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On syzygies, degree, and geometric properties of projective schemes with property $\mathbf{N}_{3,p}$

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A R T I C L E I N F O

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A B S T R A C T

Let X be a reduced, but not necessarily irreducible closed subscheme of codimension e in a projective space. One says that X satisfies property $\mathbf{N}_{d,p}$ $(d \geq 2)$ if the i-th syzygies of the homogeneous coordinate ring are generated by elements of degree $< d+i$ for $0 \leq i \leq p$ (see [10] for details). Much attention has been paid to linear syzygies of quadratic schemes $(d = 2)$ and their geometric interpretations (cf. [1,9,15–17]). However, not very much is actually known about algebraic sets satisfying property $\mathbf{N}_{d,p}$, $d \geq 3$. Assuming property $\mathbf{N}_{d,e}$, we give a sharp upper bound $\deg(X) \leq \binom{e+d-1}{d-1}$. It is natural to ask whether $\deg(X) = \binom{e+d-1}{d-1}$ implies that X is arithmetically Cohen–Macaulay (ACM) with a d-linear resolution. In case of $d = 3$, by using the elimination mapping cone sequence and the generic initial ideal theory, we show that $\deg(X) = \binom{e+2}{2}$ if and only if X is ACM with a 3-linear resolution. This is a generalization of the results of Eisenbud et al. $(d = 2)$ [9,10]. We also give more general inequality concerning the length of the finite intersection of X with a linear space of not necessary complementary dimension in terms of graded Betti numbers. Concrete examples are given to explain our results.

1. Introduction

In this paper we study geometric properties of projective algebraic sets (always reduced, but not necessarily irreducible) that follow from certain vanishing assumptions on their syzygies.

Let $R = k[x_0, \cdots, x_{n+e}]$ denote the homogeneous coordinate ring of the projective space $\mathbb{P}^{n+e}$ over an algebraically closed field k of characteristic zero, and let $I_X \subset R$ denote the homogeneous ideal of an algebraic set $X \subset \mathbb{P}^{n+e}$. The syzygy modules $B_{i,j}$ are defined by

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$$B_{i,j} = \mathrm{Tor}_i^R(R/I_X, k)_{i+j},$$

and the dimension of these modules is the Betti number $\beta_{i,j}(X) = \dim_k(B_{i,j})$. One says that X satisfies property $\mathbf{N}_{d,p}$ $(p \leq \infty)$ if

$$\beta_{i,j}(X) = 0 \quad \text{for } i \leq p \text{ and } j \geq d.$$

So, property $\mathbf{N}_{d,\infty}$ means that X is d-regular. One of the main results is as follows:

Theorem 1.1. *Let $X \subset \mathbb{P}^{n+e}$ be a non-degenerate algebraic set of dimension n. Suppose that X satisfies $\mathbf{N}_{d,e}$. Then we have*

$$\deg(X) \leq \binom{e+d-1}{d-1}.$$

There are many examples of algebraic sets satisfying the equality in the above theorem: take for instance X to be the algebraic set defined by the ideal of maximal minors of a 1-generic $d \times (e+d-1)$ matrix of linear forms (for an even more concrete example, take X to be the $(d-1)$-secant variety of a rational normal curve of degree $(e+2d-3)$; see [5, Chapter 6]).

All these examples have the property that the only non-zero Betti numbers are $\beta_{0,0}(X)$ and $\beta_{i,d-1}(X)$ for $i = 1, 2, \cdots, e$: in this case one says that X is arithmetically Cohen–Macaulay (ACM) with a d-linear resolution. It is then natural to ask

Question 1. If X is as in Theorem 1.1 with $\deg(X) = \binom{e+d-1}{d-1}$, is X necessarily ACM with a d-linear resolution?

When $d = 3$, we give an affirmative answer to this question in this paper. The extremal cases can be characterized by the combinatorial property of the syzygies of generic initial ideals.

Theorem 1.2. *Let $X \subset \mathbb{P}^{n+e}$ be a non-degenerate algebraic set of dimension n. Suppose that X satisfies $\mathbf{N}_{3,e}$. Then $\deg(X) = \binom{e+2}{2}$ if and only if X is ACM with a 3-linear resolution.*

In the case of $d = 2$, it is shown in [10, Corollary 1.8] that the condition $\mathbf{N}_{2,e}$ implies that X is 2-regular, and since X is non-degenerate, it must have a 2-linear resolution; combining this with [7, Corollary 1.11], it follows that if in addition $\deg(X) = 1 + e$, then X is ACM, so Question 1 has a positive answer when $d = 2$ as well. However, the question remains still open for $d > 3$.

In the case of $d = 3$, we prove a more general inequality than in Theorem 1.1, concerning the length of the finite intersection of X with a linear space of not necessarily complementary dimension:

Theorem 1.3. *Assume that $X \subset \mathbb{P}^{n+e}$ is a non-degenerate algebraic set of dimension n and satisfies $\mathbf{N}_{2,p}$ for some $p \geq 0$. If $\alpha \leq e$ is such that X satisfies $\mathbf{N}_{3,\alpha}$, and $L^\alpha \subset \mathbb{P}^{n+e}$ is a linear space of dimension α whose intersection with X is 0-dimensional, then*

$$\mathrm{length}\big(X \cap L^\alpha\big) \leq 1 + \alpha + \min\left\{\frac{|\alpha - p|(\alpha + p + 1)}{2}, \beta^R_{\alpha,2}(X)\right\}.$$

In the case $\alpha \leq p$, $\beta_{\alpha,2}(X) = 0$, the inequality in Theorem 1.3 becomes $\mathrm{length}(X \cap L^\alpha) \leq 1 + \alpha$, which also follows from [10, Theorem 1.1].

To achieve the result, we use the elimination mapping cone construction for graded modules and apply it to give a systematic approach to the relation between multisecants and graded Betti numbers. We also provide some illuminating examples of our main results via calculations done with *Macaulay 2* [13]. For instance, an example (suggested by F.-O. Schreyer) is given to show that condition $\mathbf{N}_{d,e}$ does not imply d-regularity in general (see Example 3.11).

2. Preliminaries

2.1. Notations and definitions

For precise statements, we begin with notations and definitions used in the subsequent sections:

- We work over an algebraically closed field k of characteristic zero.
- Unless otherwise stated, X is a non-degenerate reduced, but not necessarily irreducible closed subscheme of dimension n and codimension e in $\mathbb{P}^{n+e}$.
- For a finitely generated graded $R = k[x_0, x_1, \ldots, x_{n+e}]$-module $M = \bigoplus_{\nu \geq 0} M_\nu$, consider a minimal free resolution of M:

$$\cdots \to \oplus_j R(-i-j)^{\beta^R_{i,j}(M)} \to \cdots \to \oplus_j R(-j)^{\beta^R_{0,j}(M)} \to M \to 0$$

  where $\beta^R_{i,j}(M) := \dim_k \operatorname{Tor}^R_i(M, k)_{i+j}$. We write $\beta^R_{i,j}(M)$ as $\beta^R_{i,j}$ if it is obvious. We define the regularity of M as follows:

$$\operatorname{reg}_R(M) := \max\{j \mid \beta^R_{i,j}(M) \neq 0 \text{ for some } i\}$$

  In particular, we define the regularity of X as $\operatorname{reg}_R(I_X)$.
- The regularity has an alternate description in terms of cohomology. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{n+e}$ is said to be m-regular if $H^i(\mathbb{P}^{n+e}, \mathcal{F}(m-i)) = 0$ for all $i > 0$; the regularity $\operatorname{reg}_R(\mathcal{F})$ (in the sense of Castelnuovo–Mumford) is the smallest such m.
  In particular, if I is a saturated ideal, m-regularity of I as a homogeneous ideal is equivalent to the geometric condition that the associated ideal sheaf $\mathcal{I}$ on projective space $\mathbb{P}^{n+e}$ satisfies the condition of Castelnuovo–Mumford m-regularity, i.e. $\operatorname{reg}(I) = \operatorname{reg}(\mathcal{I})$.
- For an algebraic set X in $\mathbb{P}^{n+e}$, one says that X is m-normal if $H^1(\mathbb{P}^{n+e}, \mathcal{I}_X(m-1)) = 0$.
- One says that M satisfies *property* $\mathbf{N}^R_{d,\alpha}$ if $\beta^R_{i,j}(M) = 0$ for all $j \geq d$ and $0 \leq i \leq \alpha$ (see [16], [17]). We can also think of M as a graded $S_t = k[x_t, \ldots, x_{n+e}]$-module by an inclusion map $S_t \hookrightarrow R$. As a graded S_t-module, we say that M satisfies *property* $\mathbf{N}^{S_t}_{d,\alpha}$ if $\beta^{S_t}_{i,j}(M) := \dim_k \operatorname{Tor}^{S_t}_i(M, k)_{i+j} = 0$ for all $j \geq d$ and $0 \leq i \leq \alpha$.

2.2. Elimination mapping cone construction

For a graded R-module M, consider the natural multiplicative $S_1 = k[x_1, x_2, \ldots, x_{n+e}]$-module map $\varphi : M(-1) \xrightarrow{\times x_0} M$ such that $\varphi(m) = x_0 \cdot m$ and the induced map on the graded Koszul complex of M over S_1:

$$\overline{\varphi} : \mathbb{F}_\bullet = K^{S_1}_\bullet\big(M(-1)\big) \xrightarrow{\times x_0} \mathbb{G}_\bullet = K^{S_1}_\bullet(M).$$

Then, we have the mapping cone $(C_\bullet(\overline{\varphi}), \partial_{\overline{\varphi}})$ such that $C_\bullet(\overline{\varphi}) = \mathbb{G}_\bullet \oplus \mathbb{F}_\bullet[-1]$, and $W = \langle x_1, x_2, \ldots, x_n \rangle$;

- $C_i(\overline{\varphi})_{i+j} = [\mathbb{G}_i]_{i+j} \oplus [\mathbb{F}_{i-1}]_{i+j} = (\wedge^i W \otimes M_j) \oplus (\wedge^{i-1} W \otimes M_j)$.
- The differential $\partial_{\overline{\varphi}} : C_i(\overline{\varphi}) \to C_{i-1}(\overline{\varphi})$ is given by

$$\partial_{\overline{\varphi}} = \begin{pmatrix} \partial & \overline{\varphi} \\ 0 & -\partial \end{pmatrix},$$

where ∂ is the differential of Koszul complex $K_\bullet^{S_1}(M)$.

From the exact sequence of complexes

$$0 \longrightarrow \mathbb{G}_\bullet \longrightarrow C_\bullet(\overline{\varphi}) \longrightarrow \mathbb{F}_\bullet[-1] \longrightarrow 0 \tag{1}$$

and the natural isomorphism $H_i(C_\bullet(\overline{\varphi}))_{i+j} \simeq \mathrm{Tor}_i^R(M,k)_{i+j}$ (cf. Lemma 3.1 in [1]), we have the following long exact sequence in homology.

Theorem 2.1 *(Theorem 3.2 in [1]). For a graded R-module M, there is a long exact sequence:*

$$\begin{aligned} &\longrightarrow \mathrm{Tor}_i^{S_1}(M,k)_{i+j} \longrightarrow \mathrm{Tor}_i^R(M,k)_{i+j} \longrightarrow \mathrm{Tor}_{i-1}^{S_1}(M,k)_{i-1+j} \longrightarrow \\ &\xrightarrow{\delta=\times x_0} \mathrm{Tor}_{i-1}^{S_1}(M,k)_{i-1+j+1} \longrightarrow \mathrm{Tor}_{i-1}^R(M,k)_{i-1+j+1} \longrightarrow \mathrm{Tor}_{i-2}^{S_1}(M,k)_{i-2+j+1} \end{aligned}$$

whose connecting homomorphism δ is the multiplicative map $\times x_0$.

Corollary 2.2. *Let M be a finitely generated graded R-module and also finitely generated as an S_1-module. Then,*

$$\mathrm{proj.dim}_{S_1}(M) = \mathrm{proj.dim}_R(M) - 1.$$

Proof. Let $\ell = \mathrm{proj.dim}_R(M)$. Thus, $\beta^R_{\ell+1,j}(M) = 0$ for all $j \geq 1$ and the following map $\delta = \times x_0$ is injective for all $j \geq 1$:

$$0 = \mathrm{Tor}_{\ell+1}^R(M,k)_{\ell+1+j} \to \mathrm{Tor}_\ell^{S_1}(M,k)_{\ell+j} \xrightarrow{\delta=\times x_0} \mathrm{Tor}_\ell^{S_1}(M,k)_{\ell+j+1}.$$

But, $\mathrm{Tor}_\ell^{S_1}(M,k)_{\ell+j+1} = 0$ for $j \gg 0$ due to the finiteness of M (as an S_1-module). Therefore, $\mathrm{Tor}_\ell^{S_1}(M,k)_{\ell+j} = 0$ for all $j \geq 1$. On the other hand, $\beta^R_{\ell,j_*}(M) \neq 0$ for some $j_* > 0$. So,

$$0 = \mathrm{Tor}_\ell^{S_1}(M,k)_{\ell+j_*} \to \mathrm{Tor}_\ell^R(M,k)_{\ell+j_*} \to \mathrm{Tor}_{\ell-1}^{S_1}(M,k)_{\ell-1+j_*}$$

is injective and $\beta^{S_1}_{\ell-1,j_*}(M) \neq 0$. Consequently, we get

$$\mathrm{proj.dim}_{S_1}(M) = \mathrm{proj.dim}_R(M) - 1,$$

as we wished. □

Proposition 2.3. *Let M be a finitely generated graded R-module satisfying property $\mathbf{N}^R_{d,\alpha}$ ($\alpha \geq 1$). If M is also finitely generated as an S_1-module, then we have the following:*

(a) *M satisfies property $\mathbf{N}^{S_1}_{d,\alpha-1}$. In particular, $\mathrm{reg}_{S_1}(M) = \mathrm{reg}_R(M)$.*
(b) *$\beta^{S_1}_{i-1,d-1}(M) \leq \beta^R_{i,d-1}(M)$ for $1 \leq i \leq \alpha$.*

Proof. Suppose that M satisfies $\mathbf{N}^R_{d,\alpha}$ ($\alpha \geq 1$) and let $1 \leq i \leq \alpha$ and $j \geq d$.

(a): Consider the exact sequence from Theorem 2.1

$$\cdots \to \mathrm{Tor}_i^R(M,k)_{i+j} \to \mathrm{Tor}_{i-1}^{S_1}(M,k)_{i-1+j} \xrightarrow{\delta=\times x_0} \mathrm{Tor}_{i-1}^{S_1}(M,k)_{i-1+j+1} \to \mathrm{Tor}_{i-1}^R(M,k)_{i-1+j+1} \to \cdots$$

By the property $N_{d,\alpha}^R$, we see that $\mathrm{Tor}_i^R(M,k)_{i+j} = 0$. Hence we obtain an isomorphism

$$\mathrm{Tor}_{i-1}^{S_1}(M,k)_{(i-1)+j} \xrightarrow{\delta=\times x_0} \mathrm{Tor}_{i-1}^{S_1}(M,k)_{i+j}.$$

By the assumption that M is a finitely generated S_1-module, we conclude as in the proof of Corollary 2.2 that $\mathrm{Tor}_{i-1}^{S_1}(M,k)_{(i-1)+j} = 0$ for $1 \le i \le \alpha$ and $j \ge d$. Hence M satisfies $\mathbf{N}_{d,\alpha-1}^{S_1}$.

If $\alpha = \infty$, we have that $\mathrm{reg}_{S_1}(M) \le \mathrm{reg}_R(M)$. Conversely, if $m \ge \mathrm{reg}_{S_1}(M)$ then it follows from the following exact sequence

$$\cdots \xrightarrow{\delta=\times x_0} \mathrm{Tor}_i^{S_1}(M,k)_{i+m} \to \mathrm{Tor}_i^R(M,k)_{i+m} \to \mathrm{Tor}_{i-1}^{S_1}(M,k)_{i-1+m+1} = 0$$

that $\mathrm{reg}_{S_1}(M) \ge \mathrm{reg}_R(M)$.

(b): Note that we have the following surjection map for $1 \le i \le \alpha$

$$\mathrm{Tor}_i^R(M,k)_{i+d-1} \to \mathrm{Tor}_{i-1}^{S_1}(M,k)_{i-1+d-1} \xrightarrow{\delta=\times x_0} \mathrm{Tor}_{i-1}^{S_1}(M,k)_{i-1+d} = 0,$$

which is obtained from Theorem 2.1. This implies that for $1 \le i \le \alpha$

$$\beta_{i-1,d-1}^{S_1}(M) \le \beta_{i,d-1}^R(M)$$

as we wished. □

From Proposition 2.3(b), one obtains immediately the following result.

Corollary 2.4. *Let M be a finitely generated graded R-module satisfying property $\mathbf{N}_{d,\alpha}^R$ for some $\alpha \ge 1$. If M is also finitely generated as an $S_t = k[x_t, x_{t+1}, \ldots, x_{n+e}]$-module for every $1 \le t \le \alpha$ then M satisfies property $\mathbf{N}_{d,\alpha-t}^{S_t}$. Moreover, in the strand of $j = d-1$, we have the inequality*

$$\beta_{0,d-1}^{S_\alpha} \le \beta_{1,d-1}^{S_{\alpha-1}} \le \cdots \le \beta_{\alpha-1,d-1}^{S_1} \le \beta_{\alpha,d-1}^R.$$

Let Λ be a linear subvariety in $\mathbb{P}^{n+e}$ with homogeneous coordinates $x_0, \ldots, x_{t-1}$ and let $W = \langle x_0, \ldots, x_{t-1} \rangle$ be a vector space. Consider a projection of X from the center Λ

$$\pi_\Lambda : X \to \mathbb{P}^{n+e-t} = \mathbb{P}(W).$$

We say that π_Λ is an outer projection if $X \cap \Lambda = \emptyset$. The most interesting case for us is a projective coordinate ring $M = R/I_X$ of an algebraic set X. In this case, the elimination mapping cone theorem is naturally associated to outer projections of $X \subset \mathbb{P}^n$. Our starting point is to understand some algebraic and geometric information on X via the relation between $\mathrm{Tor}_i^R(R/I_X, k)$ and $\mathrm{Tor}_i^{S_\alpha}(R/I_X, k)$.

Let X be a non-degenerate algebraic set of dimension n in $\mathbb{P}^{n+e}$. Let $\Lambda = \mathbb{P}^{\alpha-1}$ be an $(\alpha-1)$-dimensional linear subspace with homogeneous coordinates $x_0, \ldots, x_{\alpha-1}$ ($\alpha \le e$) such that $\Lambda \cap X$ is empty. Then each point $q_i = [0:0:\cdots:1:\cdots:0]$ whose i-th coordinate is 1 is not contained in X for $0 \le i \le \alpha-1$. Therefore, there is a homogeneous polynomial $f_i \in I_X$ of the form $x_i^{m_i} + g_i$ where $g_i \in R = k[x_0, x_1, \ldots, x_{n+e}]$ is a homogeneous polynomial of degree m_i with the power of x_i less than m_i. Therefore, R/I_X is a finitely generated $S_\alpha = k[x_\alpha, x_{\alpha+1}, \ldots, x_{n+e}]$-module with monomial generators

$$\{x_0^{j_0}x_1^{j_1}\ldots x_{\alpha-1}^{j_{\alpha-1}} \mid 0 \le j_k < m_k, 0 \le k \le \alpha-1\}.$$

Note that the above generating set is not minimal. If X satisfies $\mathbf{N}^R_{d,\alpha}$ then X also satisfies $\mathbf{N}^{S_\alpha}_{d,0}$. This implies that R/I_X is generated in degree $< d$ as an S_α-module and thus $\beta^{S_\alpha}_{0,i} \le \binom{\alpha-1+i}{i}$ for $0 \le i \le d-1$. To sum up, we have the following corollary.

Corollary 2.5. *Let X be a non-degenerate algebraic set of dimension n in $\mathbb{P}^{n+e}$ and let $\Lambda = \mathbb{P}^{\alpha-1}$ be an $(\alpha-1)$-dimensional linear subspace with homogeneous coordinates $x_0, \ldots, x_{\alpha-1}$ ($\alpha \le e$) such that $\Lambda \cap X$ is empty. Suppose X satisfies the property $\mathbf{N}^R_{d,\alpha}$ and consider the following minimal free resolution of R/I_X as a graded $S_\alpha = k[x_\alpha, \ldots, x_{n+e}]$-module:*

$$\cdots \to F_1 \to F_0 \to R/I_X \to 0.$$

(a) *R/I_X satisfies the property $\mathbf{N}^{S_\alpha}_{d,0}$ as an S_α-module;*
(b) *The Betti numbers of F_0 satisfy the following:*
  (i) *$\beta^{S_\alpha}_{0,0} = 1$, $\beta^{S_\alpha}_{0,1} = \alpha$, and $\beta^{S_\alpha}_{0,i} \le \binom{\alpha-1+i}{i}$ for $2 \le i \le d-1$;*
  (ii) *Furthermore, $\beta^{S_\alpha}_{0,d-1} \le \beta^{S_{\alpha-1}}_{1,d-1} \le \cdots \le \beta^{S_1}_{\alpha-1,d-1} \le \beta^R_{\alpha,d-1}$.*
(c) *When $\alpha = e$, R/I_X is a free S_e-module if and only if X is arithmetically Cohen–Macaulay. In this case, letting $d = \operatorname{reg}(X)$,*

$$R/I_X = S_e \oplus S_e(-1)^e \oplus \cdots \oplus S_e(-d+1)^{\beta^{S_e}_{0,d-1}}$$

and $\pi_{\Lambda *}\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^e \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(-d+1)^{\beta^{S_e}_{0,d-1}}$.

Proof. Note that $\binom{\alpha-1+i}{i}$ is the dimension of the vector space of all homogeneous polynomials of degree i in $k[x_0, \ldots, x_{\alpha-1}]$ defining $\Lambda = \mathbb{P}^{\alpha-1}$. Since X is non-degenerate, $\{x_i \mid 0 \le i \le \alpha-1\}$ is contained in the minimal generating set of R/I_X as an S_α-module. So, $\beta^{S_\alpha}_{0,1} = \alpha$. The remaining part of (b) is given by Proposition 2.3 and the argument is given in Corollary 2.4 below.

For a proof of (c), first note that by Corollary 2.2 and Proposition 2.3,

$$\text{proj.dim}_{S_e}(R/I_X) = \text{proj.dim}_R(R/I_X) - e$$
$$\text{reg}_{S_e}(R/I_X) = \text{reg}_R(R/I_X).$$

Consequently, R/I_X is a free S_e-module if and only if $\text{proj.dim}_R(R/I_X) = e$, as we wished. □

Remark 2.6. If a reduced algebraic set X is arithmetically Cohen–Macaulay, then it is locally Cohen–Macaulay, equidimensional and connected in codimension one. Furthermore, as shown in Corollary 2.5,

$$\pi_{\Lambda *}\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^e \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(-d+1)^{\beta^{S_e}_{0,d-1}}.$$

However, in general, if X is locally Cohen–Macaulay and equidimensional, then $\pi_{\Lambda *}\mathcal{O}_X$ is a vector bundle of rank $r = \deg(X)$ because the map is flat (see [4, Exercise 18.17]). Furthermore, by the well-known splitting criterion due to Horrocks or Evans and Griffith [8,14], $\pi_{\Lambda *}\mathcal{O}_X$ is a direct sum of line bundles if and only if $H^i(\mathbb{P}^n, \pi_{\Lambda *}\mathcal{O}_X(j)) = H^i(X, \mathcal{O}_X(j)) = 0$ for all $1 \le i \le n-1$, $\forall j \in \mathbb{Z}$. This condition is weaker than arithmetically Cohen–Macaulayness.

Example 2.7 *(Macaulay 2 [13]).* For one's familiarity with these topics, we show the simplest examples in the following table: Let $\Lambda = \mathbb{P}^{i-1}$ be a general linear subspace with coordinates $x_0, \cdots, x_{i-1}$ and R/I is an $S_i = k[x_i, \cdots, x_{n+e}]$-module. Note that by Corollary 2.2 and Proposition 2.3,

$$\text{proj.dim}_{S_i}(R/I_X) = \text{proj.dim}_R(R/I_X) - i \quad \text{and} \quad \text{reg}_{S_i}(R/I_X) = \text{reg}_R(R/I_X).$$

	R-modules					S_1-modules				S_2-modules		
		0	1	2	3		0	1	2		0	1
A rational normal curve $C \subset \mathbb{P}^4$ in generic coordinates	0	1	0	0	0	0	1	0	0	0	1	0
	1	0	6	8	3	1	1	5	3	1	2	3
		0	1	2			0	1			0	
A generic complete intersection $S \subset \mathbb{P}^4$ of quadric and cubic	0	1	0	0		0	1	0		0	1	
	1	0	1	0		1	1	0		1	2	
	2	0	1	0		2	0	1		2	2	
	3	0	0	1		3	0	1		3	1	
		0	1	2			0	1			0	
The secant variety of a rational normal curve $\text{Sec}(C) \subset \mathbb{P}^5$ in generic coordinates	0	1	0	0		0	1	0		0	1	
	1	0	0	0		1	1	0		1	2	
	2	0	4	3		2	1	3		2	3	

In generic coordinates, the Betti table for R/I as an S_i-module can be computed with Macaulay 2 [13] as follows:

```
minresS = (I,i) -> (
        R := ring I;
        n := # gens R;
        RtoR := map(R,R,random(R^{0}, R^{numgens R:-1}));
        S := (coefficientRing R)[apply(n-i, j -> (gens R)#(j+i))];
        F := map(R,S);
        use R;
        betti res pushForward(F, coker gens RtoR I)
        );
```

3. Syzygetic properties of algebraic sets satisfying property $\mathbf{N}_{d,e}$

For an algebraic set X of dimension n in $\mathbb{P}^{n+e}$ satisfying property $\mathbf{N}_{2,p}$, it is proved by Eisenbud et al. in [10] that if Λ is a linear space of dimension $\leq p$ which intersects X in a finite scheme, then the length of the intersection is at most $\dim(\Lambda) + 1$. In addition, it is known that X satisfies property $\mathbf{N}_{2,e}$ if and only if X is an ACM scheme with 2-linear resolution. In this section, we generalize these results to the case of $\mathbf{N}_{d,\alpha}$ ($d \geq 3$ and $\alpha \leq e$). Theorem 1.1 gives us a sharp upper bound on the degree of X when X satisfies property $\mathbf{N}_{d,e}$. One might ask whether the equality holds if and only if X is an arithmetically Cohen–Macaulay scheme with d-linear resolution. In the case when $d = 3$, Theorem 1.2 gives an affirmative answer to this question. Theorem 1.3 gives a more general inequality than in Theorem 1.1, concerning the length of the finite intersection of X with a linear space of not necessarily complementary dimension.

3.1. The proof of Theorem 1.1

Let X be a non-degenerate algebraic set of dimension n in $\mathbb{P}^{n+e}$. Let $\Lambda = \mathbb{P}^{\alpha-1}$ be an $(\alpha-1)$-dimensional linear subspace with homogeneous coordinates $x_0, \dots, x_{\alpha-1}$ ($\alpha \leq e$) such that $\Lambda \cap X$ is empty. Suppose X satisfies the property $\mathbf{N}^R_{d,\alpha}$. Consider the minimal free resolution of R/I_X as a graded $S_\alpha = k[x_\alpha, \dots, x_{n+e}]$-module

$$\cdots \to S_\alpha \oplus S_\alpha(-1)^\alpha \oplus S_\alpha(-2)^{\beta^{S_\alpha}_{0,2}} \oplus \cdots \oplus S_\alpha(-d+1)^{\beta^{S_\alpha}_{0,d-1}} \to R/I_X \to 0. \tag{2}$$

Sheafifying the sequence (2), we have the following surjective morphism

$$\cdots \to \mathcal{O}_{\mathbb{P}^{n+e-\alpha}} \oplus \mathcal{O}_{\mathbb{P}^{n+e-\alpha}}(-1)^{\alpha} \oplus \mathcal{O}_{\mathbb{P}^{n+e-\alpha}}(-2)^{\beta_{0,2}^{S_\alpha}} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{n+e-\alpha}}(-d+1)^{\beta_{0,d-1}^{S_\alpha}} \stackrel{\widetilde{\varphi_\alpha}}{\to} \pi_{\Lambda_*}\mathcal{O}_X \to 0.$$

For any point $q \in \pi_\Lambda(X)$, note that $\pi_{\Lambda_*}\mathcal{O}_X \otimes k(q) \simeq H^0(\langle \Lambda, q\rangle, \mathcal{O}_{\pi_\Lambda{}^{-1}(q)})$. Thus, by tensoring $\mathcal{O}_{\mathbb{P}^{n+e-\alpha}}(d-1) \otimes k(q)$ on both sides, we have the surjection on vector spaces:

$$\big[\mathcal{O}_{\mathbb{P}^{n+e-\alpha}}(d-1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{n+e-\alpha}}(1)^{\beta_{0,d-2}^{S_\alpha}} \oplus \mathcal{O}_{\mathbb{P}^{n+e-\alpha}}^{\beta_{0,d-1}^{S_\alpha}}\big] \otimes k(q) \twoheadrightarrow H^0\big(\langle \Lambda, q\rangle, \mathcal{O}_{\pi_\Lambda{}^{-1}(q)}(d-1)\big) \to 0$$

where $[\mathcal{O}_{\mathbb{P}^{n+e-\alpha}}(d-1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{n+e-\alpha}}(1)^{\beta_{0,d-2}^{S_\alpha}} \oplus \mathcal{O}_{\mathbb{P}^{n+e-\alpha}}^{\beta_{0,d-1}^{S_\alpha}}] \otimes k(q) \subset H^0(\langle \Lambda, q\rangle, \mathcal{O}_{\langle \Lambda, q\rangle}(d-1))$. This implies that $\pi_\Lambda^{-1}(q) = \langle \Lambda, q\rangle \cap X$ is d-regular. Moreover, since we have $\beta_{0,i}^{S_\alpha} \leq \binom{\alpha-1+i}{i}$ for $0 \leq i \leq d-1$ from Corollary 2.5(b), the length of any fiber of π_Λ satisfies the following inequality:

$$\operatorname{length}\big(\langle \Lambda, q\rangle \cap X\big) \leq 1 + \alpha + \sum_{i=2}^{d-1} \beta_{0,i}^{S_\alpha} \leq \sum_{i=0}^{d-1} \binom{\alpha-1+i}{i} = \binom{\alpha+d-1}{d-1}. \tag{3}$$

Now we are ready to prove Theorem 1.1.

The proof of Theorem 1.1. Suppose that $L^\alpha \subset \mathbb{P}^{n+e}$ is a linear space of dimension α ($\alpha \leq e$) whose intersection with X is zero-dimensional. Choose a linear subspace $\Lambda \subset L^\alpha$ of dimension $\alpha - 1$ such that $X \cap \Lambda = \emptyset$. Consider a projection $\pi_\Lambda : X \to \pi_\Lambda(X) \subset \mathbb{P}^{n+e-\alpha}$ and regard $L^\alpha \cap X$ as a fiber of π_Λ at the point $\pi_\Lambda(L^\alpha \setminus \Lambda) \in \pi_\Lambda(X)$. Then it follows from (3) that

$$\operatorname{length}\big(X \cap L^\alpha\big) \leq \binom{\alpha+d-1}{d-1}.$$

In particular, when $\alpha = e$, if L^e is a general linear space then we have

$$\deg(X) \leq \binom{e+d-1}{d-1}, \tag{4}$$

which completes the proof. □

The bound in (4) is sharp because if M is a 1-generic matrix of size $d \times t$ for $t \geq d$ then the determinantal variety X defined by maximal minors of M achieves this degree bound. In this case, the minimal free resolution of I_X is a d-linear resolution, which is given by Eagon–Northcott complex.

In fact, we have proved the following result in the proof of Theorem 1.1.

Corollary 3.1. *Assume that $X \subset \mathbb{P}^{n+e}$ is a non-degenerate algebraic set of dimension n and satisfies $\mathbf{N}_{d,\alpha}$ for some $\alpha \leq e$. If $L^\alpha \subset \mathbb{P}^{n+e}$ is a linear space of dimension α whose intersection with X is 0-dimensional, then $X \cap L^\alpha$ is d-regular and*

$$\operatorname{length}\big(X \cap L^\alpha\big) \leq \binom{\alpha+d-1}{d-1}.$$

It was first proved by Eisenbud et al. [10, Theorem 1.1] that if X satisfies $\mathbf{N}_{d,\alpha}$ then every finite linear section $X \cap L^\alpha$ is d-regular.

Remark 3.2. In the proof of Theorem 1.1, if $X \subset \mathbb{P}^{n+e}$ satisfies $\mathbf{N}_{3,e}$ then we have the surjection on vector spaces:

$$\big[\mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^e \oplus \mathcal{O}_{\mathbb{P}^n}^{\beta_{0,2}^{S_e}}\big] \otimes k(q) \twoheadrightarrow H^0\big(\langle \Lambda, q\rangle, \mathcal{O}_{\pi_\Lambda{}^{-1}(q)}(2)\big)$$

where $[\mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^e \oplus \mathcal{O}_{\mathbb{P}^n}^{\beta^{S_e}_{0,2}}] \otimes k(q) \subset H^0(\langle \Lambda, q\rangle, \mathcal{O}_{\langle \Lambda, q\rangle}(2))$. Thus, $\pi_\Lambda^{-1}(q) = X \cap \langle \Lambda, q\rangle$ is 2-normal and so 3-regular. Moreover, the length of any fiber of π_Λ is at most $1 + e + \beta^{S_e}_{0,2}$. This will be used to prove Theorems 1.2 and 1.3.

3.2. The proof of Theorem 1.2

Suppose that X satisfies property $\mathbf{N}_{3,e}$. Then we have the following inequality from Theorem 1.1;

$$\deg(X) \le \binom{e+2}{2}. \tag{5}$$

Note that if X is arithmetically Cohen–Macaulay and I_X has 3-linear resolution then the degree of X is $\binom{e+2}{2}$ (see [7, Corollary 1.1]). The converse is not true in general. For example, let Y be the secant variety of a rational normal curve in $\mathbb{P}^n$ and let P be a general point in $\mathbb{P}^n$. Then the algebraic set $X = Y \cup P$ has the geometric degree $\binom{e+2}{2}$ but it does not satisfy $N_{3,e}$ because there exists a $\binom{e+2}{2} + 1$ secant e plane to X. This also implies that I_X does not have 3-linear resolution.

It is natural to ask what makes the ideal I_X have 3-linear resolution under the condition $\deg(X) = \binom{e+2}{2}$. Theorem 1.2 shows that property $\mathbf{N}_{3,e}$ is sufficient for this.

Remark 3.3. Note that the condition $\mathbf{N}_{3,e}$ is essential and cannot be weakened. For example, let S be a smooth complete intersection surface of type $(2, 3)$ in $\mathbb{P}^4$. Then the codimension e is two such that $\deg(S) = 6 = \binom{e+2}{2}$. However I_X does not have 3-linear resolution. Note that S satisfies $\mathbf{N}_{3,e-1}$ but not $\mathbf{N}_{3,e}$.

For a proof of Theorem 1.2, we need the following lemma.

Lemma 3.4. *Suppose that X satisfies property $\mathbf{N}_{3,e}$ and $\deg(X) = \binom{e+2}{2}$. Then,*

(a) *I_X has no quadric generators. This implies that I_X is 3-linear up to e-th step.*
(b) *$\binom{\alpha+1}{2} \le \beta^R_{\alpha,2}(R/I_X)$ for all $1 \le \alpha \le e$.*

Proof. Suppose that $\deg(X) = \binom{e+2}{2}$ and there is a quadric hypersurface Q containing X. For a general linear space L^e of dimension e, let $\Lambda \subset L^e$ be a linear space of dimension $e - 1$ disjoint from X with homogeneous coordinates $x_0, \dots, x_{e-1}$. By the same argument given in the proof of Theorem 1.1, we can regard $L^e \cap X$ as a fiber of a projection $\pi_\Lambda : X \to \pi_\Lambda(X)$. Since L^e is general, we may assume that the point $q = (1, 0, \cdots, 0)$ is not contained in Q. Then we have a surjective morphism $S_1 \oplus S_1(-1) \twoheadrightarrow R/I_X$ as a graded S_1-module (see the proof in [2, Theorem 4.2]). This implies that $\mathrm{Tor}_0^{S_1}(R/I_X, k)_2 = 0$. Consider the following exact sequences

$$\begin{aligned}
&\mathrm{Tor}_0^{S_e}(R/I_X, k)_1 \xrightarrow{\times x_{e-1}} \mathrm{Tor}_0^{S_e}(R/I_X, k)_2 \twoheadrightarrow \mathrm{Tor}_0^{S_{e-1}}(R/I_X, k)_2 \to 0,\\
&\mathrm{Tor}_0^{S_{e-1}}(R/I_X, k)_1 \xrightarrow{\times x_{e-2}} \mathrm{Tor}_0^{S_{e-1}}(R/I_X, k)_2 \twoheadrightarrow \mathrm{Tor}_0^{S_{e-2}}(R/I_X, k)_2 \to 0,\\
&\vdots\\
&\mathrm{Tor}_0^{S_2}(R/I_X, k)_1 \xrightarrow{\times x_1} \mathrm{Tor}_0^{S_2}(R/I_X, k)_2 \twoheadrightarrow \mathrm{Tor}_0^{S_1}(R/I_X, k)_2 = 0.
\end{aligned} \tag{6}$$

Since we see from (6) that $\beta^{S_i}_{0,2} \le \beta^{S_i}_{0,1} + \beta^{S_{i-1}}_{0,2}$ for each $2 \le i \le e$, it follows from Corollary 2.5(b) that

$$\beta^{S_e}_{0,2} \le \beta^{S_e}_{0,1} + \beta^{S_{e-1}}_{0,1} + \cdots + \beta^{S_2}_{0,1} + \beta^{S_1}_{0,2} = e + (e-1) + \cdots + 2 + 0 = \binom{e+1}{2} - 1.$$

By the same argument given in the proof of Theorem 1.1 and Remark 3.2 we have

$$\deg(X) \le 1 + e + \beta^{S_e}_{0,2} \le \binom{e+2}{2} - 1,$$

which contradicts our assumption. So, there is no quadric vanishing on X and the minimal free resolution of I_X is 3-linear up to e-th step. In addition, in the case of 3-linearity up to e-th step, there are no syzygies in degree 2 and

$$\beta^{S_e}_{0,2} = \beta^{S_e}_{0,1} + \beta^{S_{e-1}}_{0,1} + \cdots + \beta^{S_2}_{0,1} + \beta^{S_1}_{0,1} = \binom{e+1}{2} \le \beta^{R}_{e,2}(R/I_X),$$

as we wished. □

For a proof of Theorem 1.2, it suffices to show that $\deg(X) = \binom{e+2}{2}$ implies I_X has a 3-linear resolution under the condition $\mathbf{N}_{3,e}$ [7, Corollary 1.11]. Our proof is divided into four steps.

The proof of Theorem 1.2. Step *I*. First we show that if H is a general linear space of dimension i where $e \le i \le n$, then $I_{X\cap H,H}$ cannot have quadric generators.

For general linear space L of dimension e, we see from Remark 3.2 that $I_{X\cap L,L}$ is 3-regular. Since $X \cap L$ is a zero dimensional scheme of

$$\deg(X \cap L) = \deg(X) = \binom{e+2}{2} = \binom{\operatorname{codim}(X \cap L, L) + 2}{2},$$

it follows from Lemma 3.4 that $I_{X\cap L,L}$ has a 3-linear resolution and hence there is no quadric generator in the ideal $I_{X\cap L,L}$. This implies that if H is a general linear space of dimension i for some $e \le i \le n$, then $I_{X\cap H,H}$ cannot have quadric generators. In particular, if $H = \mathbb{P}^n$ then I_X does not have quadric generators and hence

$$\beta_{k,1}(R/I_X) = 0 \quad \text{for all } k \ge 0.$$

	0	1	...	e-1	e	e+1	e+2	...
0	1	0	...	0	0	0	0	...
1	0	*	...	*	*	*	*	...
2	0	*	...	*	*	*	*	...
3	0	0	...	0	0	*	*	...

$\Longrightarrow$

	0	1	...	e-1	e	e+1	e+2	...
0	1	0	...	0	0	0	0	...
1	0	0	...	0	0	0	0	...
2	0	*	...	*	*	*	*	...
3	0	0	...	0	0	*	*	...

Step *II*. The goal in this step is to show that

$$\beta_{k,3}(I_X) = \beta_{k+1,2}(R/I_X) = 0 \quad \text{for all } k \ge e.$$

	0	1	...	e-1	e	e+1	e+2	...
0	1	0	...	0	0	0	0	...
1	0	0	...	0	0	0	0	...
2	0	*	...	*	*	*	*	...
3	0	0	...	0	0	*	*	...

$\Longrightarrow$

	0	1	...	e-1	e	e+1	e+2	...
0	1	0	...	0	0	0	0	...
1	0	0	...	0	0	0	0	...
2	0	*	...	*	*	0	0	...
3	0	0	...	0	0	*	*	...

To show this, we prove that if $k \ge e$ then $\beta_{k,3}(\operatorname{gin} I_X) = 0$, where $\operatorname{gin}(I_X)$ is a generic initial ideal of I_X with respect to the reverse lexicographic monomial order. Note that $\beta_{k,3}(\operatorname{gin}(I_X)) = 0$ implies that $\beta_{k,3}(I_X) = 0$ [12, Corollary 1.21]. Let $\mathcal{G}(\operatorname{gin}(I_X))_d$ be the set of *monomial generators* of $\operatorname{gin}(I_X)$ in degree d. For each monomial T in $R = k[x_0, \ldots, x_n]$, we denote by $m(T)$

$$\max\{i \geq 0 \mid \text{a variable } x_i \text{ divides } T\}.$$

Now suppose that

$$\beta_{k,3}\big(\mathrm{gin}(I_X)\big) \neq 0 \quad \text{for some } k \geq e, \tag{7}$$

and let k be the largest integer satisfying the condition (7). By the result of Eliahou and Kervaire [11] we see that

$$\beta_{k,3}\big(\mathrm{gin}(I_X)\big) = \big|\big\{T \in \mathcal{G}\big(\mathrm{gin}(I_X)\big)_3 \mid m(T) = k\big\}\big|.$$

Since $\beta_{k,3}(\mathrm{gin}(I_X)) \neq 0$, we can choose a monomial $T \in \mathcal{G}(\mathrm{gin}(I_X))_3$ such that $m(T) = k$. This implies that T is divisible by x_k. If H is a general linear space of dimension k then it follows from [12, Theorem 2.30] that the ideal

$$\mathrm{gin}(I_{X\cap H,H}) = \left[\frac{(\mathrm{gin}(I_X), x_{k+1}, \ldots, x_n)}{(x_{k+1}, \ldots, x_n)}\right]^{\mathrm{sat}} = \left[\frac{(\mathrm{gin}(I_X), x_{k+1}, \ldots, x_n)}{(x_{k+1}, \ldots, x_n)}\right]_{x_k \to 1} \tag{8}$$

has to contain the quadratic monomial T/x_k. This means that $X \cap H$ is cut out by a quadric hypersurface, which contradicts the result in Step *I*. Hence we conclude that $\beta_{k,3}(I_X) = 0$ for all $k \geq e$.

Step *III*. We claim that

$$\mathcal{G}\big(\mathrm{gin}(I_X)\big)_3 = \mathrm{gin}(I_X)_3 = k[x_0, \ldots, x_{e-1}]_3. \tag{9}$$

By Lemma 3.4 and [12, Corollary 1.21], we see that

$$\binom{e+1}{2} \leq \beta_{e,2}(R/I_X) = \beta_{e-1,3}(I_X) \leq \beta_{e-1,3}\big(\mathrm{gin}(I_X)\big). \tag{10}$$

Since $\beta_{k,3}(\mathrm{gin}(I_X)) = 0$ for each $k \geq e$, any monomial generator $T \in \mathcal{G}(\mathrm{gin}(I_X))_3$ is not divisible by x_k for any $k \geq e$. Thanks to the result of Eliahou and Kervaire [11] again,

$$\begin{aligned}\beta_{e-1,3}\big(\mathrm{gin}(I_X)\big) &= \big|\big\{T \in \mathcal{G}\big(\mathrm{gin}(I_X)\big)_3 \mid m(T) = e-1\big\}\big| \\ &\leq \dim_k\big(x_{e-1} \cdot k[x_0, \ldots, x_{e-1}]_2\big) \\ &= \binom{e+1}{2}.\end{aligned}$$

By the dimension counting and Eq. (10), we have $\beta_{e-1,3}(\mathrm{gin}(I_X)) = \binom{e+1}{2}$ and thus

$$\big\{T \in \mathcal{G}\big(\mathrm{gin}(I_X)\big)_3 \mid m(T) = e-1\big\} = x_{e-1} \cdot k[x_0, \ldots, x_{e-1}]_2,$$

which implies that $x_{e-1}^3 \in \mathrm{gin}(I_X)$. Note that $\mathrm{gin}(I_X)$ does not have any quadratic monomial. Hence we conclude from Borel fixed property of $\mathrm{gin}(I_X)$ that

$$\mathcal{G}\big(\mathrm{gin}(I_X)\big)_3 = \mathrm{gin}(I_X)_3 = k[x_0, \ldots, x_{e-1}]_3. \tag{11}$$

Step *IV*. Finally, by the result in Step *II*, we only need to show that, for all $k \geq e$ and $j \geq 3$,

$$\beta_{k,j}(I_X) = 0.$$

	0	1	...	e-1	e	e+1	e+2	...
0	1	0	...	0	0	0	0	...
1	0	0	...	0	0	0	0	...
2	0	*	...	*	*	0	0	...
3	0	0	...	0	0	*	*	...
4	0	0	...	0	0	*	*	...

$\Longrightarrow$

	0	1	...	e-1	e	e+1	e+2	...
0	1	0	...	0	0	0	0	...
1	0	0	...	0	0	0	0	...
2	0	*	...	*	*	0	0	...
3	0	0	...	0	0	0	0	...
4	0	0	...	0	0	0	0	...

Since $\beta_{k,j}(I_X) \leq \beta_{k,j}(\text{gin}(I_X))$ (see [12, Proposition 2.11]), it is sufficient to prove that $\text{gin}(I_X)$ has no generators in degree ≥ 4. To prove this, suppose that there is a monomial generator $T \in \mathcal{G}(\text{gin}(I_X))_j$ for some $j \geq 4$. Then the monomial T can be written as a product of two monomials N_1 and N_2 such that

$$N_1 \in k[x_e, \ldots, x_n], \qquad N_2 \in k[x_0, \ldots, x_{e-1}].$$

By the result in Step *III*, if the monomial N_2 is divisible by some cubic monomial in $k[x_0, \ldots, x_{e-1}]$ then T cannot be a monomial generator of $\text{gin}(I_X)$. Hence we see $\deg(N_2)$ is at most 2. If L is a general linear space of dimension e then it follows from the similar argument given in the proof of Step *III* with Eq. (8) that $N_2 \in \text{gin}(I_{X\cap L,L})$. Hence $I_{X\cap L,L}$ has a hyperplane or a quadratic polynomial, which contradicts the result proved in Step *I*. □

Remark 3.5. The similar argument in the proof of Theorem 1.2 can also be applied to show that X satisfies property $\mathbf{N}_{2,e}$ if and only if X is an ACM scheme with 2-linear resolution.

Example 3.6. In [15], the authors have shown that if a non-degenerate reduced scheme $X \subset \mathbb{P}^n$ satisfies $\mathbf{N}_{2,p}$ for some $p \geq 1$ then the inner projection from any smooth point of X satisfies at least property $\mathbf{N}_{2,p-1}$. So it is natural to ask whether the inner projection from any smooth point of X satisfies at least property $\mathbf{N}_{3,p-1}$ when X satisfies $\mathbf{N}_{3,p}$ for some $p \geq 1$. Our result shows that this is not true in general. For example, if we consider the secant variety $X = \text{Sec}(C)$ of a rational normal curve C then the inner projection Y from any smooth point of X has the degree

$$\deg(Y) = \binom{2+e}{2} - 1 = \binom{e+1}{2} + \binom{e}{1} > \binom{2+(e-1)}{2},$$

where $e = \text{codim(X)}$ and $e-1 = \text{codim(Y)}$. This implies that X satisfies $\mathbf{N}_{3,e}$ but Y does not satisfy $\mathbf{N}_{3,e-1}$.

Example 3.7. Remark that there exists an *algebraic set* X of degree $< \binom{e+2}{2}$ whose defining ideal I_X has 3-linear resolution. For example, let $I = (x_0^3, x_0^2x_1, x_0x_1^2, x_1^3, x_0^2x_2)$ be a monomial ideal of $R = k[x_0, x_1, x_2, x_3]$. Note that the sufficiently generic distraction $D_{\mathcal{L}}(I)$ of I is of the form

$$D_{\mathcal{L}}(I) = (L_1L_2L_3, L_1L_2L_4, L_1L_4L_5, L_4L_5L_6, L_1L_2L_7),$$

where L_i is a generic linear form for each $i = 1, \ldots, 7$ (see [3] for the definition of distraction). Then the algebraic set X defined by the ideal $D_{\mathcal{L}}(I)$ is a union of 5 lines and one point such that its minimal free resolutions are given by

R-modules

	0	1	2	3
0	1	0	0	0
1	0	0	0	0
2	0	5	5	1

S_1-modules

	0	1	2
0	1	0	0
1	1	0	0
2	1	4	1

S_2-modules

	0	1
0	1	0
1	2	0
2	3	1

In this case, we see that $e = 2$, $\deg(X) = 5 < \binom{2+2}{2} = 6$ and there is a 6-secant 2-plane to X. We see that a general hyperplane section of X is contained in a quadric hypersurface from $\beta_{e+1,2}(R/I_X) \neq 0$.

3.3. The proof of Theorem 1.3

Let $X \subset \mathbb{P}^{n+e}$ be a non-degenerate algebraic set of dimension n satisfying $\mathbf{N}_{2,p}$ for some $p \geq 0$. If $\alpha \leq e$ is such that X satisfies $\mathbf{N}_{3,\alpha}$, and $L^\alpha \subset \mathbb{P}^{n+e}$ is a linear space of dimension α whose intersection with X is zero-dimensional then we have to show that

$$\operatorname{length}\big(X \cap L^\alpha\big) \leq 1 + \alpha + \min\left\{\frac{|\alpha - p|(\alpha + p + 1)}{2}, \beta^R_{\alpha,2}(R/I_X)\right\}. \tag{12}$$

The proof of Theorem 1.3. Note that $\beta^R_{\alpha,2} = 0$ if $\alpha \leq p$. In this case, the inequality (12) follows from [10, Theorem 1.1] directly. Now we assume $\alpha > p$ and $\beta^R_{\alpha,2} \neq 0$. Suppose $\dim(X \cap L^\alpha) = 0$ and choose a linear subspace $\Lambda \subset L^\alpha$ of dimension $(\alpha - 1)$ disjoint from X with homogeneous coordinates $x_0, \ldots, x_{\alpha-1}$.

By the same argument given in the proof of Theorem 1.1 and Remark 3.2, we have the following surjective morphism

$$\cdots \to \mathcal{O}_{\mathbb{P}^{n+e-\alpha}} \oplus \mathcal{O}_{\mathbb{P}^{n+e-\alpha}}(-1)^\alpha \oplus \mathcal{O}_{\mathbb{P}^{n+e-\alpha}}(-2)^{\beta^{S_\alpha}_{0,2}} \stackrel{\widetilde{\varphi_\alpha}}{\to} \pi_{\Lambda_*}\mathcal{O}_X \longrightarrow 0.$$

For any point $q \in \pi_\Lambda(X)$, note that $\pi_{\Lambda_*}\mathcal{O}_X \otimes k(q) \simeq H^0(\langle\Lambda, q\rangle, \mathcal{O}_{\pi_\Lambda{}^{-1}(q)})$. Thus, by tensoring $\mathcal{O}_{\mathbb{P}^{n+e-\alpha}}(2) \otimes k(q)$ on both sides, we have the surjection on vector spaces:

$$\big[\mathcal{O}_{\mathbb{P}^{n+e-\alpha}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n+e-\alpha}}(1)^\alpha \oplus \mathcal{O}_{\mathbb{P}^{n+e-\alpha}}^{\beta^{S_\alpha}_{0,2}}\big] \otimes k(q) \twoheadrightarrow H^0\big(\langle\Lambda, q\rangle, \mathcal{O}_{\pi_\Lambda{}^{-1}(q)}(2)\big). \tag{13}$$

Therefore, $\langle\Lambda, q\rangle \cap X$ is 3-regular and the length of any fiber of π_Λ is at most $1 + \alpha + \beta^{S_\alpha}_{0,2}$. Hence it is important to get an upper bound of $\beta^{S_\alpha}_{0,2}$.

Claim. *There are following inequalities on graded Betti numbers:*

(i) $\beta^{S_\alpha}_{0,2} \leq \beta^{S_{\alpha-1}}_{1,2} \leq \cdots \leq \beta^{S_1}_{\alpha-1,2} \leq \beta^R_{\alpha,2}, \alpha \leq e = \operatorname{codim}(X)$;
(ii) $\beta^{S_\alpha}_{0,2} \leq \frac{(\alpha-p)(\alpha+p+1)}{2}$.

Due to the claim, we have the following inequality:

$$\beta^{S_\alpha}_{0,2} \leq \min\left\{\frac{|\alpha - p|(\alpha + p + 1)}{2}, \beta^R_{\alpha,2}(R/I_X)\right\}.$$

Therefore, the length of any fiber of $\pi_\Lambda : X \to \mathbb{P}^{n+e-\alpha}$ *is at most*

$$1 + \alpha + \beta^{S_\alpha}_{0,2} \leq 1 + \alpha + \min\left\{\frac{|\alpha - p|(\alpha + p + 1)}{2}, \beta^R_{\alpha,2}(R/I_X)\right\}.$$

Since $X \cap L^\alpha$ *can be regarded as a fiber of the map* $\pi_\Lambda : X \to \mathbb{P}^{n+e-\alpha}$, *this completes the proof of Theorem 1.3.*

Now let us prove the Claim. Note that Claim (i) follows directly from Corollary 2.5(b) for $d = 3$. Hence we only need to show Claim (ii). We consider the multiplicative maps appearing in the mapping cone sequence as follows:

$$\begin{aligned}
&\operatorname{Tor}_0^{S_\alpha}(R/I_X, k)_1 \xrightarrow{\times x_{\alpha-1}} \operatorname{Tor}_0^{S_\alpha}(R/I_X, k)_2 \twoheadrightarrow \operatorname{Tor}_0^{S_{\alpha-1}}(R/I_X, k)_2 \to 0,\\
&\operatorname{Tor}_0^{S_{\alpha-1}}(R/I_X, k)_1 \xrightarrow{\times x_{\alpha-2}} \operatorname{Tor}_0^{S_{\alpha-1}}(R/I_X, k)_2 \twoheadrightarrow \operatorname{Tor}_0^{S_{\alpha-2}}(R/I_X, k)_2 \to 0,\\
&\cdots \quad \cdots \quad \cdots\\
&\operatorname{Tor}_0^{S_{p+1}}(R/I_X, k)_1 \xrightarrow{\times x_p} \operatorname{Tor}_0^{S_{p+1}}(R/I_X, k)_2 \twoheadrightarrow \operatorname{Tor}_0^{S_p}(R/I_X, k)_2 = 0.
\end{aligned} \tag{14}$$

Since R/I_X satisfies property $\mathbf{N}^{S_p}_{2,0}$ as an S_p-module by Corollary 2.5(a), we get

$$\mathrm{Tor}^{S_p}_0(R/I_X,k)_2=0.$$

From the above exact sequences, we have the following inequalities on the graded Betti numbers by dimension counting:

$$\begin{aligned}\beta^{S_\alpha}_{0,2} &\le \beta^{S_\alpha}_{0,1}+\beta^{S_{\alpha-1}}_{0,2}\le \beta^{S_\alpha}_{0,1}+\beta^{S_{\alpha-1}}_{0,1}+\beta^{S_{\alpha-2}}_{0,2}\le\cdots\le \beta^{S_\alpha}_{0,1}+\beta^{S_{\alpha-1}}_{0,1}+\cdots+\beta^{S_{p+1}}_{0,1}+\beta^{S_p}_{0,2}\\ &=\alpha+(\alpha-1)+\cdots+(p+1)=\frac{(\alpha-p)(\alpha+p+1)}{2}.\end{aligned}$$

Thus, we obtain the desired inequality

$$\beta^{S_\alpha}_{0,2}(R/I_X)\le \min\left\{\frac{(\alpha-p)(\alpha+p+1)}{2},\beta^R_{\alpha,2}(R/I_X)\right\},$$

as we claimed. □

The following result shows that if X is a nondegenerate variety satisfying $\mathbf{N}_{3,e}$ then there is some sort of rigidity toward the beginning and the end of the resolution. This means the following Betti diagrams are equivalent;

Property $\mathbf{N}_{3,e}$ and $\beta^R_{e,2}=0$

0	1	2	…	e−1	e	e+1	e+2	…
0	1	0	…	0	0	0	0	…
1	0	*	…	*	*	*	*	…
2	0	*	…	*	0	*	*	…
3	0	0	…	0	0	*	*	…
4	0	0	…	0	0	*	*	…

$\Longleftrightarrow$

X is 2-regular

0	1	2	…	e−1	e	e+1	e+2	…
0	1	0	…	0	0	0	0	…
1	0	*	…	*	*	*	*	…
2	0	0	…	0	0	0	0	…
3	0	0	…	0	0	0	0	…
4	0	0	…	0	0	0	0	…

Corollary 3.8. *Suppose $X\subset\mathbb{P}^{n+e}$ is a non-degenerate variety of dimension n and codimension e with property $\mathbf{N}_{3,e}$. Then, $\beta^R_{e,2}=0$ if and only if X is 2-regular.*

Proof. Let L^e be a linear space of dimension e and assume that $X\cap L^e$ is finite. By Theorem 1.3, $\mathrm{length}(X\cap L^e)\le 1+e+\beta^R_{e,2}$. Therefore, $\beta^R_{e,2}=0$ implies $\mathrm{length}(X\cap L^e)\le 1+e$. Since X is a nondegenerate variety this implies that X is small (i.e. for every zero-dimensional intersection of X with a linear space L, the length of $X\cap L$ is at most $1+\dim(L)$ (see [6, Definition 11])). Then it follows directly from [9, Theorem 0.4] that X is 2-regular. □

Remark 3.9. What can we say about the case $\beta^R_{\alpha,2}=0$ where $\alpha<e$? In this case, we see that if $\Lambda\cap X$ is finite for a linear subspace Λ of dimension $\le\alpha$ then $\mathrm{length}(\Lambda\cap X)\le\dim\Lambda+1$. Note that this condition is a necessary condition for property $\mathbf{N}_{2,\alpha}$. However, the converse is false in general, as for example in the case of a double structure on a line in $\mathbb{P}^3$ or the case of the plane with embedded point (see [10, Example 1.4]). We do not know if there are other cases when X is a variety.

Example 3.10 *(Macaulay 2 [13]).* (a) The two skew lines X in $\mathbb{P}^3$ satisfy $\deg(X)=2<1+e=3$. The Betti table of R/I_X is given by

	0	1	2	3	4	…
0	1	0	0	0	0	…
1	0	4	4	1	0	…
2	0	0	0	0	0	…

Note that X is 2-regular but not a CM.

(b) Let C be a rational normal curve in $\mathbb{P}^4$, which is 2-regular. If $X = C \cup P$ for a general point $P \in \mathbb{P}^4$ then $\deg(X) = 1 + e = 4$. However a general hyperplane L passing through P is 5-secant 3-plane such that $\deg(L \cap X) = 5 > 4 = 1 + e$. This implies that $\beta^R_{e,2}(R/I_X) \neq 0$. If $P \in \mathrm{Sec}(C)$ then there is a 3-secant line to X. Therefore $\beta^R_{1,2}(R/I_X) \neq 0$. For the two cases, the corresponding Betti tables for X are computed as follows [13, Macaulay 2]:

	0	1	2	3	4	5	…
0	1	0	0	0	0	0	…
1	0	5	5	0	0	0	…
2	0	1	3	4	1	0	…
3	0	0	0	0	0	0	…

Case 1: $P \in \mathrm{Sec}(C)$

	0	1	2	3	4	5	…
0	1	0	0	0	0	0	…
1	0	5	4	0	0	0	…
2	0	0	3	4	1	0	…
3	0	0	0	0	0	0	…

Case 2: $P \notin \mathrm{Sec}(C)$

Since a small algebraic set is 2-regular, if X satisfies property $\mathbf{N}_{2,e}$ then X is 2-regular. One may ask if property $\mathbf{N}_{d,e}$ implies X is d-regular. The following example (suggested by F.-O. Schreyer) shows that condition $\mathbf{N}_{d,e}$ does not imply d-regularity in general.

Example 3.11 *(F.-O. Schreyer).* Let C be a rational normal curve and Z be a set of general 4 points in $\mathbb{P}^3$.

```
i1 : R=QQ[x_0..x_3];
     C=minors(2,matrix{{x_0,x_1,x_2},{x_1,x_2,x_3}}); -- a rational normal curve
     Z=minors(2,random(R^2,R^{4:-1})); -- general 4 points
     X=intersect(C,Z);
```

Using Macaulay 2, we can compute the Betti table of $X = C \cup Z$ as follows:

```
i5 : betti res X

               0 1 2 3
o5 = total :   1 6 6 1
         0 :   1 . . .
         1 :   . . . .
         2 :   . 6 6 .
         3 :   . . . 1
```

Since the codimension e of X is two, X satisfies property $\mathbf{N}_{3,e}$. Note that X is not 3-regular. Unlike the case of $\mathbf{N}_{2,e}$, the condition $\mathbf{N}_{3,e}$ does not imply 3-regularity.

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