Reidemeister torsion and homology cylinders I

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Origin of Reidemeister torsion

- Geometric topology: classify spaces up to homeomorphism.

Origin of Reidemeister torsion

- How do we distinguish lens spaces up to homeomorphism?
- Idea: use a cellular structure, the universal cover, and the action of the deck transformations: Reidemeister torsion.
- Application: The homotopy equivalent lens spaces $L(7,1)$ and $L(7,2)$ are not homeomorphic (not simple-homotopy equivalent).

- References:
  - J. Milnor. Whitehead torsion. *Bull. Amer. Math. Soc.* **72** (1966) 358–426.
  - V. Turaev. Introduction to combinatorial torsions. Birkhäuser. (2001).

- Λ: associative ring with 1 such that for any $r \neq s \in \mathbb{N}, \Lambda^r \ncong \Lambda^s$.
- $\mathrm{GL}(\Lambda) := \cup_{n\geq 0}\mathrm{GL}(n, \Lambda)$, the infinite general linear group.
- $K_1(\Lambda) := \mathrm{GL}(\Lambda)/[\mathrm{GL}(\Lambda), \mathrm{GL}(\Lambda)]$.

Reidemeister torsion: algebraic theory

- $C = \{C_i, d_i\}_{i=0}^m$: acyclic chain complex over Λ such that C_i are fin. gen. free Λ-modules with the bases c_i.
- Suppose $B_i := \mathrm{Im}(d_i : C_{i+1} \to C_i)$ are free.
- Since C is acyclic,

$$0 \to B_i \to C_i \xrightarrow{d_{i-1}} B_{i-1} \to 0$$

  is exact.
- $b_i :=$ bases of B_i. Then $b_i b_{i-1}$ are bases of C_i.
- $(b_i b_{i-1}/c_i) :=$ the transition matrix over Λ.

Definition

$$\tau(C) := \prod_{i=0}^{m} (b_i b_{i-1}/c_i)^{(-1)^{i+1}} \in K_1(\Lambda).$$

- det : $K_1(\Lambda) \to \Lambda^*$, a surjective homomorphism.

Lemma

For a field $\mathbb{F}$, det : $K_1(\mathbb{F}) \to \mathbb{F}^*$ is an isomorphism of abelian groups

Reidemeister torsion: nonacyclic case

- Suppose $C = \{C_i, d_i\}_{i=0}^{m}$ is not acyclic.
- $H_i(C) = \mathrm{Ker} d_{i-1}/\mathrm{Im} d_i$ is based with the bases h_i.
- $b_i h_i b_{i-1}$ becomes a basis of C_i.

Definition

$$\tau(C) := \prod_{i=0}^{m} (b_i h_i b_{i-1}/c_i)^{(-1)^{i+1}} \in K_1(\Lambda).$$

Homological computation

- R:= Noetherian UFD (e.g. $R = \mathbb{Z}[t_1^{\pm}, \ldots, t_n^{\pm}]$)
- $Q(R)$:= the field of fractions of R
- $C = \{C_i, d_i\}_{i=0}^m$: based free chain complex of finite rank over R such that $C \otimes_R Q(R)$ is acyclic.

Theorem

$$\tau(C \otimes_R Q(R)) = \prod_{i=0}^{m} (ord H_i(C))^{(-1)^{i+1}}.$$

- X:= finite connected CW-complex. $\pi := \pi_1(X)$.
- $\tilde{X}$:= the universal cover of X.
- Suppose a ring homomorphism $\phi : \mathbb{Z}[\pi] \to \Lambda$ is given.
- $H_i(X;\Lambda) := H_i(C(\tilde{X}) \otimes_\phi \Lambda)$.
- Suppose $H_i(X;\Lambda) = 0$.

The Reidemeister torsion of X

$$\tau_\phi(X) := \tau(C(\tilde{X}) \otimes_\phi \Lambda) \in K_1(\Lambda)/\pm\phi(\pi).$$

- The Reidemeister torsion of X is invariant under arbitrary homeomorphisms of X.

EXAMPLE: S^1

- For S^1, let $\pi_1(S^1) = \langle t \rangle$.
- Consider $\phi : \mathbb{Z}[\pi_1(S^1)] \to \mathbb{Z}[t^{\pm}] \to \mathbb{Q}(t)$.
- S^1= 1 0-cell $\cup$ 1 1-cell, and $\tilde{S^1} = \mathbb{R}$.
- The chain complex $C(\tilde{S^1}) \otimes_\phi \mathbb{Q}(t)$ is

$$0 \to \mathbb{Q}(t) \xrightarrow{t-1} \mathbb{Q}(t) \to 0.$$

- Therefore,

$$\tau_\phi(S^1) = (t-1)^{-1} \in K_1(\mathbb{Q}(t)/\pm\phi(\pi_1(S^1)) = \mathbb{Q}(t)^* / \{\pm t^n\}_{n \in \mathbb{Z}}.$$

Example: Knot complement

- $K :=$ knot in S^3. $X(K) := S^3 - N(K)$.
- $\phi : \mathbb{Z}[\pi_1(X(K))] \to \mathbb{Z}[t^{\pm}] \to \mathbb{Q}(t)$.
- $$\mathrm{ord}(H_i(X(K); \mathbb{Z}[t^{\pm}])) = \begin{cases} t-1 & : \quad i = 0 \\ \Delta_K(t) & : \quad i = 1 \\ 1 & : \quad i > 1 \end{cases}$$
- Therefeore, $\tau_\phi(X(K)) = \Delta_K(t)/t-1 \in \mathbb{Q}(t)^*/\{\pm t^n\}_{n\in\mathbb{Z}}$. (The Milnor torsion of $X(K)$.)

Example: zero surgery on a knot

- $M_K :=$ zero surgery on K in S^3.
- $\phi : \mathbb{Z}[\pi_1(M_K)] \to \mathbb{Z}[t^{\pm}] \to \mathbb{Q}(t)$.
-
$$\mathrm{ord}(H_i(M)K; \mathbb{Z}[t^{\pm}])) = \begin{cases} t-1 & : \quad i=0 \\ \Delta_K(t) & : \quad i=1 \\ t^{-1}-1 & : \quad i=2 \\ 1 & : \quad i>2 \end{cases}$$
- Therefeore, $\tau_\phi(M_K) = \Delta_K(t)/(t-1)(t^{-1}-1) \in \mathbb{Q}(t)^*/\{\pm t^n\}_{n\in\mathbb{Z}}$.

Properties

Theorem

Let $0 \to C' \to C \to C'' \to 0$ *be a short exact sequence of free acyclic chain complexes of finite rank over* Λ. *Then*

$$\tau(C) = \pm\tau(C')\tau(C'') \in K_1(\Lambda).$$

Properties

Theorem

Let M be a compact connected orientable pl-manifold of dimension m. For a homomorphism $\phi : \mathbb{Z}[\pi_1(M)] \to \Lambda$, suppose that Λ is endowed with an involution $\lambda \mapsto \bar{\lambda}$ such that

$$\overline{\phi(\alpha)} = \phi(\alpha^{-1}) \text{ for all } \alpha \in \pi_1(M).$$

Then

$$\tau_\phi(M, \partial M) = \overline{\tau_\phi(M)}^{(-1)^{m+1}} \in K_1(\Lambda)/ \pm \phi(\pi_1(M)).$$

Applications

Theorem

Let K be a slice knot. Then

$$\Delta_K(t) \doteq f(t)f(t^{-1}) \text{ for some } f(t) \in \mathbb{Z}[t^{\pm}].$$

- $M_K :=$ zero surgery on K in S^3.
- $W :=$ slice disk complement in B^4. Then $\partial W = M_K$.
- The homomorphism $\phi : \mathbb{Z}[\pi_1(M_K)] \to \Lambda := \mathbb{Z}[t^{\pm}]$ factors through $\mathbb{Z}[\pi_1(M_K)] \to \mathbb{Z}[\pi_1(W)]$.

Applications

- From the short exact sequence

$$0 \to C(\tilde{M}_K) \otimes \Lambda \to C(\tilde{W}) \otimes \Lambda \to C(\tilde{W}, \tilde{M}_K) \otimes \Lambda \to 0,$$

  we have

$$\tau_\phi(W) = \tau_\phi(M_K)\tau_\phi(W, M_K).$$

- Therefore

$$\begin{aligned} \tau_\phi(M_K) &= \tau_\phi(W)\tau_\phi(W, M_K)^{-1} \\ &= \tau_\phi(W)\overline{\tau_\phi(W)}. \end{aligned}$$

- $\Delta_K(t) = \tau_\phi(W)(t-1)\overline{\tau_\phi(W)}\,\overline{(t-1)}.$

Twisted Reidemeister torsion

- For a space X, choose $\phi : \pi_1(X) \to \mathbb{Z} := \langle t \rangle$ and $\alpha : \pi_1(X) \to \mathrm{GL}(n, \mathbb{F})$ for a field $\mathbb{F}$.
- $\pi_1(X)$ acts on $\mathbb{F}^n[t^{\pm}] := \mathbb{F}^n \otimes \mathbb{F}[t^{\pm}]$ via $\alpha \otimes \phi$.
- $C(\tilde{X}) \otimes_{\alpha\otimes\phi} \mathbb{F}^n[t^{\pm}]$ is a right $\mathbb{F}[t^{\pm}]$-module.

The twisted Reidemeister torsion of X

$$\tau_{\alpha\otimes\phi}(X) := \tau(C(\tilde{X}) \otimes_{\alpha\otimes\phi} \mathbb{F}^n[t^{\pm}]) \in K_1(\mathbb{F}(t)) \cong \mathbb{F}^*.$$

Applications of Twisted Reidemeister torsion

- **Fiberedness of 3-manifolds and links** : Cha, Goda-Kitano-Morifuji, Goda-Pojitnov, Friedl-K, Silver-Williams, Friedl-Vidussi
- **Symplectic structure on** $S^1 \times M^3$: Friedl-Vidussi
- **Knot genus and Thurston norm of 3-manifolds** : Friedl-K
- **Knot isotopy** : Silver-Williams, Friedl-Vidussi
- **Knot and link concordance** : Kirk-Livingston, Cha-Friedl, Cha-Friedl-K
- **Periodic knots** : Hillman-Livingston-Naik
- **Groups** : Kitano-Suzuki-Wada
- **Algebraic curves** : Cogolludo-Florens