Reidemeister torsion and homology cylinders II

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Homology cylinder

- For $g, k \geq 0$, let $\Sigma_{g,k} :=$ oriented compact surface of genus g with k boundary components.
- A homology cylinder (M, i_+, i_-) over $\Sigma_{g,k}$ is a 3-manifold M together with two embeddings $i_+, i_- : \Sigma_{g,k} \to \partial M$ such that
  (1) i_+ is orientation preserving and i_- is orientation reversing,
  (2) $\partial M = i_+(\Sigma_{g,k}) \cup i_-(\Sigma_{g,k})$ and $i_+(\Sigma_{g,k}) \cap i_-(\Sigma_{g,k}) = i_+(\partial \Sigma_{g,k}) = i_-(\partial \Sigma_{g,k})$,
  (3) $i_+|_{\partial \Sigma_{g,k}} = i_-|_{\partial \Sigma_{g,k}}$,
  (4) $i_+, i_- : H_*(\Sigma_{g,k}; \mathbb{Z}) \to H_*(M; \mathbb{Z})$ are isomorphisms.

Homology cylinder: Examples

- For a diffeomorphism φ of $\Sigma_{g,k}$ which fixes $\partial\Sigma_{g,k}$ pointwise,

$$(\Sigma_{g,k} \times [0,1]/\partial\Sigma_{g,k} \times [0,1], \mathrm{id} \times 1, \varphi \times 0)$$

  is a homology cylinder.
- Let K be a knot of genus g such that $\Delta_K(t)$ is monic and such that $\deg(\Delta_K(t)) = 2g$. Let $\Sigma \subset S^3 - N(K)$ be a minimal genus Seifert surface. Then $(S^3 - N(K)) - \Sigma \times (0,1)$ is a homology cylinder over $\Sigma_{2g,1}$ in a natural way.

Homology cylinder

- Two homology cylinders (M, i_+, i_-) and (N, j_+, j_-) over $\Sigma_{g,k}$ are called isomorphic if there exists an orientation preserving diffeomorphism $f : M \to N$ satisfying $j_+ = f \circ i_+$ and $j_- = f \circ i_-$.
- $\mathcal{C}_{g,k} :=$ the monoid of all isomorphism classes of homology cylinders over $\Sigma_{g,k}$.
- The product operation on $\mathcal{C}_{g,k}$:

$$(M, i_+, i_-) \cdot (N, j_+, j_-) := (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-).$$

The identity is $(\Sigma_{g,k} \times [0,1]/\partial\Sigma_{g,k} \times [0,1], \mathrm{id} \times 1, \mathrm{id} \times 0)$.

Homology cobordism of homology cylinders

- Two homology cylinders (M, i_+, i_-) and (N, j_+, j_-) over $\Sigma_{g,k}$ are called homology cobordant if there exists a compact oriented smooth 4-manifold W such that

$$\partial W = M \cup (-N) \,/\, (i_+(x) = j_+(x), i_-(x) = j_-(x), x \in \Sigma_{g,k}),$$

  and such that the inclusion induced maps $H_*(M;\mathbb{Z}) \to H_*(W;\mathbb{Z})$ and $H_*(N;\mathbb{Z}) \to H_*(W;\mathbb{Z})$ are isomorphisms.
- $\mathcal{H}_{g,k} :=$ the group of homology cobordism classes of elements in $\mathcal{C}_{g,k}$

Homology cylinders and mapping class groups

- For a diffeomorphism φ of $\Sigma_{g,k}$ which fixes $\partial\Sigma_{g,k}$ pointwise,

$$(\Sigma_{g,k} \times [0,1]/\partial\Sigma_{g,k} \times [0,1], \mathrm{id} \times 1, \varphi \times 0)$$

  is a homology cylinder.

- $\mathcal{M}_{g,k}$:= the mapping class group of $\Sigma_{g,k}$.

Theorem (Garoufalidis-Levine)

$\mathcal{M}_{g,k}$ embeds into $\mathcal{C}_{g,k}$ and $\mathcal{H}_{g,k}$.

Homology cylinder

- If $g \geq 3$, then $\mathcal{M}_{g,k}$ is perfect.

Theorem (2009, Goda-Sakasai)

For $g \geq 1$, $\mathcal{C}_{g,1}$ surjects to $\mathbb{Z}^\infty$, hence not perfect.

- Idea of proof: rank of sutured Floer homology $SFH(M, i_+(\partial\Sigma_{g,1}))$.

Main Theorem

Question

Are $\mathcal{H}_{g,k}$ perfect? Do they have nontrivial abelian quotients?

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Theorem (Cha-Friedl-K)

Let $g, k \geq 0$ such that $b_1(\Sigma_{g,k}) > 0$. Then there exists a surjective homomorphism

$$\mathcal{H}_{g,k} \to (\mathbb{Z}/2)^\infty.$$

Ideas of proof

- Define Reidemeister torsion of $(M, i_+(\Sigma_{g,k}))$.
- Construct a homomorphism $\Phi_p : \mathcal{C}_{g,k} \to \mathbb{Z}_{\geq 0}$
- Show that Φ_p descends to $\Psi_p : \mathcal{H}_{g,k} \to \mathbb{Z}/2$
- Show that Ψ_p is nontrivial by realizing examples.

Reidemeister torsion

- For a homology cylinder (M, i_+, i_-), let $\Sigma := \Sigma_{g,k}$, $\Sigma_\pm := i_\pm(\Sigma)$, $H = H_1(\Sigma; \mathbb{Z})$.
- Consider $\varphi : \pi_1(M) \to H_1(M) \xrightarrow{\cong} H_1(\Sigma_+) \xrightarrow{i_+^{-1}} H$.
- $Q(H)$:= the field of fractions of $\mathbb{Z}[H]$.

Definition

$$\tau(M, i_+, i_-) := \tau_\varphi(M, \Sigma_+; Q(H)) \in Q(H)^* / \{\pm h\}_{h \in H}.$$

Lemma

For any homology cylinder, we have

$$\tau(M) = \operatorname{ord} H_1(M, \Sigma_+; \mathbb{Z}[H]) \in \mathbb{Z}[H]/\{\pm h\}_{h\in H}.$$

- Idea of proof :

$$\tau(M) = \prod_i (\operatorname{ord} H_i(M, \Sigma_+; \mathbb{Z}[H]))^{(-1)^{i+1}},$$

and $\operatorname{ord} H_i(M, \Sigma_+; \mathbb{Z}[H]) = 1$ for $i \neq 1$.

Product formula for torsion

- For (M, i_+, i_-), $\varphi(M)$ is defined to be the homomorphism

$$H = H_1(\Sigma;\mathbb{Z}) \xrightarrow{i_-} H_1(i_-(\Sigma);\mathbb{Z}) \xrightarrow{\cong} H_1(M;\mathbb{Z}) \xrightarrow{\cong} H_1(i_+(\Sigma);\mathbb{Z}) \xrightarrow{i_+^{-1}} H.$$

Therefore $\varphi(M) \in \mathrm{Aut}(H)$

Theorem

Let $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$ be homology cylinders over $\Sigma_{g,k}$. Then

$$\tau(M \cdot N) = \tau(M) \cdot \varphi(M)(\tau(N)) \in \mathbb{Z}[H]/\{\pm h\}_{h\in H}.$$

Product formula for torsion

- For $p, q \in \mathbb{Z}[H]$, define $p \sim q$ if $p = \phi(q)$ for some $\phi \in \mathrm{Aut}(\mathbb{Z}[H])$.

Corollary

Let $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$ be homology cylinders over $\Sigma_{g,k}$. Then

$$\tau(M \cdot N) = \tau(M) \cdot \tau(N) \in \mathbb{Z}[H]/\sim .$$

Corollary

$(M, i_+, i_-) \mapsto \tau(M)$ induces a (monoid) homomorphism

$$\mathcal{C}_{g,k} \to \mathbb{Z}[H]/\sim .$$

Construction of homomorphisms

- For a prime polynomial p in $\mathbb{Z}[H]$, define a function

$$\Phi_p : \mathbb{Z}[H]/\sim \;\to \mathbb{Z}_{\geq 0},$$

as follows:

Given $q \in \mathbb{Z}[H] \setminus \{0\}$, we write $q = q_1 \cdot \cdots \cdot q_k$ where $q_1, \ldots, q_k$ are prime elements in $\mathbb{Z}[H]$. Define

$$\Phi_p(q) := \#\{i \mid q_i \sim p\}.$$

- For each prime $p \in \mathbb{Z}[H]$, $\Phi_p(M) := \Phi_p(\tau(M))$ induces a (monoid) homomorphism

$$\Phi_p : \mathcal{C}_{g,k} \to \mathbb{Z}_{\geq 0}.$$

Torsion and homology cobordism

Theorem

Let $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$ be homology cobordant homology cylinders. Then

$$\tau(M) = \tau(N) \cdot p \cdot \overline{p} \in \mathbb{Z}[H]/\sim$$

for some $p \in \mathbb{Z}[H]^*$.

- Idea of proof:
    - For $0 \to C' \to C \to C'' \to 0$,

$$\tau(C) = \pm\tau(C')\tau(C'').$$

    - For an n-manifold X,

$$\tau_\phi(X, \partial X) = \overline{\tau_\phi(X)}^{(-1)^{n+1}}.$$

- $p \in \mathbb{Z}[H]$ is defined to be self-dual if $p \sim \overline{p}$

THEOREM

Let $p \in \mathbb{Z}[H]$ be a self-dual prime polynomial. Then $\Phi_p : \mathcal{C}_{g,k} \to \mathbb{Z}_{\geq 0}$ descends to a group homomorphism

$$\Psi_p : \mathcal{H}_{g,k} \to \mathbb{Z}/2.$$

Nontriviality: realization

- For a polynomial $p \in \mathbb{Z}[H]$, define

$$C_p := \{\text{nontrivial coefficients of } p\}.$$

- For each $i \geq 1$, choose a knot K_i with the Alexander polynomial $\Delta_i(t)$ such that $C_{\Delta_i} \neq C_{\Delta_j}$ for $i \neq j$.
- Let $E_i := S^3 - N(K_i)$ for $i \geq 0$, where $K_0 :=$ unknot.
- Choose an embedding $f : S^1 \times D^2 \cong E_0 \to M := \Sigma \times [0,1]$ representing $g \neq 0 \in H$. For $i \geq 1$, define

$$M_i := \left(M - f(\operatorname{int} E_0)\right) \bigcup_{f(\partial E_0) = \partial E_i} E_i.$$

Nontriviality: realization

Lemma

$$\tau(M_i) = \Delta_i(g) \in \mathbb{Z}[H].$$

Theorem

If $\beta_1(\Sigma) > 0$, the image of $\bigoplus_{[p]} \Psi_p$ contains infinitely many $\mathbb{Z}/2$ summands.

- Idea of proof:

$$\Psi_{\Delta_i(g)}(M_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Integral-valued homomorphisms

Theorem (Cha-Friedl-K)

If either $\left\{ \begin{array}{c} k > 2 \text{ or} \\ k = 2 \text{ and } g > 0 \end{array} \right\}$, *then* $\mathcal{H}_{g,k}$ *surjects to* $\mathbb{Z}^\infty$.

- Idea: Over $\Sigma_{g,k}$ satisfying the assumption, there exists a homology cylinder (M, i_+, i_-) such that $\overline{\tau(M, \Sigma_+)} = \tau(M, \Sigma_-) \not\approx \tau(M, \Sigma_+)$.