal{O}_{C,p} = 1$ implies that:

(a) $\varphi^{-1}(p)$ consists of smooth two points of $\widetilde{C}$.
(b) $(\varphi_* \mathcal{O}_{\widetilde{C}})_p / \mathcal{O}_{C,p}$ is a simple $\mathcal{O}_{C,p}$-module.

Then it follows from [5, Lemma 3.2 (b)] that $\dim \mathrm{Tan}_p(C) = 2$, as we wished. □

Theorem 3.5. *Let I_C be the defining ideal of an integral curve C of degree d in $\mathbb{P}^3$, with* $\dim \mathrm{Tan}_p(C) = 2$ *for every singular point $p \in C$, then*:

(a) $M(I_C) = \max\{d, 1 + \binom{d-1}{2} - \rho_a(C)\}$.
(b) *$M(I_C)$ can be obtained from one of the following two monomial generators*

$$x_1^d, \quad x_0 x_2^{\binom{d-1}{2} - \rho_a(C)}.$$

Proof. Note that by Theorem 3.5 in [1],

$$M(I_C) = \max_{k \geqslant 0}\big\{\mathrm{reg}\big(\mathrm{Gin}\big(K_k(I_C)\big)\big) + k\big\}.$$

Let $s = \max\{d, 1 + \binom{d-1}{2} - \rho_a(C)\}$. Since $K_0(I_C)$ defines a plane curve $\pi(C)$ of degree d and $K_1(I_C)$ defines a set of points of degree $\binom{d-1}{2} - \rho_a(C)$,

$$\mathrm{reg}\big(\mathrm{Gin}\big(K_0(I_C)\big)\big) = d$$

and

$$\mathrm{reg}\big(\mathrm{Gin}\big(K_1(I_C)\big)\big) = \binom{d-1}{2} - \rho_a(C) \quad \text{(Theorem 3.3)}.$$

This means that $M(I_C) \geqslant s$.

Conversely, to prove that $M(I_C) \leqslant s$ it suffices to show that

$$\mathrm{reg}\big(\mathrm{Gin}\big(K_t(I_C)\big)\big) \leqslant s - t \quad \text{for all } t \geqslant 2.$$

Let $\bar{R}_t = \bar{R}/K_t(I_C)$ for each $t \geqslant 0$. We know that $\bar{R}_t$ is an Artinian ring for $t \geqslant 2$ and from the definition of regularity using local cohomology, that $\operatorname{reg}(K_t(I_C)) = \min\{m \mid H(\bar{R}_t, m) = 0\}$. Now, we will prove that if $m \geqslant s$ then $H(\bar{R}_t, m-t) = 0$, for all $t \geqslant 2$. It is enough to show that for all $m \geqslant s$

$$H(R/I, m) = H(\bar{R}_0, m) + H(\bar{R}_1, m-1).$$

By the regularity bound,

$$H(R/I, m) = P_C(m) \quad \text{if } m \geqslant s \geqslant d. \tag{6}$$

Note that $Y_0(C)$ is a plane curve of degree d in $\mathbb{P}^2$ and $Y_1(C)$ is a reduced set of points of degree $\binom{d-1}{2} - \rho_a(C)$.

Thus if $m \geqslant s$ then $m \geqslant \operatorname{reg} Y_i(C)$, $i = 0, 1$, and thus,

$$H(\bar{R}_0, m) = P_{Y_0(C)}(m),$$
$$H(\bar{R}_1, m-1) = P_{Y_1(C)}(m-1) = \binom{d-1}{2} - \rho_a(C).$$

Consequently, we have that if $m \geqslant s$ then

$$\begin{aligned} H(R/I, m) &= P_S(m) = P_{Y_0(C)}(m) + P_{Y_1(C)}(m-1) \\ &= H(\bar{R}_0, m) + H(\bar{R}_1, m-1). \end{aligned}$$

We now prove part (b). Since a generic projection of C is a hypersurface of degree d in $\mathbb{P}^2$, we have that $\operatorname{Gin}(K_0(I_C)) = (x_1^d)$ by the Borel fixed property. Furthermore we can consider all monomial generators of the form $x_0 \cdot h_j(x_1, x_2, x_3)$ in $\operatorname{Gin}(I_C)$. Then, $\{h_j(x_1, x_2, x_3)\}$ is a minimal generating set of $\operatorname{Gin}(K_1(I_C))$ by Proposition 2.5. Recall that $K_1(I_C)$ defines $\binom{d-1}{2} - \rho_a(C)$ distinct nodes in $\mathbb{P}^2$. Thus $\operatorname{Gin}(K_1(I_C))$ should contain the monomial $x_2^{\binom{d-1}{2} - \rho_a(C)}$. Therefore, $\operatorname{Gin}(I_C)$ contains monomials x_1^d, $x_0 x_2^{\binom{d-1}{2} - \rho_a(C)}$. □

Remark 3.6. Let $C \subset \mathbb{P}^r$, $r \geqslant 4$ with $\dim \operatorname{Tan}_p(C) = 2$ for every singular point $p \in C$. Consider the generic projection π_Λ from a generic $(r-4)$-dimensional linear subvariety $\Lambda \subset \mathbb{P}^r$. Since $\dim \operatorname{Tan}_p(C) = 2$ for every singular point $p \in C$ we know that a generic projection $\pi_\Lambda : C \to \mathbb{P}^3$ is an isomorphism and $M(I_C) = M(I_{\pi_\Lambda(C)})$ by Remark 3.1(b). Thus we may assume that $I_{\pi_\Lambda(C)} \subset k[x_{r-3}, \dots, x_r]$ and $M(I_C)$ can be obtained from one of the following two monomial generators

$$x_{r-2}^d, \quad x_{r-3} x_{r-1}^{\binom{d-1}{2} - \rho_a(C)}.$$

Therefore we get the following Corollary 3.7.

Corollary 3.7. *Let I_C be the defining ideal of an integral curve C of degree d in $\mathbb{P}^r$, $r \geqslant 4$ with* $\dim \operatorname{Tan}_p(C) = 2$ *for every singular point $p \in C$, then*:

(a) $M(I_C) = \max\{d, 1 + \binom{d-1}{2} - \rho_a(C)\}$.
(b) *$M(I_C)$ can be obtained from one of the following two monomial generators*

$$x_{r-2}^d, \quad x_{r-3} x_{r-1}^{\binom{d-1}{2} - \rho_a(C)}.$$

Proposition 3.8. *Let C be a non-degenerate integral curve of degree d and arithmetic genus $p_a(C)$ in $\mathbb{P}^r$, $r \geqslant 3$, with* $\dim \mathrm{Tan}_p(C) = 2$ *for every singular point $p \in C$. Then*

$$M(I_C) = \begin{cases} 3 & \text{if } d = 3, \text{ i.e. } C \text{ is a rational curve of minimal degree;} \\ 4 & \text{if } d = 4, \text{ i.e. } C \text{ is of next to minimal degree;} \\ 1 + \binom{d-1}{2} - \rho_a(C) & \text{for } d \geqslant 5. \end{cases}$$

Proof. From Remark 3.6, we can reduce the case of an integral curve C in $\mathbb{P}^3$. By Theorem 3.5,

$$M(I_C) = \mathrm{reg}\big(\mathrm{Gin}_{\mathrm{GLex}}(I_C)\big) = \max\left\{d, 1 + \binom{d-1}{2} - \rho_a(C)\right\}.$$

Applying the genus bound in the Montreal lecture note of Eisenbud and Harris (1982) [11] to a non-degenerate integral curve $C \subset \mathbb{P}^3$, we get

$$\rho_a(C) \leqslant \pi(d,3) = \begin{cases} (\frac{d}{2} - 1)^2 & \text{if } d \text{ is even;} \\ (\frac{d-1}{2})(\frac{d-3}{2}) & \text{if } d \text{ is odd} \end{cases}$$

and for all $d \geqslant 5$, we have the following inequality:

$$\rho_a(C) \leqslant \pi(d,3) \leqslant 1 + \binom{d-1}{2} - d. \tag{7}$$

Thus,

$$d \leqslant 1 + \binom{d-1}{2} - \rho_a(C)$$

and by Theorem 3.5, for $d \geqslant 5$,

$$M(I_C) = 1 + \binom{d-1}{2} - \rho_a(C).$$

For special two cases of $d = 3$ and $d = 4$, it is very easy to compute $M(I_C)$.

If $d = 3$ then C is a rational normal curve and $1 + \binom{d-1}{2} - \rho_a(C) = 2 < 3 = \deg(C)$. Therefore, $M(I_C) = 3$. On the other hand, when $d = 4$, we get the inequality $\rho_a(C) \leqslant \pi(4,3) = 1$. Since $1 + \binom{d-1}{2} - \rho_a(C) = 3$ or 4, we have $M(I_C) = 4$. □

Finally, we provide some illuminating examples of our results. These are based on computations in *Macaulay* 2 and `Singular` [10,16]. Note that several generic initial ideals computed in the following examples are very likely the real Gin.

Example 3.9. (`Singular` [10], *Macaulay* 2 [16].) We revisit the example [9, Example 4.3] introduced by A. Conca and J. Sidman. $I_C = (x^3 - yz^2, y^3 - z^2t)$ defines an irreducible complete intersection curve C of the arithmetic genus $\rho_a(C) = 10$ in $\mathbb{P}^3$ with only one singular point $q = [0,0,0,1]$. Note that this singular point is neither node nor ordinary cusp and $\delta_q = 10$. We can compute the defining ideal of the normalization of a curve C and delta invariant δ_q using `Singular`. Furthermore, since $\dim \mathrm{Tan}_q(C) = 3$, $\pi(q)$ is contained in the zero locus of $K_1(I_C)$. Thus we cannot apply our results. In fact, $\mathrm{Gin}(K_1(I_C))$ is

$$\big(y^4, y^3z^2, y^2z^5, yz^8, \mathbf{z^{15}}, y^2z^4t, y^3zt^2, y^2z^3t^2, yz^7t^2,$$
$$y^3t^3, y^2z^2t^4, yz^6t^4, y^2zt^5, yz^5t^6, y^2t^7, yz^4t^8, yz^3t^{10}\big).$$

Therefore, $M(I_C) = 1 + M(K_1(I_C)) = 16$ which is not equal to $1 + \binom{9-1}{2} - \rho_a(C) = 1 + 28 - 10 = 19$.

Example 3.10. (Singular [10], *Macaulay* 2 [16].) Consider the ideal $I_C = (x^4 - yz^3, y^2 - zt) \subset k[x, y, z, t]$. This defines an irreducible complete intersection curve C of $\rho_a(C) = 10$ in $\mathbb{P}^3$ with one singular point $q = [0, 0, 0, 1]$. The delta invariant δ_q is 9 by Singular. Since $\dim \operatorname{Tan}_q(C) = 2$, we can compute by our formula, $M(I_C) = 1 + \binom{8-1}{2} - 9 = 13$. In fact, $\operatorname{Gin}(I)$ is

$$\big(x^2, xy^3, y^8, xy^2z^2, xyz^5, \mathbf{xz^{12}}, xy^2zt^2, xyz^4t^2, xy^2t^4, xyz^3t^4, xyz^2t^6, xyzt^8, xyt^{10}\big).$$

Example 3.11. (*Macaulay* 2 [16].) Let C be a rational normal curve in $\mathbb{P}^5$ and C_1 be a projection curve in $\mathbb{P}^4$ with center $q \in \operatorname{Sec}(C) \setminus C$. Then C_1 has one singular point as a node. Consider a singular curve C_2 in $\mathbb{P}^3$ which is a generic isomorphic projection of C_1. In fact, C_2 is a singular curve of degree 5 and the arithmetic genus $\rho_a(C_2) = 1$. Thus,

$$M(I_{C_1}) = M(I_{C_2}) = 1 + \binom{d-1}{2} - \rho_a(C_2) = 6.$$

On the other hand, we can compute $\operatorname{Gin}(I_{C_2})$ using *Macaulay* 2.

$$\operatorname{Gin}(I_{C_2}) = \big(x_2^3, x_2^2x_3, x_2x_3^3, x_3^5, x_2^2x_4, x_2x_3x_4^2, \mathbf{x_2x_4^5}, x_2x_3x_4x_5, x_2x_3x_5^2\big).$$

Remark 3.12. Let X be an irreducible reduced projective variety of dimension n and codimension two. It is still open to compute or estimate $M(I_X)$ for $\dim(X) \geqslant 2$ (cf. [2]). However, if X is smooth or has mild singularities, then it is expected that $M(I_X)$ is determined by the degree complexity of the double point locus under a generic projection. Thus, by the induction on the dimension of the double point locus, we expect asymptotically that

$$M(I_X) \sim 2(d/2)^{2^n}.$$

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