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Toric Topology Workshop
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LECTURES ON TORIC TOPOLOGY

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Contents

Preface

Toric topology is a new and actively developing field gaining a constantly increasing interest from the specialists in algebraic topology and related fields (see [27]). An invitational overview to the subject can be found in [9].

These lecture notes aim on introducing the reader to several aspects of toric topology where a significant progress has been achieved very recently. The exposition appeals to a broad audience and contains the necessary definitions, statements of results, and examples. Proofs are given only when they are short and illustrative. Many details may be found in [3].

I take this opportunity to express my gratitude to the Korea Advanced Institute of Science and Technology (KAIST) and particularly to Professor Dong Youp Suh for organising a Toric Topology Workshop in June 2008, where these lectures were delivered, and providing excellent work conditions. The atmosphere during the workshop was especially creative and inspiring, and the numerous informal discussions of talks greatly influenced the contents of these lecture notes.

I am grateful to Dong Youp for his work on preparing these lecture notes for publication.

My thanks also go to Taras Panov for the valuable discussions of these lectures and preparing Appendix A on certain important problems in toric topology.

Lecture I. Face-polynomials of simple polytopes and applications

[**Abstract**] A convex n-dimensional polytope is called simple if there are exactly n facets meeting at every vertex. For many decades simple polytopes have been studied in convex geometry and combinatorics. Recently it has become clear that they play important role in algebraic and symplectic geometry, in applications to physics. They are also main objects in toric topology. There is a commutative associative ring $\mathscr{P}$ generated by simple polytopes. The ring $\mathscr{P}$ possesses a natural derivation d, which comes from the boundary operator. We shall describe a ring homomorphism from the ring $\mathscr{P}$ to the ring of polynomials $\mathbb{Z}[\alpha, t]$ transforming a simple polytope to the face-polynomial and the operator d to the partial derivative $\partial/\partial t$.

This result opens way to a relation between theory of polytopes and differential equations. As it has turned out, certain important series of polytopes (including some recently discovered) lead to fundamental non-linear differential equations in partial derivatives.

We shall discuss constructions of important series of simple polytopes, and transformations of these series into non-linear differential equations. Particular examples of the transformations link Stasheff polytopes (also known as associahedra) to the E.Hopf equation, and Bott-Taubes polytopes (cyclohedra) to the Burgers equation.

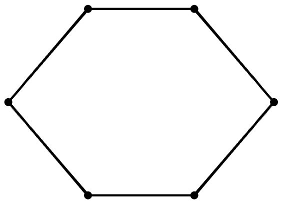
In the next series of lectures, I will discuss in details many ideas from this lecture.

1. Introduction

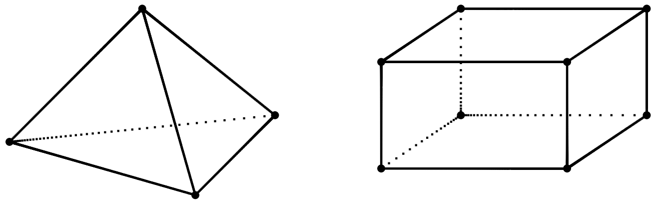
Definition 1.1. A convex polytope P^n of dimension n is said to be *simple* if every vertex of P is the intersection of exactly n facets, i.e. faces of dimension $(n-1)$.

It is easy to check that each k-face of a simple polytope P^n is again a simple polytope and is the intersection of exactly $(n-k)$ facets.

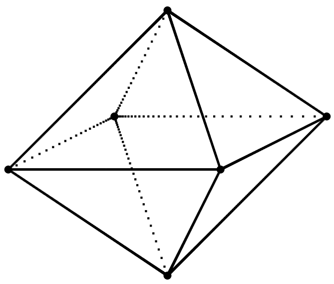
$\underline{n = 2.}$ Any polygon (2-polytope) is simple



$\underline{n = 3.}$ Simple polytopes



Non simple polytope



Definition 1.2. Two polytopes P_1 and P_2 of the same dimension are said to be *combinatorially equivalent* if there is a bijection between their sets of faces that preserves the inclusion relation.

Definition 1.3. A combinatorial polytope is a class of combinatorially equivalent geometrical polytopes.

The collection of all n-dimensional combinatorial simple polytopes is denoted by $\mathscr{P}_{2n}$. An Abelian group structure on $\mathscr{P}_{2n}$ is induced by the disjoint union of polytopes. The zero element of the group $\mathscr{P}_{2n}$ is the empty set. The direct product $P_1^n \times P_2^m$ of simple polytopes P_1^n and P_2^m is a simple polytope as well.

The weak direct sum

$$\mathscr{P} = \sum_{n \geqslant 0} \mathscr{P}_{2n}$$

yields a *graded* commutative associative ring. The unit element is a single point.

Let m be the number of facets. Set $\nu(P^n) = m - n$ and $\nu(P_1^{n_1} + P_2^{n_2}) = \max\Big(\nu(P_1^{n_1}), \nu(P_2^{n_2})\Big)$. We obtain a *multiplicative filtration* in $\mathscr{P}$:

$$\mathscr{P}^0 \subset \mathscr{P}^1 \subset \mathscr{P}^2 \subset \cdots \subset \mathscr{P}^k \subset \cdots,$$

where $\nu(P^0) = 0$ and $P^n \in \mathscr{P}^k$, if $\nu(P^n) \leqslant k$. We have $\mathscr{P}^k \mathscr{P}^l \subset \mathscr{P}^{k+l}$ because

$$\nu(P_1^{n_1} \times P_2^{n_2}) = \nu(P_1^{n_1}) + \nu(P_2^{n_2}).$$

Let $P^n \in \mathscr{P}_{2n}$ be a simple polytope. Denote by $dP^n \in \mathscr{P}_{2(n-1)}$ the *disjoint* union of all its facets.

Lemma 1.4. *We have a linear operator of degree* -2

$$d : \mathscr{P} \longrightarrow \mathscr{P},$$

such that

$$d(P_1^n P_2^m) = (dP_1^n)P_2^m + P_1^n(dP_2^m).$$

and $\nu(P^n) \geqslant \nu(dP^n)$*, where* $\nu(dP^n) = \max \nu(P^{n-1})$*. Here* P^{n-1} *runs over all facets of* P^n*.*

Example 1.5.

$$d\Delta^n = (n+1)\Delta^{n-1}, \quad \nu(\Delta^n) = 1,$$

$$dI^n = n(dI)I^{n-1} = 2nI^{n-1}, \quad \nu(I^n) = n,$$

where Δ^n is the standard n-simplex and $I^n = I \times \cdots \times I$ is the standard n-cube.

2. Face-polynomial

Consider the linear map

$$F\colon \mathscr{P} \longrightarrow \mathbb{Z}[t, \alpha],$$

which send a simple polytope P^n to the homogeneous *face-polynomial*

$$F(P^n) = \alpha^n + f_{n-1,1}\alpha^{n-1}t + \cdots + f_{1,n-1}\alpha t^{n-1} + f_{0,n}t^n,$$

where $f_{k,n-k} = f_{k,n-k}(P^n)$ is the number of its k-dimensional faces. Thus, $f_{n-1,1}$ is the number of facets, and $f_{0,n}$ is the number of vertices.

Theorem 2.1. *The mapping* F *is a ring homomorphism such that*

$$F(dP^n) = \frac{\partial}{\partial t} F(P^n).$$

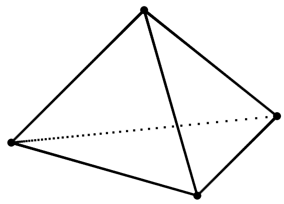
Corollary 2.2. *The face polynomials of the standard n-cube I^n and the standard n-simplex are*

$$F(I^n) = (\alpha + 2t)^n$$

and

$$F(\Delta^n) = \frac{(\alpha + t)^{n+1} - t^{n+1}}{\alpha}.$$

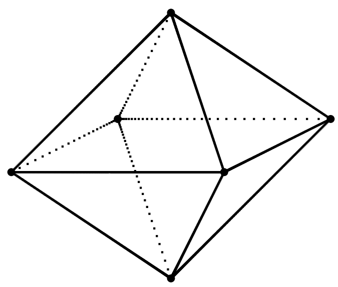
Example 2.3. (1) Simple polytope P_1^3 (tetrahedron)



$$dP_1^3 = 4\Delta^2$$

$$F(P_1^3) = \alpha^3 + 4t\alpha^2 + 6t^2\alpha + 4t^3$$
$$\frac{\partial}{\partial t}F(P_1^3) = 4\alpha^2 + 12t\alpha + 12t^2$$
$$F(dP_1^3) = 4F(\Delta^2) = 4(\alpha^2 + 3t\alpha + 3t^2)$$
$$F(dP_1^3) = \frac{\partial}{\partial t}F(P_1^3)$$

(2) Non simple polytope P_2^3 (octahedron)



$$dP_2^3 = 8\Delta^2$$

$$F(P_2^3) = \alpha^3 + 8t\alpha^2 + 12t^2\alpha + 6t^3$$
$$\frac{\partial}{\partial t}F(P_2^3) = 8\alpha^2 + 24t\alpha + 18t^2$$
$$F(dP_2^3) = 8F(\Delta^2) = 8(\alpha^2 + 3t\alpha + 3t^2)$$
$$F(dP_2^3) \neq \frac{\partial}{\partial t}F(P_2^3)$$

Set

$$U(t,x;\alpha,I)=\sum_{n\geqslant 0}F(I^n)x^{n+1},$$

$$U(t,x;\alpha,\Delta)=\sum_{n\geqslant 0}F(\Delta^n)x^{n+2}.$$

Lemma 2.4. *We have the following:*

(1) *The function $U(t,x;\alpha,I)$ is the solution of the equation*

$$\frac{\partial}{\partial t}U(t,x)=2x^2\frac{\partial}{\partial x}U(t,x)\quad \textit{with}\quad U(0,x)=\frac{x}{1-\alpha x}.$$

(2) *The function $U(t,x;\alpha,\Delta)$ is the solution of the equation*

$$\frac{\partial}{\partial t}U(t,x)=x^2\frac{\partial}{\partial x}U(t,x)\quad \textit{with}\quad U(0,x)=\frac{x^2}{1-\alpha x}.$$

We have

$$U(t,x;\alpha,I)=\frac{x}{1-(\alpha+2t)x},$$

$$U(t,x;\alpha,\Delta)=\frac{x^2}{(1-tx)(1-(\alpha+t)x)}.$$

Theorem 2.5. *Let $\widetilde{F}\colon\mathscr{P}\to\mathbb{Z}[t,\alpha]$ be a* linear *map such that*

$$\widetilde{F}(dP^n)=\frac{\partial}{\partial t}\widetilde{F}(P^n)\ \textit{and}\ \widetilde{F}(P^n)|_{t=0}=\alpha^n.$$

Then $\widetilde{F}(P^n)=F(P^n)$.

Proof. We have $\widetilde{F}(P^0)=1=F(P^0)$. Let, by induction, it be true for all $k\leqslant n$. Then

$$\widetilde{F}(dP^{n+1})(\alpha,t)=F(dP^{n+1})(\alpha,t)$$

for all simple $(n+1)$-dim polytopes. We obtain

$$\frac{\partial}{\partial t}\widetilde{F}(P^{n+1})=\frac{\partial}{\partial t}F(P^{n+1})$$

Thus,

$$\widetilde{F}(P^{n+1})(\alpha,t)=F(P^{n+1})(\alpha,t)+c(\alpha).$$

Setting $t=0$, we obtain

$$\alpha^{n+1}=\alpha^{n+1}+c(\alpha),$$

i.e. $c(\alpha)=0$. $\square$

3. Dehn–Sommerville relations

Theorem 3.1. *For any simple polytope P^n we have*

$$F(P^n)(\alpha,t)=F(P^n)(-\alpha,\alpha+t).$$

Proof. We have

$$F(P^0)(\alpha,t)=1=F(P^0)(-\alpha,\alpha+t).$$

Let, by induction, it be true for all $k\leqslant n$. Then

$$F(dP^{n+1})(\alpha,t)=F(dP^{n+1})(-\alpha,\alpha+t).$$

Thus

$$\frac{\partial}{\partial t}F(P^{n+1})(\alpha,t)=\frac{\partial}{\partial t}F(P^{n+1})(-\alpha,\alpha+t).$$

Hence,

$$F(P^{n+1})(\alpha,t)-F(P^{n+1})(-\alpha,\alpha+t)=c(\alpha).$$

Setting $t=0$, we obtain

$$\alpha^{n+1}\Big[1-\Big((-1)^{n+1}+(-1)^n f_{n,1}+\cdots+f_{0,n+1}\Big)\Big]=c(\alpha).$$

The boundary ∂P^{n+1} of a simple polytope P^{n+1} is n-dim sphere S^n. Using the classical Euler–Poincaré formula for the sphere S^n we obtain

$$f_{0,n+1}-f_{1,n}+\cdots+(-1)^n f_{n,1}=1+(-1)^n,$$

i.e. $c(\alpha)=0$. □

The classical Dehn–Sommerville relations

$$f_{k,n-k}=\sum_{j=k}^{n}(-1)^{n-j}\binom{j}{k}f_{j,n-j}$$

are equivalent to the formula

$$F(P^n)(\alpha,t)=F(P^n)(-\alpha,\alpha+t).$$

Thus we obtain a new proof of Dehn–Sommerville relations.

Corollary 3.2. *The function $F(P^n)\big(\alpha,\frac{1}{2}(z-\alpha)\big)$ is an even function of α.*

Proof.

$$\begin{aligned}F(P^n)\left(\alpha,\frac{1}{2}(z-\alpha)\right)&=F(P^n)\left(-\alpha,\alpha+\frac{1}{2}(z-\alpha)\right)\\&=F(P^n)\left(-\alpha,\frac{1}{2}(z+\alpha)\right).\end{aligned}$$

□

Example 3.3. $\underline{n=3}$

$$8F(P^3)\big(\alpha, \frac{1}{2}(z-\alpha)\big) = 8\alpha^3 + 4f_{2,1}(z-\alpha)\alpha^2 + \\ + 2f_{1,2}(z-\alpha)^2\alpha + f_{0,3}(z-\alpha)^3.$$

Coefficient at α^k, $k=1$ and 3 must be zero. Hence,

$$\begin{aligned} 2f_{1,2} &= 3f_{0,3} && \text{for } k=1,\\ 8-4f_{2,1}+2f_{1,2}-f_{0,3} &= 0 && \text{for } k=3. \end{aligned}$$

Thus $f_{1,2} = 3(f_{2,1}-2)$, $f_{0,3} = 2(f_{2,1}-2)$.

Set

$$h(P^n)(\alpha,t) = \alpha^n + h_1 t\alpha^{n-1} + \dots + h_{n-1}t^{n-1}\alpha + t^n,$$

where

$$h(P^n)(\alpha,t) = F(P^n)(\alpha - t, t).$$

For example,

$$h(I^n)(\alpha,t) = (\alpha+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k\alpha^{n-k},$$

$$h(\Delta^n)(\alpha,t) = \frac{\alpha^{n+1}-t^{n+1}}{\alpha - t} = \sum_{k=0}^{n} t^k\alpha^{n-k}.$$

From Dehn–Sommervile relations we obtain

$$h(P^n)(\alpha,t) = h(P^n)(t,\alpha).$$

The mapping

$$h\colon \mathscr{P} \longrightarrow Z[\alpha,t] \;:\; P^n \longrightarrow h(P^n)(\alpha,t)$$

is the ring homomorphism such that

$$h(dP^n) = \partial h(P^n), \quad h(P^n)\big|_{t=0} = \alpha^n,$$

where $\partial = \frac{\partial}{\partial\alpha} + \frac{\partial}{\partial t}$.

Theorem 3.4. *The ring homomorphism h satisfies the following:*

(1) *Image of h is generated by*

$$h(\Delta^1) = \alpha + t \quad \textit{and} \quad h(\Delta^2) = \alpha^2 + \alpha t + t^2.$$

(2) *Let*

$$H\colon \mathscr{P} \longrightarrow Z[\alpha,t]$$

be a linear mapping such that

$$H(dP^n) = \partial H(P^n), \quad H(P^n)\big|_{t=0} = \alpha^n,$$

where $\partial = \frac{\partial}{\partial\alpha} + \frac{\partial}{\partial t}$. Then $H(P^n) = h(P^n)$ for any simple polytope P^n.

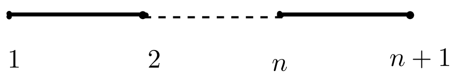
4. Graph-associahedra

The faces of a polytope P of all dimensions form a poset with respect to inclusion.

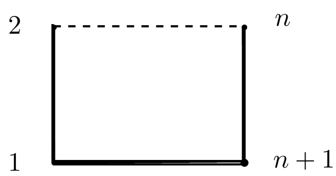
Given a finite graph Γ. The graph-associahedron $P(\Gamma)$ is a simple polytope whose poset is based on the connected subgraph of Γ (see details in Lecture III and [33]).

When Γ is:

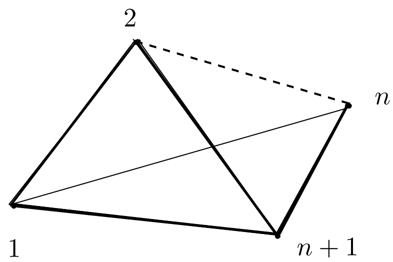
a path



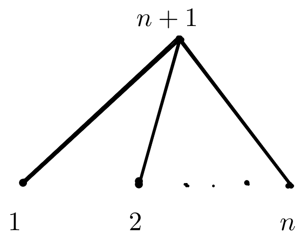
a cycle



a complete graph



an n-star graph

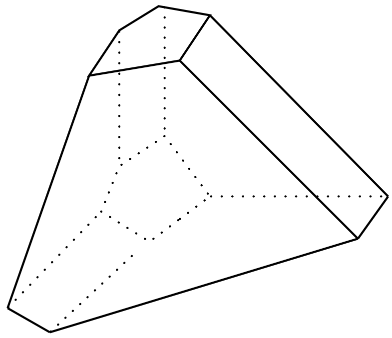


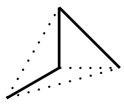
The polytope $P(\Gamma)$ results in the:
associahedron (Stasheff polytope) As^n; cyclohedron (Bott–Taubes polytope) Cy^n; permutohedron Pe^n; stellohedron St^n, respectively.

$As^2 = St^2$ is 5-gon and $Cy^2 = Pe^2$ is 6-gon.

Associahedron As^3 and 3-path

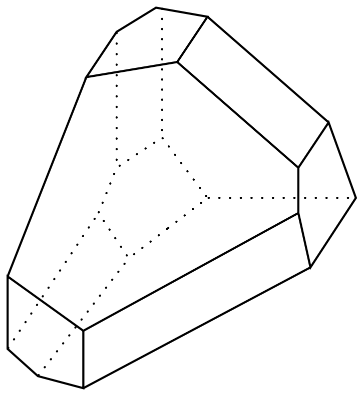
The Stasheff polytope K_5

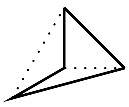




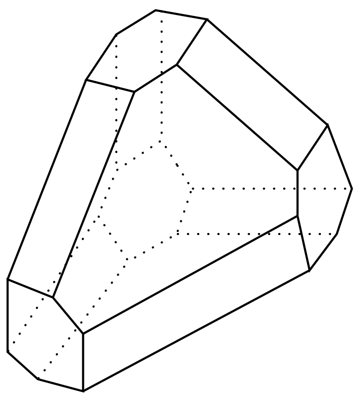
Cyclohedron Cy^3 and 3-cycle

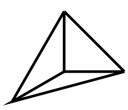
Bott–Taubes polytope



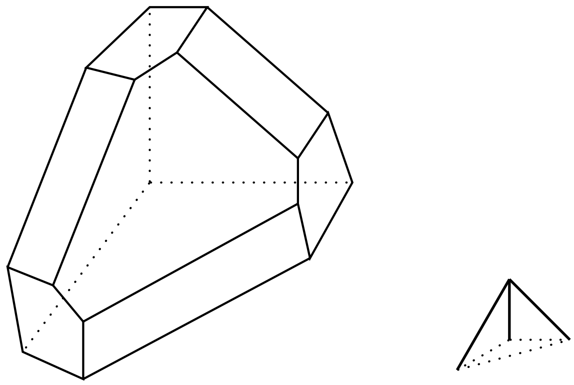


Permutohedron Pe^3 and 3-complete graph





Stellohedron St^3 and 3-star graph



Theorem 4.1 (see [24, 33]). *For a connected graph Γ on $n+1$ nodes, we have*

$$dP(\Gamma) = \sum_G P(\Gamma_G) \times P(\overline{\Gamma}_{G^c})$$

where

(1) $G \subsetneq \{1, \ldots, n+1\}$,
(2) Γ_G *is the subgraph of* Γ *with the vertex set* G,
(3) $\overline{\Gamma}_{G^c}$ *is the graph with the vertex set* $\{1, \ldots, n+1\} \backslash G$ *and arcs between two vertices* i *and* j *if they are path connected in* $\Gamma_{G \cup \{i,j\}}$,
(4) G *runs over all proper subsets of* $\{1, \ldots, n+1\}$ *such that* Γ_G *is connected.*

We have these formulas for $d\big(P(\Gamma)\big)$ (see [24])

$$dAs^n = \sum_{i+j=n-1} (i+2) As^i \times As^j$$

$$dCy^n = (n+1) \sum_{i+j=n-1} As^i \times Cy^j$$

$$dPe^n = \sum_{i+j=n-1} \binom{n+1}{i+1} Pe^i \times Pe^j$$

$$dSt^n = n \cdot St^{n-1} + \sum_{i=0}^{n-1} \binom{n}{i} St^i \times Pe^{n-i-1}$$

For example (see pictures)

$$dAs^3 = 2As^0 \times As^2 + 3As^1 \times As^1 + 4As^2 \times As^0$$

$$dCy^3 = 4(As^0 \times Cy^2 + As^1 \times Cy^1 + As^2 \times Cy^0)$$

$$dPe^3 = 4Pe^0 \times Pe^2 + 6Pe^1 \times Pe^1 + 4Pe^2 \times Pe^0$$

$$dSt^3 = 3St^2 + St^0 \times Pe^2 + 3St^1 \times Pe^1 + 3St^2 \times Pe^0$$

Application to the associahedra.

Consider the series of Stasheff polytopes (the associahedra)

$$As = \{As^n = K_{n+2}, n \geqslant 0\}.$$

Set

$$U(t, x; \alpha, As) = \sum_{n \geqslant 0} F(As^n) x^{n+2}.$$

Using that

$$\frac{\partial}{\partial t} F(As^n) = \sum_{i+j=n-1} (i+2) F(As^i) F(As^j)$$

we obtain

Theorem 4.2 (see [5]). *The function $U(t, x; \alpha, As)$ is the solution of the Hopf equation*

$$\frac{\partial}{\partial t} U(t, x) = U(t, x) \frac{\partial}{\partial x} U(t, x) \quad \text{with} \quad U(0, x) = \frac{x^2}{1 - \alpha x}.$$

The function $U(t, x; \alpha, As)$ satisfies the equation

$$t(\alpha + t) U^2 - (1 - (\alpha + 2t)x) U + x^2 = 0.$$

5. Quasilinear Burgers–Hopf Equation and solitons

The Hopf equation (Eberhard F.Hopf, 1902–1983) is the equation

$$U_t + f(U) U_x = 0.$$

The Hopf equation with $f(U) = U$ is a limit case of the following equations:

$$U_t + U U_x = \mu U_{xx} \quad \text{(the Burgers equation)},$$

$$U_t + U U_x = \varepsilon U_{xxx} \quad \text{(the Korteweg–de Vries equation)}.$$

The Burgers equation (Johannes M.Burgers, 1895–1981) occurs in various areas of applied mathematics (fluid and gas dynamics, acoustics, traffic flow). It is used for describing of wave processes with velocity u and viscosity coefficient μ. The case $\mu = 0$ is a prototype of equations whose solution can develop discontinuities (shock waves).

K-d-V equation (Diederik J.Korteweg, 1848–1941 and Hugo M. de Vries, 1848–1935) was introduced in 1895 as equation for the long waves over water. It appears also in plasma physics. Today K-d-V equation is one of the most famous equation in soliton theory. The discovery that gave birth to the modern study of solitons was made in 1834 by John Scott Russell (1808–1882).

Let us consider the Burgers equation

$$U_t = U U_x - \mu U_{xx}.$$

Set $U = U_0 + \sum\limits_{k\geqslant 1} \mu^k U_k$. Then

$$U_{0,t}+\sum_{k\geqslant 1}\mu^k U_{k,t} = \left(U_0 + \sum_{k\geqslant 1}\mu^k U_k\right)\left(U_{0,x} + \sum_{k\geqslant 1}\mu^k U_{k,x}\right) - \mu U_{0,xx} - \sum_{k\geqslant 1}\mu^{k+1}U_{k,xx}.$$

Thus we obtain:

$$U_{0,t} = U_0 U_{0,x},$$

$$U_{1,t} = (U_0U_1)_x - U_{0,xx}.$$

Consider the series of Bott–Taubes polytopes (the cyclohedra)

$$Cy = \{Cy^n \,:\, n \geqslant 0\}.$$

Set

$$U(t,x;\alpha,Cy) = \sum_{n\geqslant 0} F(Cy^n)x^n.$$

Using that

$$\frac{\partial}{\partial t}F(Cy^n) = (n+1)\sum_{i+j=n-1} F(Cy^i)F(As^j),$$

we obtain

Theorem 5.1. *The function $U(t,x;\alpha,Cy)$ is the solution of the equation*

$$\frac{\partial}{\partial t}U_1 = \frac{\partial}{\partial x}(U_0U_1) \quad \textit{with} \quad U_{1,0}(0,x) = \frac{1}{1-\alpha x},$$

where U_0 is the solution of the Hopf equation

$$\frac{\partial}{\partial t}U_0 = U_0\frac{\partial}{\partial x}U_0 \quad \textit{with} \quad U_0(0,x) = \frac{x^2}{1-\alpha x}.$$

Lecture II. Toric Topology of Stasheff Polytopes

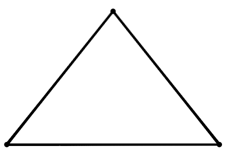
[Abstract] The Stasheff polytopes K_n, $n > 2$, appeared in the Stasheff paper [35] (1963) as the spaces of homotopy parameters for maps determining associativity conditions for a product $a_1...a_n$ for $n > 2$. Stasheff polytopes are in the limelight of several research areas. Nowadays they have become well-known due to applications of operad theory in physics. We will describe geometry and combinatorics of Stasheff polytopes using several different constructions of these polytopes and the methods of toric topology.

1. Euler's polygon division problem

In his letter (1751) to C.Goldbach (1690–1764) L.Euler (1707–1783) provides a guessing formula for computing the number of triangulations of a polygon with $n + 2$ sides. In 1756 J.Segner (1704–1777) gave the solution of this problem by the recurrence formula: $E_n = E_2E_{n-1} + E_3E_{n-2} + \cdots + E_{n-1}E_2, n > 3$, with $E_1 = E_2 = E_3 = 1$ and E_{n+2} is the number C_n of triangulation of $(n + 2)$-gon G_n. However Segner's method did not establish the validity (or invalidity) of Euler's guessing formula. This problem was posed as an open challenge to the mathematicians in the late 1830's by J.Liouville (1809–1882). He received a lot of papers with solutions. The most elegant from them (by Liouville's opinion) was communicated in a paper of G.Lame (1795–1870) in 1838.

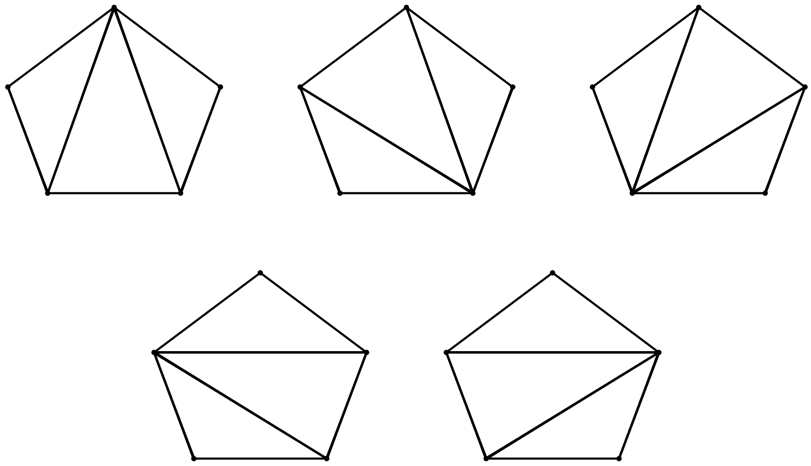
In how many ways can one partition a convex $(n + 2)$-gon G_n into triangles by means of diagonals?

$n = 1$



$n = 2$

$n = 3$



The number of triangulation of $(n+2)$*-gon* G_n is equal to the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

For example,

$$C_1 = 1,\ C_2 = 2,\ C_3 = 5,\ C_4 = 14, \dots .$$

The sequence $\{C_n\}$ is named in honour of E. Catalan (1814–1894) who discovered in 1844 the connection to bracketings of $(n+1)$-monomials.

$$
\begin{array}{llll}
n = 1 & & a_1 a_2 & \\
n = 2 & (a_1 a_2) a_3 & & a_1 (a_2 a_3) \\
n = 3 & \big((ab)c\big)d \quad \big(a(bc)\big)d & & (ab)(cd) \\
 & a\big((bc)d\big) & a\big(b(cd)\big) &
\end{array}
$$

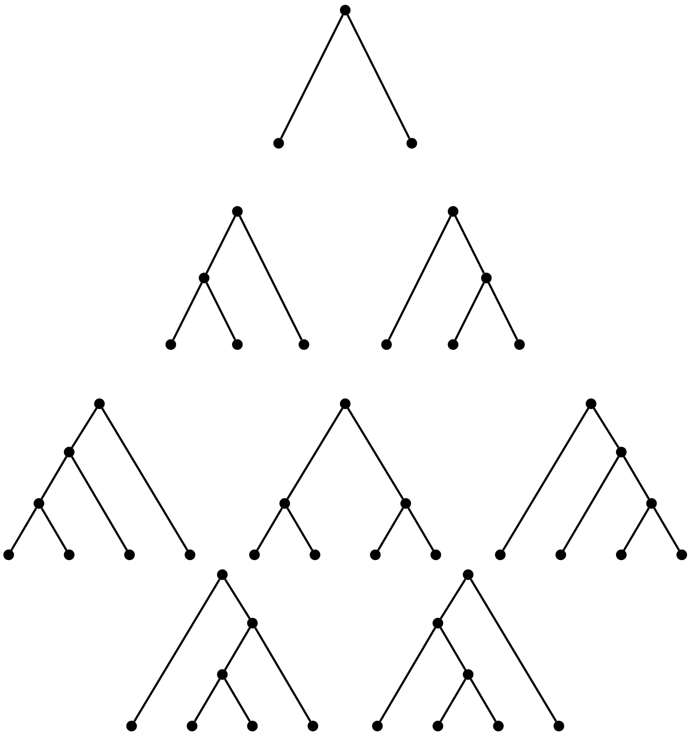
Plane trees

The connection between bracketing and plane binary trees with one root and $(n+1)$ end points was known to A. Cayley (1821–1895) (see [12]).

$n = 1$

$n = 2$

$n = 3$



The number of sequences of whole numbers $a_1, \dots, a_n$ such that $1 \leqslant a_1 \leqslant \cdots \leqslant a_n$, where $a_i \leqslant i$, is equal to C_n

$n = 1$		1	
$n = 2$	$1, 1$		$1, 2$
$n = 3$	$1, 1, 1$	$1, 1, 2$	$1, 1, 3$
	$1, 2, 2$		$1, 2, 3$

and such that $a_1 = 0$ and $0 \leqslant a_{i+1} \leqslant a_i + 1$ is equal to C_n

The number of ways to obtain a product of $n+1$ *factors* in a non-commutative and non-associative algebra is equal to C_n

$$
\begin{array}{lccc}
n=1 & & 0 & \\
n=2 & 0,0 & & 0,1 \\
n=3 & 0,0,0 & 0,0,1 & 0,1,0 \\
 & 0,1,1 & & 0,1,2
\end{array}
$$

$n=1$

$$a_1, a_2 \longrightarrow a_1 \cdot a_2$$

$n=2$

$$
\begin{tikzcd}
 & a_1, a_2, a_3 \arrow[dl] \arrow[dr] & \\
a_1, a_2 \cdot a_3 \arrow[d] & & a_1 \cdot a_2, a_3 \arrow[d] \\
a_1 \cdot (a_2 \cdot a_3) & & (a_1 \cdot a_2) \cdot a_3
\end{tikzcd}
$$

$n=3$

$$
\begin{tikzcd}
 & & a,b,c,d \arrow[dl] \arrow[d] \arrow[dr] & & \\
 & ab,c,d \arrow[dl] \arrow[dr] & a,bc,d \arrow[dl] \arrow[dr] & a,b,cd \arrow[dl] \arrow[dr] & \\
(ab)c,d \arrow[d] & a(bc),d \arrow[d] & ab,cd \arrow[d] & a,(bc)d \arrow[d] & a,b(cd) \arrow[d] \\
((ab)c)d & (a(bc))d & (ab)(cd) & a \cdot ((bc)d) & a(b(cd))
\end{tikzcd}
$$

You can find a lot other examples where arise C_n in [34].

2. Operads and Stasheff polytopes

A geometric realization of an A_{n-2}-operad is given by the *Stasheff polytope* $K_n,\ n \geqslant 2$, where $\dim K_n = (n-2)$ and the number of vertices of K_n is the Catalan number C_{n-2}. The boundary of $K_n,\ n \geqslant 3$, has the subdivision

$$\partial K_n = \bigcup_{i=2}^{n-1} \bigcup_{k=1}^{i} K_n^{i,k},$$

where $\dim K_n^{i,k} = n-3$ and $K_n^{i,k} = K_i \times K_{n-i+1} \subset \partial K_n$.

Example 2.1. (1) K_2 is a point.

(2) K_3 is an interval I and $\partial K_3 = K_3^{2,1} \bigcup K_3^{2,2}$.

(3) K_4 is a 5-gon and $\partial K_4 = \bigcup\limits_{k=1}^{2} K_4^{2,k} \bigcup \bigcup\limits_{k=1}^{3} K_4^{3,k}$.

A_n**-structure.**

An A_{n-2}-structure on a topological space X is a sequence of continuous maps $\mu_2, \dots, \mu_n$, where each $\mu_n \colon K_n \times X^n \longrightarrow X$ is appropriately defined on $\partial K_n \times X_n$ in terms of μ_i for $i < n$.

For example, an A_0-space is an H-space X with

$$\mu_2 = \mu \colon X^2 \longrightarrow X; \qquad K_2 = (\text{point}).$$

An A_1-space is a homotopy associative H-space X with

$$\mu_2 = \mu \colon X^2 \longrightarrow X \quad \text{and} \quad \mu_3 \colon I \times X^3 \longrightarrow X,$$

where $I = [0,1] = K_3$ such that

$$\begin{tikzcd}
{[0] \times X \times X \times X} \arrow[r, "\subset"] \arrow[d, "\mu\times 1"] & I \times X \times X \times X \arrow[d, "\mu_3"] & {[1] \times X \times X \times X} \arrow[l, "\supset"'] \arrow[d, "1\times\mu"] \\
X \times X \arrow[r, "\mu"] & X & X \times X \arrow[l, "\mu"']
\end{tikzcd}$$

that is

$$\mu_3(0; x_1, x_2, x_3) = \mu(\mu(x_1, x_2), x_3),$$

$$\mu_3(1; x_1, x_2, x_3) = \mu(x_1, \mu(x_2, x_3)).$$

So, μ_3 gives the usual condition of the homotopy equivalence of the maps $\mu(\mu \times 1)$ and $\mu(1 \times \mu)$.

In general, for $n \geqslant 3$ the restriction of $\mu_n \colon K_n \times X^n \to X$ on the $\partial K_n \times X^n$ gives the maps

$$\mu_n^{i,k} \colon K_i \times K_{n-i+1} \times X^n \longrightarrow X,\ 2 \leqslant i \leqslant n-1,\ 1 \leqslant k \leqslant i,$$

such that

$$\begin{aligned}
&\mu_n^{i,k}(t, \tau; x_1, \dots, x_n) \\
&\qquad = \mu_i(t; x_1, \dots, x_{k-1}, \mu_{n-i+1}(\tau; x_k, \dots, x_{n-i+k}), x_{n-i+k+1}, \dots, x_n),
\end{aligned}$$

where $t \in K_i$ and $\tau \in K_{n-i+1}$.

For example, in the case $n = 4$ we obtain the conditions on five functions:

$$\begin{aligned}
\mu_4^{2,1}(t,\tau;x_1,\dots,x_4) &= \mu_2\big(t;\mu_3(\tau;x_1,x_2,x_3),x_4\big),\\
\mu_4^{2,2}(t,\tau;x_1,\dots,x_4) &= \mu_2\big(t;x_1,\mu_3(\tau;x_2,x_3,x_4)\big),\\
\mu_4^{3,1}(t,\tau;x_1,\dots,x_4) &= \mu_3\big(t;\mu_2(\tau;x_1,x_2),x_3,x_4\big),\\
\mu_4^{3,2}(t,\tau;x_1,\dots,x_4) &= \mu_3\big(t;x_1,\mu_2(\tau;x_2,x_3),x_4\big),\\
\mu_4^{3,2}(t,\tau;x_1,\dots,x_4) &= \mu_3\big(t;x_1,x_2,\mu_2(\tau;x_3,x_4)\big).
\end{aligned}$$

3. Equivalent ways to describe Stasheff polytopes

We will use *four equivalent ways* to describe the combinatorics and toric topology of Stasheff polytopes K_n: bracketing, polygon dissection, plane trees and intervals.

The language of brackets.

Definition 3.1. The set Γ_i, $0 \leqslant i < n-2$, of i-dimensional faces of the Stasheff polytope K_n of dimension $n-2$ is the set of correct bracketings of the monomial $a_1 \cdot \ldots \cdot a_n$ with $n-2-i$ pairs of brackets. The outer pair of brackets $(a_1 \cdot \ldots \cdot a_n)$ is not taken into account.

The incidence relation is defined as follows. Let $\gamma \in \Gamma_k$ and $\delta \in \Gamma_l$, where $k > l$. The cell δ lies at the boundary of the cell γ (i.e., $\delta \subset \partial\gamma$ geometrically) if $\gamma \subset \delta$ (as sets of bracketings).

The set of 0-dimensional faces of the polytope K_n, i.e., the set of its vertices, is the set of correct bracketings of the monomial $a_1 \cdot \ldots \cdot a_n$ with $n-2$ pairs of brackets.

Two vertices in K_n are joined by an edge if and only if the bracketing corresponding to one vertex can be obtained from the bracketing corresponding to the other vertex by deleting a pair of brackets and inserting, in a unique way, another pair of brackets different from the deleted one. For example, in the case K_3:

$$(a_1a_2)a_3 \quad \bullet\!\!\longrightarrow\!\!\!\!\!\!-\!\!\bullet \quad a_1(a_2a_3)$$

The language of diagonals.

Definition 3.2. Consider a convex $(n+1)$-gon G_{n-1}. The set Γ_i, $0 \leqslant i < n-2$, of i-dimensional faces of the Stasheff polytope K_n of dimension $n-2$ is the set of all distinct sets of $n-i-2$ disjoint diagonals of G_{n-1}. (That is, each face of K_n is associated with a set of disjoint diagonals of G_{n-1}, and vice versa.)

The incidence relation is defined in the same way as in the preceding definition. Let $\gamma \in \Gamma_k$ and $\delta \in \Gamma_l$, where $k > l$. The cell δ lies at the boundary of γ (i.e., $\delta \subset \partial\gamma$ geometrically) if $\gamma \subset \delta$ (as sets of diagonals).

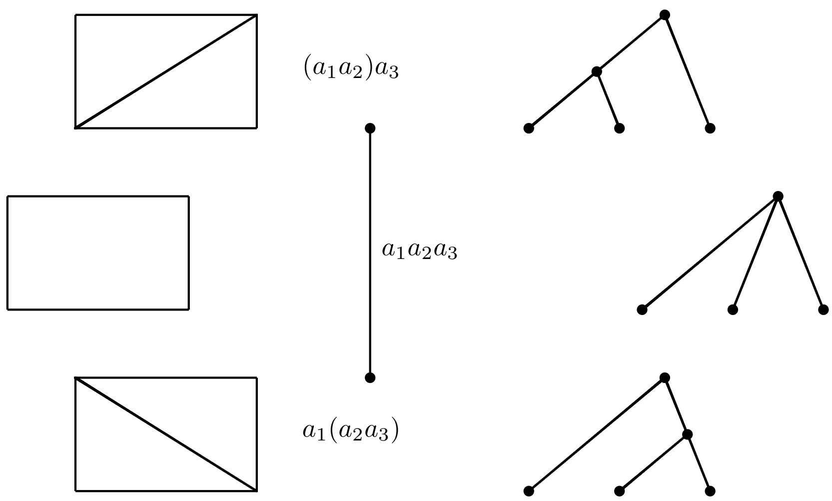
Corollary 3.3. *The dihedral group D_{n+1} of symmetries of a regular $(n+1)$-gon G_{n-1} is the transformation group of the Stasheff polytope K_n.*

The number of diagonals

$$\frac{(n-2)(n+1)}{2} = \binom{n+1}{2} - (n+1)$$

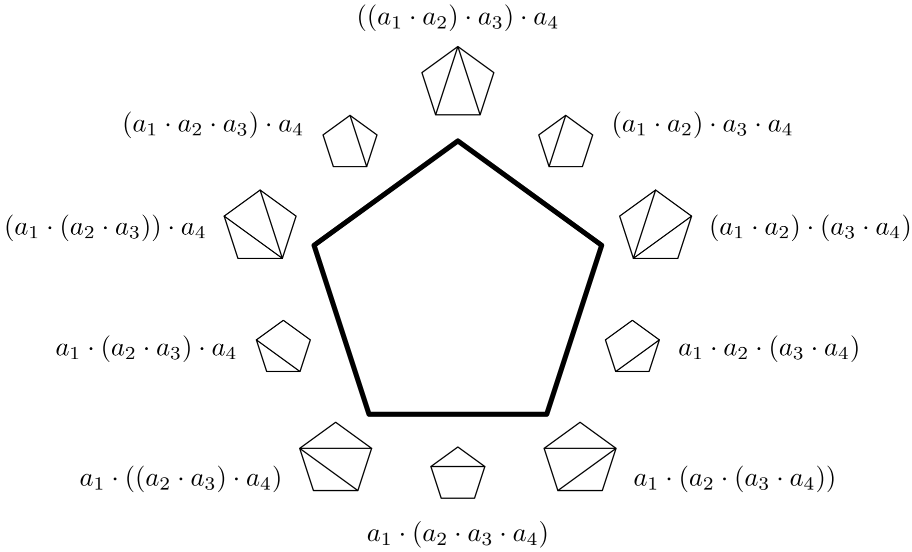
of G_{n-1} is equal to the number m of $(n-3)$-dimensional faces (facets) of K_n. Thus $\nu(K_n) = m - \dim K_n = \binom{n-1}{2}$ for $n \geqslant 3$.

Example 3.4. (The Stasheff polytope K_3)



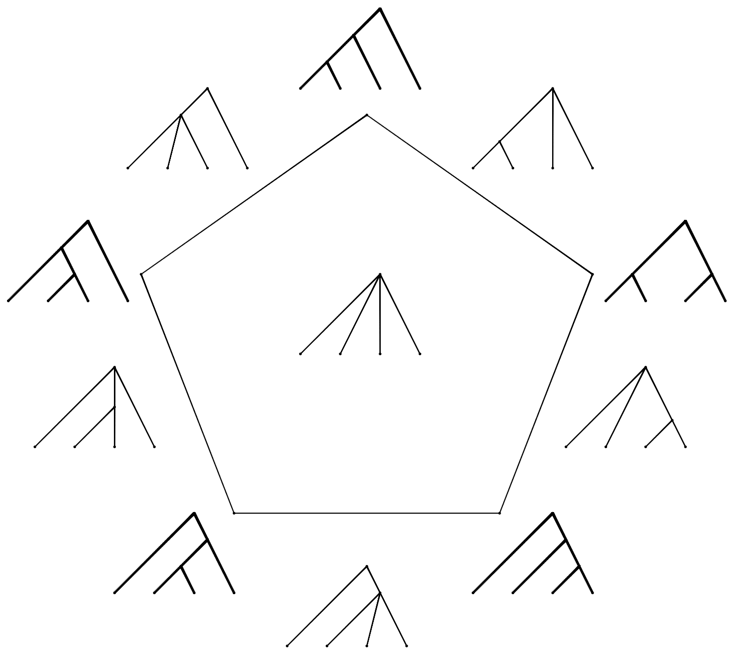
The languages: diagonals, brackets and plane trees.

Example 3.5. (The Stasheff polytope K_4)



The languages: correct bracketings and disjoint diagonals.

Example 3.6. (The Stasheff polytope K_4)



The language of plane trees.

The language of intervals.

To each pair of brackets of the form

$$a_1 \cdots a_i(a_{i+1} \cdots a_{i+l+1})a_{i+l+2} \cdots a_{n+1}$$

we assign the interval $I_{i,l} = [i+1, \cdots, i+l] \subset [1, \cdots, n]$, where $0 \leqslant i \leqslant n-l$ and $1 \leqslant l \leqslant n-1$.

For example:

$$\begin{array}{lll} K_3 & (a_1 \cdot a_2) \cdot a_3 \longrightarrow I_{0,1}, & a_1 \cdot (a_2 \cdot a_3) \longrightarrow I_{1,1}. \\ K_4 & (a_1 \cdot a_2 \cdot a_3) \cdot a_4 \to I_{0,2}, & a_1 \cdot (a_2 \cdot a_3) \cdot a_4 \to I_{1,1}, \\ & a_1 \cdot (a_2 \cdot a_3 \cdot a_4) \to I_{1,2}, & a_1 \cdot a_2 \cdot (a_3 \cdot a_4) \to I_{2,1}, \\ & (a_1 \cdot a_2) \cdot a_3 \cdot a_4 \to I_{0,1}. & \end{array}$$

4. A realization of the Stasheff polytope K_{n+1} as a simple polytope in $\mathbb{R}^n$ with integer vertices lying in a hyperplane

Consider the formal monomial $a_1 \cdot \ldots \cdot a_{n+1}$. Let us label all multiplication signs "$\cdot$" in this monomial from left to right with the numbers $1, 2, \ldots, n$, so that the i-th multiplication sign, $1 \leqslant i \leqslant n$, is between a_i and a_{i+1}, i.e.

$$a_1 \overset{1}{\cdot} a_2 \cdots a_i \overset{i}{\cdot} a_{i+1} \cdots a_n \overset{n}{\cdot} a_{n+1}.$$

To each correct bracketing of this monomial with $n-1$ pairs of brackets, we assign the n-dimensional vector $J = (m_1, \ldots, m_n)$ whose coordinates m_i are defined as follows: each multiplication sign stands for the multiplication of two smaller monomials. Set $m_i = l_i r_i$, where l_i and r_i are the lengths of the right and left monomials corresponding to the i-th multiplication sign. For example, in the case $n = 3$, the bracketing

$$a_1 \overset{1}{\cdot} ((a_2 \overset{2}{\cdot} a_3) \overset{3}{\cdot} a_4)$$

gives rise to the vector $(3, 1, 2)$, because $m_1 = 1 \cdot 3$, $m_2 = 1 \cdot 1$, and $m_3 = 2 \cdot 1$; and the bracketing $(a_1 \cdot a_2) \cdot (a_3 \cdot a_4)$ gives rise to the vector $(1, 4, 1)$.

This defines a map from the set of vertices of the $(n-1)$-dimensional Stasheff polytope K_{n+1} into $\mathbb{R}^n$. Extending it by linearity, we obtain a mapping

$$J\colon K_{n+1} \to \mathbb{R}^n.$$

For example, in the case $n = 2$ we obtain

$$J\colon K_3 \to \mathbb{R}^2.$$

$$J\Big((a_1 \cdot a_2) \cdot a_3\Big) = (1,2), \quad J\Big(a_1 \cdot (a_2 \cdot a_3)\Big) = (2,1).$$

Let $1 \leqslant l \leqslant n-1, \quad 0 \leqslant i \leqslant n-l$. Take the linear function $p_{i,l} : \mathbb{R}^n \to \mathbb{R}$, where for $1 \leqslant l \leqslant n-1$

$$\begin{aligned} p_{0,l}(\mathbf{x}) &= \frac{1}{l}(x_1 + \cdots + x_l) - \frac{1}{n-l}(x_{l+1} + \cdots + x_n), \\ p_{n-l,l}(\mathbf{x}) &= \frac{1}{l}(x_{n-l+1} + \cdots + x_n) - \frac{1}{n-l}(x_1 + \cdots + x_{n-l}) \\ &= -p_{0,n-l}(\mathbf{x}); \end{aligned}$$

and for $0 < l < n-1,\ 0 < i < n-l$

$$p_{i,l}(\mathbf{x}) = \frac{1}{l}(x_{i+1} + \cdots + x_{i+l}) - \frac{1}{n-l}(x_1 + \cdots + x_i + x_{i+l+1} + \cdots + x_n).$$

Set

$$L_{i,l} = \{\mathbf{x} \in \mathbb{R}^n : p_{i,l}(\mathbf{x}) + \tfrac{1}{2}n \geqslant 0\}.$$

Theorem 4.1. *The mapping $J : K_{n+1} \to \mathbb{R}^n$ is an embedding. Its image is the intersection of the hyperplane*

$$H = \{\mathbf{x} \in \mathbb{R}^n : \frac{1}{n}(x_1 + \cdots + x_n) = \frac{n+1}{2}\}$$

with the half-spaces $L_{i,l},\ 1 \leqslant l \leqslant n-1,\ 0 \leqslant i \leqslant n-l$.

For each vertex of K_{n+1}, its image lies in the intersection of the $n-1$ half-spaces $L_{i,l}$ determined by the pairs of brackets occurring in the correct bracketing corresponding to this vertex. This result is some improvement of the main result of J.-L. Loday (see [29]), who used the language of plane binary trees.

Set $B = \{\mathbf{x} \in \mathbb{R}^n :\ -\frac{n}{2} \leqslant p_{0,l}(\mathbf{x}) \leqslant \frac{n}{2},\ l = 1, \ldots, n-1\}$.

Corollary 4.2. *The image of K_{n+1} in $\mathbb{R}^n$ is the intersection of the $(n-1)$-dimensional* cube $H \cap B$ *with the half spaces $L_{i,l}$, where $0 < l < n-1,\ 0 < i < n-l$. Thus, K_{n+1} is a truncated $(n-1)$-dimensional cube with $\binom{n-1}{2}$ truncations. N.B. $\binom{n-1}{2} = \nu(K_n)$.*

5. A realization of the Stasheff polytope K_{n+1} as a simple polytope with integer vertices in $\mathbb{R}^{n-1}$

Set

$$ny_l = l(n-l)p_{0,l}(\mathbf{x}) + \frac{1}{2}nl(n-l),\ l = 1, \ldots, n-1.$$

We have for $\mathbf{x} \in K_{n+1} \subset \mathbb{R}^n$:

$$0 \leqslant y_l \leqslant l(n-l).$$

Let

$$z_l = x_1 + \cdots + x_l,\ l = 1, \ldots, n.$$

Using that $z_n = \frac{1}{2}n(n+1)$ for $\mathbf{x} \in K_{n+1}$, we obtain:

$$\begin{aligned} ny_l &= (n-l)z_l - l(z_n - z_l) + \frac{1}{2}nl(n-l) \\ &= n\Big(z_l - \frac{1}{2}l(l+1)\Big). \end{aligned}$$

So,

$$y_l = z_l - \binom{l+1}{2}.$$

We have for $0 < l < n-1,\ 0 < i < n-l$

$$\begin{aligned} l(n-l)p_{i,l}(\mathbf{x}) &= (n-l)(z_{i+l} - z_i) - l(z_i + z_n - z_{i+l}) \\ &= n[z_{i+l} - z_i - \frac{1}{2}l(n+1)]. \end{aligned}$$

Thus

$$\begin{aligned} l(n-l)p_{i,l}(\mathbf{x}) + \frac{1}{2}nl(n-l) &= n[z_{i+l} - z_i - \frac{1}{2}(l+1)l] \\ &= n[y_{i+l} - y_i + il]. \end{aligned}$$

Theorem 5.1. *There is the embedding*

$$J\colon K_{n+1} \longrightarrow \mathbb{R}^{n-1}$$

with the image

$$\{y = (y_1, \ldots, y_{n-1})\ :\ 0 \leqslant y_l \leqslant l(n-l),\ y_i - y_{i+l} \leqslant il\},$$

where $l = 1, \ldots, n-1,\ i = 1, \ldots, n-l-1.$

Set $y_0 = y_n = 0$ and take

$$\mathscr{L}_{i,l} = \{y \in \mathbb{R}^{n-1} : y_i - y_{i+l} \leqslant il\},\ 1 \leqslant l \leqslant n-1,\ 0 \leqslant i \leqslant n-l.$$

Any vertex $v_q \in K_{n+1},\ q = 1, \ldots, C_n$, gives a set $\{I_{i,l},\ (i,l) \in s(q)\}$ of intervals determined by the pairs of brackets occurring in the bracketing corresponding to this vertex v_q.

Corollary 5.2. *We have*

$$\mathscr{L}_{i,l} \cap J(K_{n+1}) \simeq K_{l+1} \times K_{n-l+1}.$$

For each vertex $v_q \in K_{n+1}$ we have

$$J(v_q) = \underset{(i,l)\in s(q)}{\cap} \partial \mathscr{L}_{i,l}.$$

Example 5.3. (*The Stasheff polytope K_3*)

$n = 2 \implies l = 1,\ K_3 = \{y \in \mathbb{R}^1 \ :\ 0 \leqslant y \leqslant 1\}$,

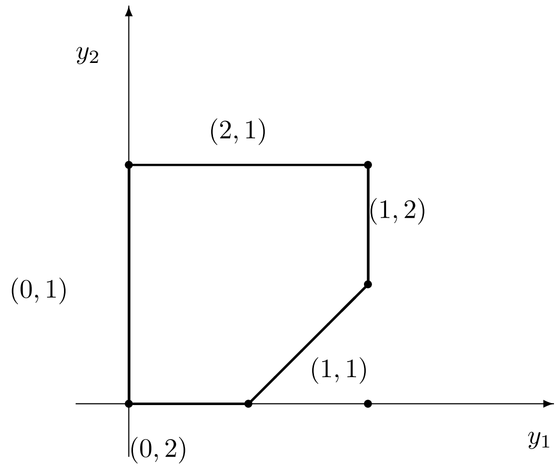
$$(a_1 \cdot a_2) \cdot a_3 \longrightarrow I_{0,1}, \quad a_1 \cdot (a_2 \cdot a_3) \longrightarrow I_{1,1}$$

$$v_1 = \{0 \in \mathbb{R}^1\}, \quad v_2 = \{1 \in \mathbb{R}^1\}.$$

Example 5.4. (*The Stasheff polytope K_4*)

$n = 3 \quad \implies \quad 1 \leqslant l \leqslant 2,\ 0 \leqslant i \leqslant 3 - l$

$$K_4 = \{(y_1, y_2) :\ 0 \leqslant y_i \leqslant 2,\ i = 1, 2,\ y_1 - y_2 \leqslant 1\},$$



$$K_4 \simeq I^2 \cap \mathscr{L}_{1,1}$$

$$\Big((a_1 \cdot a_2) \cdot a_3\Big) \cdot a_4 \Longrightarrow v_1 = \partial\mathscr{L}_{0,1} \cap \partial\mathscr{L}_{0,2},$$

$$\Big(a_1 \cdot (a_2 \cdot a_3)\Big) \cdot a_4 \Longrightarrow v_2 = \partial\mathscr{L}_{0,2} \cap \partial\mathscr{L}_{1,1},$$

$$a_1 \cdot \Big((a_2 \cdot a_3) \cdot a_4\Big) \Longrightarrow v_3 = \partial\mathscr{L}_{1,1} \cap \partial\mathscr{L}_{1,2},$$

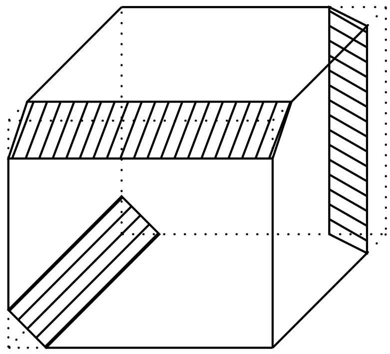
$$a_1 \cdot \Big(a_2 \cdot (a_3 \cdot a_4)\Big) \Longrightarrow v_4 = \partial\mathscr{L}_{1,2} \cap \partial\mathscr{L}_{2,1},$$

$$(a_1 \cdot a_2) \cdot (a_3 \cdot a_4) \Longrightarrow v_5 = \partial\mathscr{L}_{2,1} \cap \partial\mathscr{L}_{0,1}.$$

Example 5.5. (*The Stasheff polytope K_5*)

$n = 4 \implies 1 \leqslant l \leqslant 3,\ 0 \leqslant i \leqslant 4 - l$

Let $I^3 = \{(y_1, y_2, y_3) :\ 0 \leqslant y_i \leqslant 3,\ i = 1, 3,\ 0 \leqslant y_2 \leqslant 4\}$



$$K_5 \simeq I^3 \cap \mathscr{L}_{1,1} \cap \mathscr{L}_{1,2} \cap \mathscr{L}_{2,1},$$

where

$$\mathscr{L}_{1,1} = \{y \in \mathbb{R}^3\ :\ y_1 - y_2 \leqslant 1\},$$
$$\mathscr{L}_{1,2} = \{y \in \mathbb{R}^3\ :\ y_1 - y_3 \leqslant 2\},$$
$$\mathscr{L}_{2,1} = \{y \in \mathbb{R}^3\ :\ y_2 - y_3 \leqslant 2\}.$$

Consider the 5-monomial $a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5$. We have 9 pairs of brackets

$$(a_1 \cdot a_2) \to I_{0,1}; (a_2 \cdot a_3) \to I_{1,1}; (a_3 \cdot a_4) \to I_{2,1}; (a_4 \cdot a_5) \to I_{3,1};$$
$$(a_1 \cdot a_2 \cdot a_3) \to I_{0,2};\ (a_2 \cdot a_3 \cdot a_4) \to I_{1,2};\ (a_3 \cdot a_4 \cdot a_5) \to I_{2,2};$$
$$(a_1 \cdot a_2 \cdot a_3 \cdot a_4) \to I_{0,3}; \quad (a_2 \cdot a_3 \cdot a_4 \cdot a_5) \to I_{1,3}$$

and C_4 vertices $v_1, \dots, v_{14}$.

For example

$$\Big(\big((a_1 \cdot a_2) \cdot a_3\big) \cdot a_4\Big) \cdot a_5 \Longrightarrow \partial\mathscr{L}_{0,3} \cap \partial\mathscr{L}_{0,2} \cap \partial\mathscr{L}_{0,1},$$
$$\Big((a_1 \cdot a_2) \cdot (a_3 \cdot a_4)\Big) \cdot a_5 \Longrightarrow \partial\mathscr{L}_{0,3} \cap \partial\mathscr{L}_{0,1} \cap \partial\mathscr{L}_{2,1},$$
$$a_1 \cdot \Big(\big((a_2 \cdot a_3) \cdot a_4\big) \cdot a_5\Big) \Longrightarrow \partial\mathscr{L}_{1,1} \cap \partial\mathscr{L}_{1,2} \cap \partial\mathscr{L}_{1,3}.$$

Proposition 5.6. *The Stasheff polytopes are simple.*

You can find all details about simple polytopes in [3]

Theorem 5.1 provides an explicit description of the $(n-1)$ facets whose intersection is a given vertex of the $(n-1)$-dimensional polytope K_{n+1}.

Definition 5.7. An n-dimensional polytope P^* is said to be *dual to* P if for each i, $0 \leqslant i \leqslant n-1$, there exists an one-to-one correspondence between the i-dimensional faces γ_i of P and the $(n-i-1)$-dimensional faces γ^*_{n-i-1} of P^* such that the embedding $\gamma^*_{n-j-1} \subset \gamma^*_{n-i-1}$ corresponds to the embedding $\gamma_i \subset \gamma_j$.

Each facet of As^n is $As^i \times As^j,\ i \geqslant 0,\ i+j=n-1$, where embedding $\iota_k \colon As^i \times As^j \to \partial As^n,\ 1 \leqslant k \leqslant i+2$, corresponds to the pairing

$$(a_1 \cdots a_{i+2}) \times (b_1 \cdots b_{j+2}) \longrightarrow a_1 \cdots a_{k-1}(b_1 \cdots b_{j+2})a_{k+1} \cdots a_{i+2}.$$

Lemma 5.8. *The boundary of the associahedra is*

$$\begin{aligned} dAs^n &= \sum_{i+j=n-1} \sum_{k=1}^{i+2} \iota_k(As^i \times As^j) \\ &= \sum_{i+j=n-1} (i+2)(As^i \times As^j). \end{aligned}$$

Each pair of brackets in the monomial $a_1 \cdot \ldots \cdot a_{n+1}$ determines a facet of the polytope K_{n+1}. Thus, in terms of the dual polytope, it corresponds to a vertex of the polytope K^*_{n+1}. The number of vertices of K^*_{n+1} is $\frac{(n-1)(n+2)}{2}$.

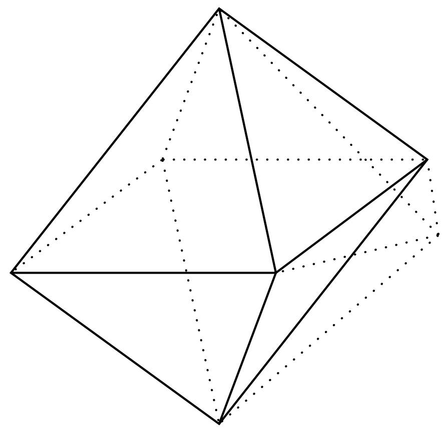
Definition 5.9. A polytope S is said to be *simplicial* if every face of S is a simplex.

The dual P^* of a simple polytope P is simplicial and vice versa.

Proposition 5.10. *The dual K^*_n of a Stasheff polytope K_n is a simplicial polytope.*

Proposition 5.11. *The boundary of the polytope K^*_n is a triangulation of the $(n-3)$-dimensional sphere.*

Example 5.12. *Construction K^*_5 (the fragment) via stellar subdivision* (see [3]).



Octahedron $(I^3)^*$ is dual to cube I^3.

Definition 5.13. A polytope P is called a *flag polytope* if each set of vertices of P pairwise joined by edges forms a simplex.

Proposition 5.14. *The dual K_n^* of a Stasheff polytope K_n is a flag polytope.*

Proof. To each set of k vertices of K_n^*, there corresponds a set of k diagonals of a convex $(n+1)$-gon G_{n-1}. These vertices are pairwise joined by edges if and only if the corresponding diagonals are disjoint. By definition, this collection of diagonals determines a face of K_n and hence a face of K_n^*. Since K_n^* is a simplicial polytope, it follows that this face is a simplex. □

6. Stanley–Reisner ring of Stasheff polytopes

Let P be a simple polytope with m facets $F_1, \dots, F_m$. Fix a commutative ring $\mathbf{k}$ with unit. Let $\mathbf{k}[v_1, \dots, v_m]$ be a polynomial graded $\mathbf{k}$-algebra, $\deg v_i = 2$.

Definition 6.1. The *face ring* $\mathbf{k}(P)$ (or the *Stanley–Reisner ring*) of a simple polytope P is the quotient ring

$$\mathbf{k}(P) = \mathbf{k}[v_1, \dots, v_m]/J_P,$$

where J_P is the ideal generated by all square-free monomials $v_{i_1} \cdot v_{i_2} \cdots v_{i_s}$, $i_1 < \cdots < i_s$, such that $F_{i_1} \cap \cdots \cap F_{i_s} = \emptyset$ in P.

Corollary 6.2. $\mathbf{k}(K_n) = \mathbf{k}[v_1, \dots, v_m]/J_{K_n}$ *where the set* $\{v_1, \dots, v_m\}$ *corresponds to the set of diagonals* $\{d_1, \dots, d_m\}$, $m = \frac{(n-2)(n+1)}{2}$ *of a convex* $(n+1)$-*gon* G_{n-1} *and* J_{K_n} *is the ideal generated by all monomials* $v_i v_j$, $i < j$, *such that* $d_i \cap d_j \neq \emptyset$ *in* G_{n-1}.

Example 6.3. $\mathbf{k}(K_3) = \mathbf{k}[v_1, v_2]/(v_1 v_2)$.

Corollary 6.4. *A generator in J_{K_n} corresponds to a set of four vertices in a convex $(n+1)$-gon G_{n-1}, that is the number of generators δ in J_{K_n}, $n \geqslant 3$, is $\binom{n+1}{4}$.*

For example,

$$n = 4 \ : \ m = 5, \ \delta = 5$$

$$n = 5 \ : \ m = 9, \ \delta = 15$$

Let $\mathbf{k}[D_{n+1}]$ be the group ring over $\mathbf{k}$ of the dihedral group D_{n+1}.

Corollary 6.5. *The face ring $\mathbf{k}(K_n)$ is a $\mathbf{k}[D_{n+1}]$-module.*

Example 6.6.

$$\mathbf{k}(K_3) = \mathbf{k}[v, \tau v]/(v \cdot \tau v),$$

where τ is the generator in $\mathbb{Z}_2$.

$$\mathbf{k}(K_4) = \mathbf{k}[\tau^i v, \ i = 0, \dots, 4]/(\tau^i w, \ i = 0, \dots, 4),$$

where τ is a generator in $\mathbb{Z}_5$ and $w = v \cdot \tau v$.

$$\mathbf{k}(K_5) = \mathbf{k}[\tau^i v_1, \ i = 0, \dots, 5, \ \tau^i v_2, \ i = 0, 1, 2]/J_{K_5},$$

where τ is a generator in $\mathbb{Z}_6$, $J_{K_5} = (\tau^i w_j, \ i = 0, \dots, 5, \ j = 1, 2, \ \tau^i w_3, \ i = 0, 1, 2)$, and $w_1 = v_1 \cdot \tau v_1, \ w_2 = v_1 \cdot v_2, \ w_3 = v_2 \cdot \tau v_2$.

7. The solution of the E.Hopf equations

It follows from the theory of partial differential equations that the quasilinear Hopf equation

$$U_t + f(U)U_x = 0$$

with the initial condition $U(0, x) = \varphi(x)$ has the solution $U = \varphi(\xi)$, where $\xi = \xi(t, x)$ is determined by the relation $x = \xi + f(\varphi(\xi))t$.

We consider only the case $f(U) = -U$. The transformation $t \to -t$ takes this equation to the equation $U_t + UU_x = 0$. For the initial condition $\varphi(x) = x^2/(1-x)$, the function $\xi(t, x)$ is given by the quadratic equation

$$(t+1)\xi^2 - (1+x)\xi + x = 0.$$

By solving this equation, we obtain a closed-form expression for the solution of the Cauchy problem in a neighborhood of the point $(0, 0)$: $U(t, x) = \frac{\xi^2}{1-\xi}$, where $\xi = \frac{2x}{x+1+\sqrt{(x+1)^2-4(t+1)x}}$.

For a general initial condition, the relation $x = \xi - \varphi(\xi)t$ implies that

$$\varphi(\xi) = \frac{1}{t}(\xi - x).$$

Thus, $\xi(t,x) = tU(t,x) + x$; i.e., we can eliminate the function $\xi(t,x)$ from the equation $U = \varphi(\xi)$.

Corollary 7.1. *The solution of the equation $U_t = UU_x$ with $U(0,x) = \varphi(x)$ is a solution of the* functional equation *(equation on the characteristics)*

$$U = \varphi(x + tU).$$

In particular, if $\varphi(x)$ is a rational function, then $U(t,x)$ satisfies an algebraic functional equation of the form

$$\sum_{k=0}^{n} a_k(t,x)U^k = 0,$$

where $a_k(t,x)$ are polynomials in t and x.

8. The case related with the family of Stasheff polytopes

In our case, $\varphi(x) = x^2/(1-x)$, and the function $U(t,x)$ satisfies the equation

$$t(1+t)U^2 + (2xt + x - 1)U + x^2 = 0.$$

It can readily be seen from this equation that U has the symmetry

$$U(t,x) = U(-(t+1), -x).$$

Let us treat $\xi(t,x)$ as a function of x with parameter t. Then it is the inverse of the function $x - \varphi(x)t$. Hence we can apply the classical Lagrange formula for computing the inverse function:

$$\begin{aligned}\xi(t,x) &= \frac{1}{2\pi i}\int_{|z|=\varepsilon} -\ln\left(1 - \frac{x}{z}\left(1 - \frac{\varphi(z)}{z}\,t\right)^{-1}\right)dz \\ &= \sum \frac{x^n}{n}\left[\left(1 - \frac{\varphi(z)}{z}\,t\right)^{-n}\right]_{n-1},\end{aligned}$$

where $[\gamma(z)]_k$ is the coefficient of z^k in the series $\gamma(z)$. By substituting the initial condition $\varphi(x) = x^2/(1-x)$ into this formula, we obtain

$$\xi(t,x) = \sum_{n\geq 1}\frac{x^n}{n}\left[\left(1 + \frac{tz}{1-(t+1)z}\right)^n\right]_{n-1}.$$

Hence

$$U(t,x) = \sum_{n\geq 2} V_n(t)x^n,$$

where

$$V_n(t) = \frac{1}{n}\sum_{l=0}^{n-2}\binom{n}{l+1}\binom{n-2}{l}t^l(1+t)^{n-2-l}.$$

Note that this formula readily implies the identity

$$U(t,x) = U(-(t+1), -x).$$

Moreover, if we use the identity

$$\sum_{l=0}^{k} \binom{n}{l+1}\binom{k}{l} = \binom{n+k}{k+1}, \quad 0 \le k \le n-2,$$

then this formula for $V_n(t)$ implies the classical result

$$f_{k-1}(K_n) = \frac{1}{n}\binom{n-2}{k}\binom{n+k}{k+1}, \quad 0 \le k \le n-2.$$

Here $f_{k-1}(K_n)$ is the number of $(n-k-2)$-dimensional faces of the Stasheff polytope K_n.

Another way to obtain the solution is to consider *conservation* laws. Let $U(t,x)$ be the solution of the Cauchy problem for the Hopf equation

$$U_t = UU_x, \qquad U(0,x) = \varphi(x).$$

This equation has the conservation laws

$$\left(\frac{U^{k+1}}{k+1}\right)_x = \left(\frac{U^k}{k}\right)_t, \qquad k = 1, 2, \dots .$$

Hence for any k and l, $1 \le k \le l$, $l = 1, 2, \dots$,
we have

$$\frac{d^k}{dx^k}\left(\frac{U^{l+1}}{l+1}\right) = \frac{d^{k-1}}{dx^{k-1}}\left(\frac{U^l}{l}\right)_t = \frac{d^k}{dt^k}\left(\frac{U^{l-k+1}}{l-k+1}\right).$$

Let us define $U_k(x)$ as the coefficient of t^k in the expansion

$$U(t,x) = \sum_n \sum_k u_{k,n} t^k x^n = \sum U_k(x) t^k.$$

Then

$$\frac{d^k}{dt^k} U\Big|_{t=0} = k!\, U_k(x) = \frac{d^k}{dx^k}\left(\frac{U_0^{k+1}(x)}{k+1}\right)$$

for $l = k$. Therefore,

$$U_k(x) = \frac{1}{(k+1)!}\, \frac{d^k}{dx^k}\, \varphi^{k+1}(x).$$

By using the binomial expansion

$$(1-x)^{-(k+1)} = 1 + (k+1)x + \dots + \frac{(k+l)\cdots(k+1)}{l!}\, x^l + \cdots ,$$

we obtain

$$u_{k,n} = \frac{1}{n}\binom{n-2}{k}\binom{n+k}{k+1} = f_{k-1}(K_n).$$

Thus we have computed the number

$$f_{k-1}(K_n),\ n \geqslant 3,\ 1 \leqslant k < n-2$$

with the help of conservation laws for the Hopf equation.

We can find the first computation of this number in the Cayley's paper (1891), where he also used the function $U_k(x)$. Note that Cayley obtained the above form of $U_k(x)$ by using the recursion formula

$$f(k,n) = \frac{n}{2k} \sum_{l+m=n+2} \sum_{i+j=k-1} f(i,l)f(j,m),$$

where $f(k,n) = u_{k,n-1} = f_{k-1}(K_{n-1})$ (see details in [25]).

Lecture III. Minkowski sum and simple polytopes

1. Minkowski sum

Let M_1 and M_2 be subsets in $\mathbb{R}^n$.

Definition 1.1. The *Minkowski*[1] *sum* of M_1 and M_2 is the set

$$M_1 + M_2 = \{x \in \mathbb{R}^n \,:\, x = x_1 + x_2,\ x_1 \in M_1,\ x_2 \in M_2\}.$$

Lemma 1.2. *If M_1 and M_2 are convex polytopes then $M_1 + M_2$ is again a convex polytope.*

The collection of all convex polytopes in $\mathbb{R}^n$ is denoted by $\mathscr{M}_n$. The Minkowski sum gives *an abelian monoid structure* on $\mathscr{M}_n$, where zero 0 is *the point* $0 = (0,\dots,0) \in \mathbb{R}^n$.

Proposition 1.3. *The following hold:*

(1) *There is the canonical homomorphism $\mathbb{R}^n \to \mathscr{M}_n$:*
the image of a vector $v \in \mathbb{R}^n$ is the one point polytope.

(2) *$\mathscr{M}_n$ has the structure of $\mathbb{R}$-module: for given $\lambda \in \mathbb{R}$ and $M \in \mathscr{M}_n$*

$$\lambda M = \{\lambda x \in \mathbb{R}^n,\, x \in M\}.$$

(3) *In $\mathscr{M}_n$ we have $M_1 + M = M_2 + M \ \Rightarrow\ M_1 = M_2$ for any $M \in \mathscr{M}_n$.*

(4) *Any linear map $L\colon \mathbb{R}^n \to \mathbb{R}^k$ induces the homomorphism*

$$L_*\colon \mathscr{M}_n \longrightarrow \mathscr{M}_k.$$

Denote by $\operatorname{conv}(v_1,\dots,v_N)$ the convex hull of the points $v_1,\dots,v_N$ in $\mathbb{R}^n$.

Lemma 1.4. *The Minkowski sum of convex hulls is the convex hull:*

$$\operatorname{conv}(v_1,\dots,v_k) + \operatorname{conv}(w_1,\dots,w_l) = \operatorname{conv}(v_1+w_1,\dots,v_i+w_j,\dots,v_k+w_l).$$

Proof. Set $M_1 = \operatorname{conv}(v_1,\dots,v_k)$ and $M_2 = \operatorname{conv}(w_1,\dots,w_l)$. Then $v_i + w_j \in M_1 + M_2$ for any $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$. Take

$$x_1 = \sum_{i=1}^{k} t_i v_i, \quad x_2 = \sum_{j=1}^{l} \tau_j w_j,$$

[1] Hermann Minkowski, 1864–1909

where $\sum\limits_{i=1}^{k} t_i = 1,\ t_i \geqslant 0$ and $\sum\limits_{j=1}^{l} \tau_j = 1,\ \tau_j \geqslant 0$. Then

$$\begin{aligned} x_1 + x_2 &= \sum_{i=1}^{k} t_i \Big(\sum_{j=1}^{l} \tau_j \Big) v_i + \sum_{j=1}^{l} \tau_j \Big(\sum_{i=1}^{k} t_i \Big) w_j \\ &= \sum_{i=1}^{k} \sum_{j=1}^{l} \xi_{ij} (v_i + w_j), \end{aligned}$$

where $\xi_{ij} = t_i \tau_j \geqslant 0$ and $\sum\limits_{i=1}^{k} \sum\limits_{j=1}^{l} \xi_{ij} = 1$. □

2. Minkowski sums in $\mathbb{R}^1$ and $\mathbb{R}^2$

For example, when $M_1 = [-1, 2] \subset \mathbb{R}^1, \quad M_2 = [1, 2] \subset \mathbb{R}^1$,

$$\begin{aligned} M_1 + M_1 &= [-2, 4] = 2M_1, \\ M_1 + (-M_1) &= [-1, 2] + [-2, 1] = [-3, 3], \\ M_1 + M_2 &= [0, 4], \text{ and} \\ M_1 + (-M_2) &= [-3, 1]. \end{aligned}$$

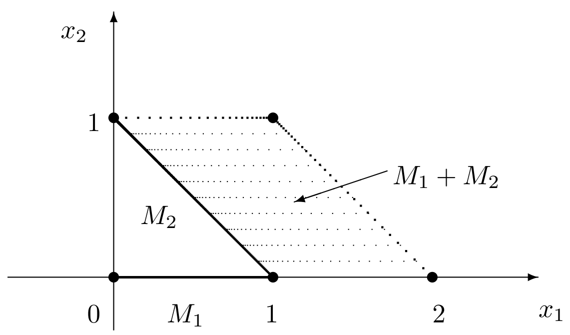
In general, if $M_i = [a_i, b_i],\ a_i \leqslant b_i,\ i = 1, 2$,

$$\begin{aligned} M_1 + M_2 &= [a, b], \text{where } a = a_1 + a_2,\ b = b_1 + b_2, \\ M_1 - M_2 &= [a', b'], \text{where } a' = a_1 - b_2,\ b' = b_1 - a_2. \end{aligned}$$

So,

$$M_1 - M_1 = (b_1 - a_1)[-1, 1].$$

Let $M_1 = \operatorname{conv}\Big((0,0), (1,0)\Big) \subset \mathbb{R}^2$ and $M_2 = \operatorname{conv}\Big((0,1), (1,0)\Big) \subset \mathbb{R}^2$.



$M_1 + M_2$ is the convex hull of the set $\{(1,0),\ (0,1),\ (2,0),\ (1,1)\}$.

Set $e_0 = (0,0),\ e_1 = (1,0),\ e_2 = (0,1)$. Let $M_1 = \operatorname{conv}(e_0, e_1, e_2)$ and $M_2 = \operatorname{conv}(e_0, x)$, where $x \in \mathbb{R}^2,\ x \neq 0$. Then

$$(M_1 + M_2)[x] = \operatorname{conv}(e_0, e_1, e_2, x, e_1 + x, e_2 + x).$$

For example, we obtain 5-gon if $x = e_1 + e_2$ and 4-gon if $x = e_1$. Moreover, if M_2 is parallel to one of the edges of M_1 then $M_1 + M_2$ is 4-gon, otherwise $M_1 + M_2$ is 5-gon.

3. Support functions

The support function of $M \in \mathscr{M}_n$ is the function

$$s_M\colon \mathbb{R}^n \longrightarrow \mathbb{R}\ :\ s_M(x) = \max_{y \in M} < x, y >,$$

where $< x, y >= \sum\limits_{k=1}^{n} x_k y_k$ is the scalar product.

It is easy to chek that:

- $s_M(\lambda x) = \lambda s_M(x)$ for any non negative $\lambda \in \mathbb{R}$. So, if $|x| \neq 0$, then $s_M(x) = |x| s_M\left(\frac{x}{|x|}\right)$.
- s_M is a piece linear function.
- s_M is a linear function iff M is a point in $\mathbb{R}^n$.
- s_M is a convex (concave up) function, that is

  $$s_M(tx_1 + (1-t)x_2) \leqslant t s_M(x_1) + (1-t) s_M(x_2)$$

  for any $x_1,\ x_2$ and $t \in [0,1]$.

Lemma 3.1. *For any $M_1,\ M_2 \in \mathscr{M}$ we have*

$$s_{M_1 + M_2} = s_{M_1} + s_{M_2}.$$

Proof. Let $M(x)$ be the image of M on mapping $\mathbb{R}^n \to \mathbb{R}\ :\ y \mapsto < x, y >$. We have $M(x) = [a, b]$, where $a = \max\limits_{y \in M} < x, y >= s_M(x)$ and $b = \min\limits_{y \in M} < x, y >$. It is clear that

$$M_1(x) + M_2(x) = (M_1 + M_2)(x).$$

Using that in $\mathbb{R}^1$

$$[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2],$$

we obtain the proof. □

4. Fan of a convex polytope

Consider the set $\{v_1, \dots, v_N\}$ of all vertices of convex polytope M. For any $x \in \mathbb{R}^n$ there exists such v_i that $s_M(x) =< x, v_i >$. Set

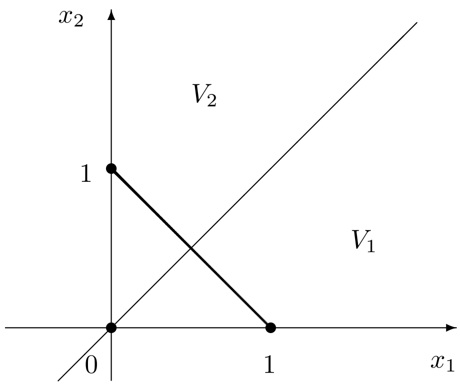
$$V_i = \{x \in \mathbb{R}^n \,:\, s_M(x) =< x, v_i >\}.$$

We have $\mathbb{R}^n = \bigcup\limits_{i=1}^{N} V_i$ and any V_i is the convex polyhedral cone with vertex $O \in \mathbb{R}^n$. The set $\{V_i,\ i = 1, \dots, N\}$ gives the fan of convex polytope M.

Example 4.1. $M = \operatorname{conv}\big((1,0),(0,1)\big) \subset \mathbb{R}^2$. Then

$$V_i = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, \max_{y \in M} < x, y >= x_i\},$$

$$\max_{y \in M} < x, y >= \max_{t \in [0,1]} \big(tx_1 + (1-t)x_2\big).$$



5. Minkowski sum of simple polytopes

A *zonohedron* is a convex polyhedron in $\mathbb{R}^3$ where every face is a polygon with point symmetry or, equivalently, symmetry under rotations through 180°. Any zonohedron may equivalently be described as the Minkowski sum of line segments in $\mathbb{R}^3$, or as the 3-dim projection of a hypercube. Zonohedra were originally defined and studied by E. S. Fedorov (1853–1919), a Russian crystallographer.

More generally, in $\mathbb{R}^n$ the Minkowski sum of line segments forms a polytope known as a *zonotope*. Set $v_0 = (0,0,0),\ v_1 = (\sqrt{3},0,0),\ v_2 = (0,\sqrt{3},0),\ v_3 = (1,1,1),\ v_4 = (1,1,-1)$. Let $M_i = \operatorname{conv}(v_0, v_i)$. Then $M_1 + M_2 + M_3 + M_4$ is the well known *rhombic dodecahedron*. It is a convex polyhedron with 12 rhombic faces, 24 edges and 14 vertices. Some minerals such as garnet form a rhombic dodecahedral crystal habit. Honeybees use the geometry of rhombic dodecahedra to form honeycomb. It gives an example when a Minkowski sum of the simple polytopes forms a nonsimple polytope.

Problem 5.1. When does Minkowski sum of simple polytopes give a simple polytope?

Let $\Delta^1 = I^1 = \{(t_1, t_2),\ t_i \geqslant 0,\ t_1+t_2 = 1\}$. Consider a collection $\{(v_{1,1}, v_{1,2}), \dots, (v_{N,1}, v_{N,2})\}$ of pairs of vectors in $\mathbb{R}^n$ and the mapping

$$\varphi\colon I^N \longrightarrow \mathbb{R}^n\ :\ \varphi\Big((t_{1,1}, t_{1,2}), \dots, (t_{N,1}, t_{N,2})\Big) = \sum_{i=1}^{N} (t_{i,1}v_{i,1} + t_{i,2}v_{i,2}).$$

The image of φ is a *zonotope* in $\mathbb{R}^n$.

Problem 5.2. When does φ give a simple polytope?

For more general problem, consider a collection $\{(v_{1,1}, \dots, v_{1,i_1+1}), \dots, (v_{N,1}, \dots, v_{N,i_N+1})\}$ of the finite sets of vectors in $\mathbb{R}^n$ and the mapping

$$\Phi\colon \Delta^{i_1} \times \dots \times \Delta^{i_N} \longrightarrow \mathbb{R}^n\ :\ \Phi(t_1, \dots, t_N) = \sum_{k=1}^{N} \sum_{j=1}^{i_k+1} t_{k,j} v_{k,j},$$

where $\Delta^{i_k} = \{t_k = (t_{k,1}, \dots, t_{k,i_k+1}),\ t_{k,j} \geqslant 0,\ \sum_{j=1}^{i_k+1} t_{k,j} = 1\}$.

Problem 5.3. When does Φ give a simple polytope?

6. Building sets

Consider a collection B of non-empty subsets of the set $[n] = \{1, \dots, n\}$. Let $e_i,\ i = 1, \dots, n$, be the endpoints of the standard basis vectors in $\mathbb{R}^n$. For any $I \in B$ set $\Delta_I = \operatorname{conv}(e_i | i \in I)$ and $P_B = \sum\limits_{I \in B} \Delta_I$. Convex polytope P_B is the image of the map $\varphi_B\colon \prod\limits_{I \in B} \Delta_I \longrightarrow \mathbb{R}^n$.

Problem 6.1. When does φ_B give a simple polytope?

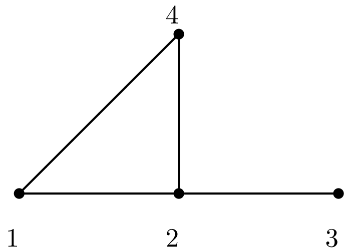
Definition 6.2. A collection B of non-empty subsets of the set $[n] = \{1, \dots, n\}$ is called a *building set* if:

- $I, J \in B$ and $I \bigcap J \neq \emptyset \Rightarrow I \cup J \in B$
- $\{i\} \in B$ for all $i \in [n]$.

Theorem 6.3 ([33])**.** *The convex polytope $P_B \subset \mathbb{R}^n$ is a simple polytope.*

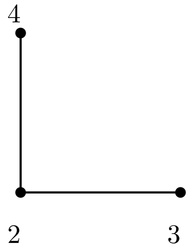
Definition 6.4. Let Γ be a graph with the vertex set $[n] = \{1, \dots, n\}$ and no loops or multiple edges. The *graphical building set* $B(\Gamma)$ is the set of all non-empty subsets $I \subset [n]$ such that the graph $\Gamma|_I$ is connected.

For example, for the graph Γ

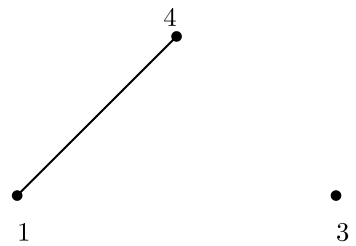


we have

$\Gamma|_{\{2,3,4\}}$



$\Gamma|_{\{1,3,4\}}$



$$\text{that is} \quad \{2,3,4\} \in B(\Gamma), \qquad \text{and} \qquad \{1,3,4\} \notin B(\Gamma).$$

It is easy to obtain the following result.

Lemma 6.5. *The graphical building set $B(\Gamma)$ is a building set.*

You can find a lot of results concerning the building sets and graphical building set in [33].

Given a finite graph Γ. The graph-associahedron $P(\Gamma)$ is a simple polytope $P_{B(\Gamma)}$. We have obtained (see Lecture I):

- associahedron (Stasheff polytope) As^n is the graph-associahedron $P(\Gamma)$ where Γ is an n-path;
- cyclohedron (Bott–Taubes polytope) Cy^n is $P(\Gamma)$ where Γ is an n-cycle;
- permutohedron Pe^n is $P(\Gamma)$ where Γ is an n-complete graph;
- stellohedron St^n is $P(\Gamma)$ where Γ is an n-star graph.

Lemma 6.6. *Let $[n] \in B$. Then for any $I_k \in B$ such that $I_k \neq [n]$, the equation*

$$\sum_{i \in I_k} x_i = \mu(I_k) \tag{$*$}$$

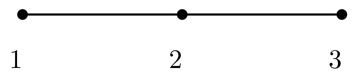
gives a facet of P_B in the hyperplane $\sum_{i=1}^{n} x_i = \mu([n])$, where $\mu(I_k)$ is the number of $I_l \in B$ such that $I_l \subseteq I_k$. Any facet of P_B can be described by equation $(*)$ for some $I_k \in B$.

Corollary 6.7. *The polytope P_B is result of successive truncations of the simplex $\{x \in \mathbb{R}^n \,:\, \sum_{i=1}^{n} x_i = \mu([n])\}$.*

Example 6.8.

(1)

Γ :



$P(\Gamma) = As^2 = St^2$ is 5-gon

The graphical building set $B(\Gamma)$ is $B = \{\, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,2,3\} \,\}$.

We obtain that the equations

$$x_1 = 1;\; x_2 = 1;\; x_3 = 1;\; x_1 + x_2 = 3;\; x_2 + x_3 = 3$$

give the facets of $As^2 \subset L$, where L is the hyperplane $\{x \in \mathbb{R}^3 \,:\, x_1 + x_2 + x_3 = 6\}$.

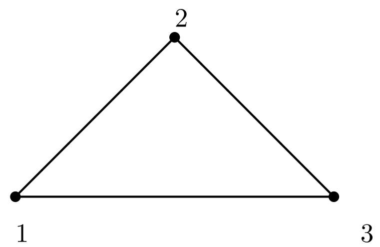
Thus, the equations

$$x_1 = 1;\; x_2 = 1;\; x_1 + x_2 = 5;\; x_1 + x_2 = 3;\; x_1 = 3$$

give the facets of $As^2 \subset \mathbb{R}^2$.

(2)

Γ :



$P(\Gamma) = Cy^2 = Pe^2$ is 6-gon

The graphical building set $B(\Gamma)$ is $B = \{\, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\} \,\}$.

We obtain that the equations

$$x_1 = 1;\; x_2 = 1;\; x_3 = 1;\; x_1 + x_2 = 3;\; x_2 + x_3 = 3;\; x_1 + x_3 = 3$$

give the facets of $Cy^2 \subset L$, where L is the hyperplane $\{x \in \mathbb{R}^3 \,:\, x_1 + x_2 + x_3 = 7\}$.

Thus, the equations

$$x_1 = 1;\; x_2 = 1;\; x_1 + x_2 = 6;\; x_1 + x_2 = 3;\; x_1 = 4;\; x_2 = 4$$

give the facets of $Cy^2 \subset \mathbb{R}^2$.

Corollary 6.9. *The equations*

$$x_i + \ldots + x_{i+k} = \binom{k+2}{2}, \quad i = 1, \ldots, n+1,\ k = 0, \ldots, n+1-i,$$

give the facets of $As^n \subset L$, where L is the hyperplane

$$\left\{ x \in \mathbb{R}^{n+1} : x_1 + \ldots + x_{n+1} = \binom{n+2}{2} \right\}.$$

Lecture IV. Moment-angle complexes and applications

1. Simplicial complexes and maps.

An abstract simplicial *complex* on a set $V = \{v_1, \ldots, v_m\}$ is a collection

$$\mathscr{K} = \{\sigma\}$$

of subsets in V such that for every $\sigma \in \mathscr{K}$ all subsets in σ (including $\emptyset$) also belong to $\mathscr{K}$. By definition:

- $\sigma \in \mathscr{K}$ — (abstract) simplex.
- One-element subsets — vertices.
- $\dim \sigma = |\sigma| - 1$, where $|\sigma|$ — the number of elements.
- $\dim \emptyset = -1$.
- $\dim \mathscr{K} = \max\limits_{\sigma \in K} \dim \sigma$.

A simplicial map $\varphi : \mathscr{K}_1 \to \mathscr{K}_2$ is a mapping $V_1 \to V_2$ such that the image of every simplex from $\mathscr{K}_1$ is a simplex in $\mathscr{K}_2$.

A simplicial map φ is said to be *non-degenerate* if $|\varphi(\sigma)| = |\sigma|$ for any $\sigma \in \mathscr{K}_1$.

Geometric *simplices* Δ^k in $\mathbb{R}^n$ for $n > k$ are *convex* hulls of sets of *affinely* independent points $\alpha_1, \ldots, \alpha_{k+1} \in \mathbb{R}^n$

$$x = \sum_{j=1}^{k+1} x^j \alpha_j, \qquad \sum_{j=1}^{k+1} x^j = 1, \quad x^j \geqslant 0,$$

where $(x^1, \ldots, x^{k+1})$ are called *barycentric* coordinates of a point $x \in \Delta^k$.

A *face* of Δ^k is the simplex determined by a subset of vertices $\alpha_1, \ldots, \alpha_{k+1}$. The empty subset of vertices determine the empty face. A face of $\dim(k-1)$ (a facet) is given by $\Delta_j^{k-1} = (\alpha_1, \ldots, \widehat{\alpha_j}, \ldots, \alpha_{k+1})$.

A geometrical simplicial complex is a set K of geometric simplices of arbitrary dimensions lying in $\mathbb{R}^n$ such that every face of a simplex from K lies in K and intersection of any two simplices from K is a face of each of them.

Example 1.1. The boundary of an n-dim simplex Δ^n is the union $\cup_j \Delta_j^{n-1}$ of its $(n-1)$-dim facets, together with all their faces. This is an $(n-1)$-dim simplicial complex, the *standard* simplicial subdivision of the sphere S^{n-1}.

A *map of simplices*

$$\Delta_1^n \to \Delta_2^m$$

is a map from the vertices of Δ_1^n to the vertices of Δ_2^m extended linearly to the whole of Δ_1^n. A *simplicial map*

$$f\colon K_1 \to K_2$$

of simplicial complexes is a map whose restriction to each simplex is a map of simplices.

Example 1.2. Let K be any simplicial complex on the vertex set $\{v_1, \ldots, v_m\}$ and Δ^{m-1} the standard simplex on the vertices $\{\alpha_1, \ldots, \alpha_m\}$. Then there exists a canonical simplicial map (inclusion)

$$f : K \hookrightarrow \Delta^{m-1}, \quad f(v_j) = \alpha_j.$$

Let S be an arbitrary partially ordered set. Order complex $\operatorname{ord}(S)$ is the set of all chains $s_1 < s_2 < \cdots < s_k,\ s_i \in S$. Complex $\operatorname{ord}(S)$ is a abstract simplicial complex.

The *barycentric subdivision* K' of a simplicial complex K is defined as order complex $\operatorname{ord}(K \backslash \emptyset)$ of the partially ordered (with respect to inclusion) set of nonempty simplices of the complex K.

The barycentric subdivision of an abstract simplicial complex K is the simplicial complex K' on the set $\{\sigma \in K\}$ of simplices of K whose simplices are chains of embedded simplices of K. That is $\{\sigma_1, \ldots, \sigma_r\} \in K'$ if and only if $\sigma_1 \subset \sigma_2 \subset \ldots \subset \sigma_r$ (after possible re-ordering).

Example 1.3. For any $(n-1)$-dimensional simplicial complex K^{n-1} on $\{v_1, \ldots, v_m\}$ there is a non-degenerate simplicial map $K' \to \Delta^{n-1}$ defined on the vertices by $\sigma \to |\sigma| + 1,\ \sigma \in K^{n-1}$. Here $\sigma \in K$ is regarded as a vertex of K' and Δ^{n-1} as the standard simplex on the set $\{1, \ldots, n\}$.

2. Main construction.

Let K be a simplicial complex with the vertex set $V = \{v_1, \ldots, v_m\}$. For a pair (X, W) of topological spaces $W \subset X$ and a subset $\sigma \subset V$ set

$$\mathscr{Z}_\sigma = \{\xi \in X^V \mid \xi(v_k) = x_k \in W, \text{ if } v_k \in V \setminus \sigma\}.$$

Here, $X^V = X^m$ is the space of maps from V into X.

Introduce

$$\mathscr{Z}_K(X, W) = \bigcup_\sigma \mathscr{Z}_\sigma \subset X^m,$$

where σ ranges over all simplices in K.

Example 2.1. Let $D^n = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum x_i^2 \leqslant 1\right\}$ and $S^{n-1} = \partial D^n$. Then

$$\begin{aligned}
&\mathscr{Z}_{\Delta^1}(D^n, S^{n-1}) = D^n \times D^n, \\
&\mathscr{Z}_{\partial\Delta^1}(D^n, S^{n-1}) = D^n \times S^{n-1} \cup S^{n-1} \times D^n = \partial(D^n \times D^n) = S^{2n-1}.
\end{aligned}$$

In general case

$$\mathscr{Z}_{\Delta^{m-1}}(X,W) = X^m,$$
$$\mathscr{Z}_{\partial\Delta^1}(X,W) = (X\times W)\cup(W\times X).$$

Example 2.2. We have

$$\mathscr{Z}_K(W,\emptyset) = \begin{cases} \emptyset\,, & \text{if } k\neq\Delta^{m-1},\\ W^m, & \text{if } k=\Delta^{m-1}. \end{cases}$$

and

$$\mathscr{Z}_K(W,W) = W^m.$$

The canonical inclusion $(W,W)\hookrightarrow(X,W)$ induced the inclusion $W^m\hookrightarrow\mathscr{Z}_K(X,W)$. Thus the construction of $\mathscr{Z}_K$ and map G_K for a fixed simplicial complex K with m vertexes gives rise to a covariant functor $(X,W)\to\big(\mathscr{Z}_K(X,W),W^m\big)$ on the category of pairs of spaces.

Let $g\colon(X_1,W_1)\to(X_2,W_2)$ be a map of pairs of spaces. Then we have the induced map

$$G_K\colon\mathscr{Z}_K(X_1,W_1)\longrightarrow\mathscr{Z}_K(X_2,W_2),$$

where $G(\xi)\colon V\xrightarrow{\xi}X_1\xrightarrow{g}X_2$.

Let K_1 and K_2 be simplicial complexes with the vertex set $V_1=\{v_1,\dots,v_m\}$ and $V_2=\{v_1',\dots,v_{m'}'\}$, respectively. Let us fix a point $w_*\in W$. A simplicial inclusion $\varphi:K_1\to K_2$ induces the inclusion

$$\Phi\colon(X)^m\longrightarrow(X)^{m'},\quad\Phi(x_1,\dots,x_m)=(y_1,\dots,y_{m'}),$$

where

$$y_k=\begin{cases} w_*, & \varphi^{-1}(v_k')=\emptyset,\\ x_j, & \varphi^{-1}(v_k')=v_j. \end{cases}$$

Thus we obtain the inclusion $\varphi_{\mathscr{Z}}:\mathscr{Z}_{K_1}\to\mathscr{Z}_{K_2}$ such that

$$\begin{tikzcd} \mathscr{Z}_{K_1} \arrow[r, "\subset"] \arrow[d, "\varphi_{\mathscr{Z}}"'] & X^m \arrow[d, "\Phi"] \\ \mathscr{Z}_{K_2} \arrow[r, "\subset"] & X^{m'} \end{tikzcd}$$

The inclusion $\mathscr{Z}_K\subset X^m=\mathscr{Z}_{\Delta^{m-1}}$ is the map of moment-angle complexes induced by the canonical inclusion $f\colon K\hookrightarrow\Delta^{m-1}$.

3. Moment-angle complexes

Let $D^2 = \{z \in \mathbb{C} : |z| \leqslant 1\}$ be a unit disk in $\mathbb{C}$. The space

$$\mathscr{Z}_K = \mathscr{Z}_K(D^2, \partial D^2) \subset (D^2)^m$$

is called a *moment-angle complex.* The action of $S^1 = \partial D^2 = \{z \in \mathbb{C} : |z| = 1\}$ on D^2 (multiplication of complex numbers) defines (coordinate-wise) the action of torus T^m on $(D^2)^m$ and induces the *canonical action* on $\mathscr{Z}_K$. Many important examples of manifolds in topology and geometry are factor spaces of $\mathscr{Z}_K$ over a free action of $T^k \subset T^m$, for some k.

Example 3.1. Sphere

$$S^{2n+1} = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1}, \sum |z_k|^2 = 1\}$$

is $\mathscr{Z}_{\partial\Delta^n}$, where Δ^n is the n-simplex.

<u>For $n = 1$</u>, Δ^1 is an interval I^1 with boundary $\partial I^1 = S^0 = \alpha_0 \cup \alpha_1$. We have

$$S^3 = (D^2 \times S^1) \cup (S^1 \times D^2) \subset D^2 \times D^2$$

and

$$\mathbb{CP}(1) = S^3/T^1, \qquad \mathbb{CP}(1) \xrightarrow{T^1} \Delta^1.$$

<u>For $n = 2$</u>, we have

$$S^5 = (D^2 \times D^2 \times S^1) \cup (D^2 \times S^1 \times D^2) \cup (S^1 \times D^2 \times D^2) \subset (D^2)^3$$

and

$$\mathbb{CP}(2) = S^5/T^1, \qquad \mathbb{CP}(2) \xrightarrow{T^2} \Delta^2.$$

Let K_1 and K_2 be simplicial complexes with the vertex set $V_1 = \{v_1, \dots, v_m\}$ and $V_2 = \{v'_1, \dots, v'_{m'}\}$, respectively. A simplicial map $\varphi : K_1 \to K_2$ induces the map

$$\Phi\colon (D^2)^m \longrightarrow (D^2)^{m'}, \quad \Phi(x_1, \dots, x_m) = (y_1, \dots, y_{m'}),$$

where

$$y_k = \begin{cases} 1, & \varphi^{-1}(v'_k) = \emptyset \\ \prod\limits_{j \in J} x_j, & \varphi^{-1}(v'_k) = \{v_j,\ j \in J\} \end{cases}$$

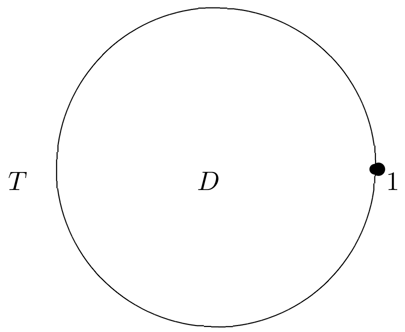
Thus we obtain the map $\varphi_{\mathscr{Z}} : \mathscr{Z}_{K_1} \to \mathscr{Z}_{K_2}$ such that

$$\begin{tikzcd} \mathscr{Z}_{K_1} \arrow[r, "\subset"] \arrow[d, "\varphi_{\mathscr{Z}}"'] & (D^2)^m \arrow[d, "\Phi"] \\ \mathscr{Z}_{K_2} \arrow[r, "\subset"] & (D^2)^{m'} \end{tikzcd}$$

The construction of $\mathscr{Z}_K$ and map $\varphi_{\mathscr{Z}}\colon \mathscr{Z}_{K_1} \to \mathscr{Z}_{K_2}$ gives rise to a *covariant* functor from the category of simplicial complexes to the category of T^m-spaces and equivariant maps.

4. Bigraded cellular complexes

The T^1-space $\mathscr{Z}_{\Delta^0} = D^2$ has the following bigraded cell decomposition:



Cells: [1], $[T]$, $[D]$

Set the *bidegree* (bdg) of the generators by

$$\operatorname{bdg}[1] = (0,0), \quad \operatorname{bdg}[T] = (-1,2), \quad \operatorname{bdg}[D] = (0,2);$$

$$\partial[1] = 0, \qquad \partial[T] = 0, \qquad \partial[D] = [T].$$

Then

$$C_*((D^2)^m;\partial) = \bigotimes_{i=1}^{m} C_*(D^2;\partial),$$

and $\mathscr{Z}_K \subset (D^2)^m$ is a *bigraded cellular subcomplex*!

Thus the cellular chains $C_*(\mathscr{Z}_K)$ are defined.

The functor $K \mapsto \mathscr{Z}_K$ induces a homomorphism between the standard simplicial chain complex of a simplicial pair (K, K') and the bigraded cellular chain complex of $(\mathscr{Z}_K, \mathscr{Z}_{K'})$.

5. Proporties of the functor $K \mapsto \mathscr{Z}_K$

The functor takes a simplicial Lefschetz pair (K, K') (i.e. a pair such that $K \setminus K'$ is an open manifold) to another Lefschetz pair (of moment-angle complexes) in such a way that the fundamental cycle is mapped to the fundamental cycle. For instance, if K is a triangulated manifold, then the simplicial pair $(K, \varnothing)$ is mapped to the pair $(\mathscr{Z}_K, \mathscr{Z}_\varnothing)$, where $\mathscr{Z}_\varnothing \cong T^m$ and $\mathscr{Z}_K \setminus \mathscr{Z}_\varnothing$ is an open manifold. Studying the functor $K \mapsto \mathscr{Z}_K$, one interprets the combinatorics of simplicial complexes in terms of the bigraded cohomology rings of moment-angle complexes. In the case when K

is a triangulated manifold, the important additional information is provided by the bigraded Poincaré duality for the Lefschetz pair $(\mathscr{Z}_K, \mathscr{Z}_\varnothing)$.

Let us consider a map

$$g\colon (D^2, S^1) \longrightarrow (X, W)$$

of pairs and a simplicial map

$$\varphi\colon K_1 \longrightarrow K_2$$

of simplicial complexes. Then we obtain the commutative diagram of the induced maps:

$$\begin{array}{ccc} \mathscr{Z}_{K_1} & \longrightarrow & \mathscr{Z}_{K_2} \\ \big\downarrow & & \big\downarrow \\ \mathscr{Z}_{K_1}(X,W) & \longrightarrow & \mathscr{Z}_{K_2}(X,W) \end{array}$$

<u>The important case:</u> Consider the map

$$\rho\colon (D^2, S^1) \longrightarrow (I, 1) \ : \ \rho(z) = |z|^2,$$

where $I = \{t \in \mathbb{R},\ 0 \leqslant t \leqslant 1\}$. Thus the canonical inclusion $f\colon K \to \Delta^{m-1}$ gives the commutative diagram:

$$\begin{array}{ccc} \mathscr{Z}_{K} & \longrightarrow & (D^2)^m \\ \big\downarrow & & \big\downarrow \\ \mathscr{Z}_{K}(I,1) & \longrightarrow & I^m \end{array}$$

Lemma 5.1. (see ch. 4 in [3]) *The space $\mathscr{Z}_K(I,1)$ is*

$$cc(K') = cone(K'),$$

i.e. the cubical subcomplex of I^m, where K' is the barycentric subdivision of K, and $(1,1,\dots,1)$ is the vertex of the cone K'.

Example 5.2.

$$\mathscr{Z}_{\partial\Delta^1}(I,1) = (I \times 1) \cup (1 \times I) = cc(\partial\Delta^1) = cone(\partial\Delta^1).$$

Let P be a *simple convex polytope* of dimension n with m facets and $K = (\partial P)^*$ is the $(n-1)$-dimensional simplicial m-vertex complex dual to the boundary ∂P of P (see Lecture II, Definition 5.7). Set $\mathscr{Z}_P(X,W) = \mathscr{Z}_{(\partial P)^*}(X,W)$.

Lemma 5.3. (see ch. 4 in [3]) *The space $\mathscr{Z}_P(I,1)$ is the canonical cubical subdivision of the simple polytope P into cubes, one for each vertex $v \in P$. The resulting cubical complex embeds canonically into canonical cubical subdivision of the boundary of I^m.*

Corollary 5.4. (see [3]) *There is the commutative diagram*

$$\begin{array}{ccc} \mathscr{Z}_P & \longrightarrow & (D^2)^m \\ {\scriptstyle\pi}\Big\downarrow & & \Big\downarrow \\ P & \longrightarrow & I^m \end{array}$$

where π is the projection map on the orbit space $\mathscr{Z}_P/T^m = P$ of the canonical action of T^m on $\mathscr{Z}_P$.

Theorem 5.5. (see [7]) *The following hold:*

(1) *$\mathscr{Z}_P$ is an $(n+m)$-dimensional smooth manifold with the action of T^m.*
(2) *The quotient $\mathscr{Z}_P/T^m$ is P.*
(3) *There is a realization of $\mathscr{Z}_K$ as a complete intersection of **real quadratic hypersurfaces**.*

Theorem 5.6. (see [7]) *Manifold $\mathscr{Z}_P$ is the following framed $(m+n)$-dimensional submanifold of $\mathbb{R}^{2m}$:*

$$\{\big((x_1,y_1),\dots,(x_m,y_m)\big)\ :\ \sum_{k=1}^{m} c_{j,k}(x_k^2+y_k^2-b_k)=0,\ 1\leqslant j\leqslant m-n\}.$$

The importance of the real quadratic viewpoint has been emphasized in the work of Bosio and Meersseman [1] who considered a specific class of complete intersections of real quadrics in $\mathbb{C}^m$, called *link*. They shown that all links (taking products with a circle in odd-dimensional cases) can be endowed with the structure of a non-Kähler complex manifold, generalizing the wellknown class of Hopf and Calabi–Ekmann manifolds (see [13]).

Theorem 5.7. (see [1]) *The class of links coincides with the class of moment-angle complex $\mathscr{Z}_P$ arising from simple polytopes.*

6. Buchstaber's invariant of simplicial complexes

Let K be a simplicial complex with the vertex set $\{v_1,\dots,v_m\}$. Define $s=s(K)$ to be the *maximal dimension* for which there exists a subgroup $H\cong T^s$ in T^m acting freely on the moment-angle complex $\mathscr{Z}_K$.

The number $s(K)$ is a *combinatorial invariant* of K. This invariant was introduced by the author in 2002 (see [3]).

Problem 6.1. Find a combinatorial description of $s(K)$.

Lemma 6.2. *Let K be an $(n-1)$-dim simplicial complex with m vertices. Then $1\leqslant s(K)\leqslant m-n$.*

Proof. Every subtorus $H = T^s$ of T^m for $s > m - n$ intersects non-trivially with any n-dimensional isotropy subgroup, and therefore cannot act freely on $\mathscr{Z}_K$. Thus $s(K) \leqslant m - n$. $\square$

Let $S_d := \{(e^{2\pi i\varphi}, \dots, e^{2\pi i\varphi}) \in T^m\}$, $\varphi \in \mathbb{R}$, be the diagonal circle subgroup. Every isotropy subgroup for $\mathscr{Z}_K$ is a coordinate subgroup, and therefore intersects S_d only at the unit. Thus S_d acts freely on any $\mathscr{Z}_K$. Thus $s(K) \geqslant 1$.

Let P^n be a simple convex polytope with m facets. Set $s(P) = s(K)$, where $K = (\partial P)^*$ is the $(n-1)$-dim simplicial complex with m vertices dual to the boundary ∂P of P.

Let $\mathscr{F} = \{F_1, \dots, F_m\}$ be the set of facets of P^n. A surjective map $\varrho : \mathscr{F} \to [k]$ (where $[k] = \{1, \dots, k\}$) is called a *regular k-paint coloring* of P^n if $\varrho(F_i) \neq \varrho(F_j)$ whenever $F_i \cap F_j \neq \emptyset$.

The *chromatic number* $\gamma(P^n)$ is the minimal k for which there exists a regular k-paint coloring of P^n. Thus $\gamma(P^n) \geqslant n$ and the equality is achieved if and only if every 2-face of P^n is an evengon. Note also that $\gamma(P^3) \leqslant 4$ by the Four Color Theorem.

Definition 6.3. A simple polytope P is called k-neighborly if any k facets of P have non-empty intersection.

Example 6.4. Suppose P^n is a 2-neighborly simple polytope with m facets. Then $\gamma(P^n) = m$.

Proposition 6.5 (I.Izmestev)**.** *The following inequality holds:*

$$s(P^n) \geqslant m - \gamma(P^n).$$

Proof. The map $\varrho : \mathscr{F} \to [k]$ defines an epimorphism of tori $\widetilde{\varrho} : T^m \to T^k$. It is easy to see that if ϱ is a regular coloring, then $\operatorname{Ker} \widetilde{\varrho} \cong T^{m-k}$ acts freely on $\mathscr{Z}_P$. $\square$

Let P^n be a simple convex polytope with m facets and $\nu(P) = m - n$.

Theorem 6.6 (see [22, 23])**.** *The invariant $s(P)$ satisfies the following:*

(1) $s(P) + s(Q) \leqslant s(P \times Q) \leqslant s(P) + s(Q) + \min(\nu(P) - s(P), \nu(Q) - s(Q))$.
(2) $s(P \sharp Q) \geqslant s(P) + s(Q)$, *where $\sharp$ is the connected sum of polytopes along the vertices.*
(3) $s(P) = 1 \Leftrightarrow P = \Delta^n$.
(4) $s(P) \geqslant m - \gamma + s(\Delta^{\gamma-1}_{n-1})$, *where $\gamma = \gamma(P)$ is the chromatic number and Δ^l_k is k-dim skeleton of l-dim simplex Δ^l.*

(5) $s(P) - \nu(P) \leqslant s(F) - \nu(F) + 1$, *where F is a facet of P.*

(6) *One can not calculate $s(P)$ if it is known only f-vector $f(P)$ and $\gamma(P)$.*

7. Bigraded cohomology ring of moment-angle complexes

Let $\mathbf{k}[v_1, \ldots, v_m]$ denote the graded polynomial algebra over a commutative ring $\mathbf{k}$ with unit, $\deg v_i = 2$.

Definition 7.1. The *face ring* (or the *Stanley–Reisner ring*) of a simplicial complex K on the vertex set $V = \{v_1, \ldots, v_m\}$ is the quotient ring

$$\mathbf{k}(K) = \mathbf{k}[v_1, \ldots, v_m]/\mathscr{I}_K,$$

where $\mathscr{I}_K$ is the homogeneous ideal generated by all square-free monomials $v_\sigma = v_{i_1} v_{i_2} \cdots v_{i_s}$ $(i_1 < \cdots < i_s)$ such that $\sigma = \{i_1, \ldots, i_s\}$ *is not* a simplex of K.

Theorem 7.2 (see [2])**.** *Let $\mathbf{k}$ be a field and K, K' be two simplicial complexes with vertices $V = \{v_1, \ldots, v_m\}, V' = \{v'_1, \ldots, v'_{m'}\}$. Suppose $\mathbf{k}(K)$ and $\mathbf{k}(K')$ are isomorphic as $\mathbf{k}$-algebras. Then there exists a bijective mapping $V \to V'$ which induces an isomorphism $\varphi\colon K \to K'$.*

Problem 7.3. Find description of $s(K)$ in the term of $\mathbf{k}(K)$.

Let K be a simplicial complex on m vertices.

Theorem 7.4 (Hochster, see [3])**.** *The following additive isomorphism holds:*

$$\mathrm{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}\big(\mathbf{k}(K), \mathbf{k}\big) \cong \oplus \widetilde{H}^*(K_\omega; \mathbf{k}),$$

where ω ranges over the subsets of $[m]$ and the K_ω are induced subcomplexes.

Theorem 7.5 (Buchstaber-Panov, see [3])**.** *The following isomorphism of algebras holds:*

$$H^{*,*}(\mathscr{Z}_K; \mathbf{k}) \cong \mathrm{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}\big(\mathbf{k}(K), \mathbf{k}\big).$$

In particular,

$$H^p(\mathscr{Z}_K; \mathbf{k}) \cong \sum_{-i+2j=p} \mathrm{Tor}^{-i,2j}_{\mathbf{k}[v_1,\ldots,v_m]}\big(\mathbf{k}(K), \mathbf{k}\big).$$

The following additive isomorphism holds:

$$H^*\big(\Omega\mathscr{Z}_K; \mathbf{k}\big) \otimes \Lambda[u_1, \ldots, u_m] \cong \mathrm{Tor}_{\mathbf{k}(K)}(\mathbf{k}, \mathbf{k}).$$

Rely on theorems 7.2 and 7.5 we put

Problem 7.6. Suppose $H^{*,*}(\mathscr{Z}_K; \mathbf{k})$ and $H^{*,*}(\mathscr{Z}_{K'}; \mathbf{k})$ are isomorphic as bigraded $\mathbf{k}$-algebras. When there exists a bijective mapping $V \to V'$ which induces an isomorphism $\varphi\colon K \to K'$?

The above question arose during our discussion with Dong Youp Suh of the recent work [16], where the bigraded cohomology of moment-angle complexes plays an important role in the study of cohomological rigidity of simple polytopes. In the preparation of these lectures for publication I asked Taras Panov to write a survey of different rigidity properties of polytopes and quasitoric manifolds. It is now included to the text (see Appendix A).

Theorem 7.5 was used also in [20] for computation of the ranks of the homotopy groups of $\mathscr{Z}_K$ in terms of the homological algebra of $\mathbf{k}(K)$.

8. Toral rank conjecture

A T^k-action on a topological space X is *almost free* if all isotropy subgroups are finite. The *toral rank* of X is the largest k for which there exists an almost free T^k-action on X. (Denote it by $\operatorname{trk}(X)$.)

Proposition 8.1. *If K is an $(n-1)$-dimensional simplicial complex on m vertices, then $\operatorname{trk}\mathscr{Z}_K \geqslant m-n$.*

S. Halperin in 1985 *conjectured* that

$$\dim H^*(X_+;\mathbb{Q}) \geqslant 2^{\operatorname{trk}(X)}$$

for any finite dimensional space X.

Corollary 8.2 (see [4])**.** *Assuming that the toral rank conjecture is true, we come to the following inequality:*

$$\dim \oplus \widetilde{H}^*(K_\omega;\mathbb{Q}) \geqslant 2^{m-n}-1,$$

where ω ranges over the subsets of $[m]$ and the K_ω are induced subcomplexes.

When K is the boundary of Δ^n, the right hand side is 1 and equality holds. When K is the boundary of an m-gon, then

$$\dim H^*(\mathscr{Z}_K;\mathbb{Q}) = (m-4)2^{m-2}+4 \geqslant 2^{m-2}.$$

Define the *moment curve* in $\mathbb{R}^n$ by

$$\boldsymbol{x}\colon \mathbb{R} \longrightarrow \mathbb{R}^n, \qquad t \mapsto \boldsymbol{x}(t) = (t, t^2, \dots, t^n) \in \mathbb{R}^n.$$

For any $m > n$ define the *cyclic polytope* $C^n(t_1,\dots,t_m)$ as the convex hull of m distinct points $\boldsymbol{x}(t_i)$, $t_1 < t_2 < \ldots < t_m$, on the moment curve. $C^n(m) = C^n(t_1,\dots,t_m)$ is a simplicial n-polytope. From all simplicial n-polytopes S with m vertices the cyclic polytope $C^n(m)$ has the maximal number of j-faces, $0 \leqslant j \leqslant n-1$. Computer calculations confirmed the exponential growth required by the corollary for cyclic polytopes (Gadjikurbanov).

Lecture V. **Quasitoric manifolds**

[**Abstract**] Our aim is to bring geometric and combinatorial methods to bear on the study omnioriented toric manifolds M, in the context of stably complex manifolds with compatible torus action. We interpret M in terms of combinatorial data (P, Λ), where P is the combinatorial type of an oriented simple polytope, and Λ is an integral matrix whose properties are controlled by P (see [7]).

1. Notation and agreements

We denote by $\mathbb{R}^n$ the standard real n-dimensional Euclidean space with the *standard basis* consisting of vectors $e_j = (0, \dots, 1, \dots, 0)$ with 1 on the j-th place, for $1 \leqslant j \leqslant n$; and similarly for $\mathbb{Z}^n$ and $\mathbb{C}^n$. The standard basis gives rise to the *canonical orientation* of $\mathbb{R}^n$.

We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by means of the real vector space isomorphism $\mathbb{C}^n \to \mathbb{R}^{2n}$ sending e_j to e_{2j-1} and $(\sqrt{-1})e_j$ to e_{2j} for $1 \leqslant j \leqslant n$. This provides the *canonical orientation* for $\mathbb{C}^n$.

Since $\mathbb{C}$-linear maps from $\mathbb{C}^n$ to $\mathbb{C}^n$ preserve the canonical orientation, we may also regard an arbitrary complex vector space as canonically oriented.

We denote by $\mathbb{T}^n$ the standard n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$ which we identify with the product of n unit circles in $\mathbb{C}^n$:

$$\mathbb{T}^n = \{(e^{2\pi\sqrt{-1}\varphi_1}, \dots, e^{2\pi\sqrt{-1}\varphi_n}) \in \mathbb{C}^n\},$$

where $(\varphi_1, \dots, \varphi_n)$ runs over $\mathbb{R}^n$. The torus $\mathbb{T}^n$ is also *canonically oriented.*

Let $H \subset \mathbb{T}^m$ be a subgroup of dimension $r \leqslant m - n$. Choosing a basis, we can write it in the form

$$H = \big\{(e^{2\pi i(s_{11}\varphi_1 + \dots + s_{1r}\varphi_r)}, \dots, e^{2\pi i(s_{m1}\varphi_1 + \dots + s_{mr}\varphi_r)}) \in \mathbb{T}^m\big\},$$

where $\varphi_i \in \mathbb{R}$, $i = 1, \dots, r$. The integer $(m \times r)$-matrix $S = (s_{ij})$ defines a monomorphism $\mathbb{Z}^r \to \mathbb{Z}^m$. For any subset $\{i_1, \dots, i_n\} \subset [m]$ denote by $S_{\hat{i}_1, \dots, \hat{i}_n}$ the $(m-n) \times r$ submatrix of S obtained by deleting the rows $i_1, \dots, i_n$.

Write each vertex $v \in P^n$ as an intersection of n facets.

Lemma 1.1. *Subgroup H acts freely on $\mathscr{Z}_P$ if and only if for every vertex $v = F_{i_1} \cap \ldots \cap F_{i_n}$ in P^n the $(m-n) \times r$ submatrix $S_{\hat{i}_1, \dots, \hat{i}_n}$ defines a monomorphism $\mathbb{Z}^r \hookrightarrow \mathbb{Z}^{m-n}$ to a direct summand.*

Proof. The orbits of $\mathbb{T}^m$-action on $\mathscr{Z}_P$ corresponding to the vertices of P^n have maximal (rank n) isotropy subgroups. The isotropy subgroup corresponding to a vertex $v = F_{i_1} \cap \ldots \cap F_{i_n}$ is the coordinate subtorus $\mathbb{T}^n_{i_1,\ldots,i_n} \subset \mathbb{T}^m$. Subgroup H acts freely on $\mathscr{Z}_P$ if and only if it intersects each isotropy subgroup only at the unit.

This is equivalent to the condition that the map

$$H \times \mathbb{T}^n_{i_1,\ldots,i_n} \to \mathbb{T}^m$$

is injective for any $v = F_{i_1} \cap \ldots \cap F_{i_n}$. This map is given by the integer $m \times (n+r)$ matrix obtained by adding n columns $(0,\ldots,0,1,0,\ldots,0)^t$ (with 1 at the place i_j, $j = 1,\ldots,n$) to S. The map is injective if and only if this enlarged matrix defines a direct summand in $\mathbb{Z}^m$. The latter holds if and only if each matrix $S_{\hat{i}_1,\ldots,\hat{i}_n}$ defines a direct summand. □

Corollary 1.2. *The subgroup H of rank $r = m - n$ acts freely on $\mathscr{Z}_P$ if and only if for any vertex $v = F_{i_1} \cap \ldots \cap F_{i_n}$ of P^n holds*

$$\det S_{\hat{i}_1\ldots\hat{i}_n} = \pm 1.$$

Definition 1.3. An $(m \times n)$ matrix Λ gives a characteristic map

$$\ell\colon \{F_1,\ldots,F_m\} \longrightarrow \mathbb{Z}^n$$

for a given simple polytope P^n with facets $\{F_1,\ldots,F_m\}$ if the columns $\lambda_{j_1},\ldots,\lambda_{j_n}$ of Λ corresponding to any vertex $F_{j_1} \cap \ldots \cap F_{j_n}$ form a basis for $\mathbb{Z}^n$.

Theorem 1.4. *A simple polytope P^n admits a characteristic map if and only if $s(P^n) = m - n$.*

We consider a simple n-dimensional polytope P given as a bounded intersection of m closed half-spaces in $\mathbb{R}^n$:

$$P = \{x \in \mathbb{R}^n \colon \langle a_i, x\rangle + b_i \geqslant 0 \ \text{ for } \ 1 \leqslant i \leqslant m\},$$

where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$.

We assume that there are no redundant inequalities, that is, every hyperplane bounding a half-space intersects P at an $(n-1)$-dimensional *facet*. It follows that there are m facets $F_1,\ldots,F_m$ in total; and we further assume that they are *finely ordered*, in the sense that $F_1 \cap \cdots \cap F_n$ defines the *initial vertex* v_1 of P.

We associate to a simple n-dim polytope P an integral $(n\times m)$-matrix of the form

$$\varLambda \;=\; \begin{pmatrix} 1 & 0 & \dots & 0 & \lambda_{1,n+1} & \dots & \lambda_{1,m} \\ 0 & 1 & \dots & 0 & \lambda_{2,n+1} & \dots & \lambda_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \lambda_{n,n+1} & \dots & \lambda_{n,m} \end{pmatrix},$$

in which the column $\lambda_j=(\lambda_{1j},\dots,\lambda_{nj})$ corresponds to the facet $F_j,\ j=1,\dots,m$, and the columns $\lambda_{j_1},\dots,\lambda_{j_n}$ corresponding to any vertex $F_{j_1}\cap\cdots\cap F_{j_n}$ are required to form a basis for $\mathbb{Z}^n$.

In other words, the associated $(n\times n)$-submatrices $(\lambda_{j_1},\dots,\lambda_{j_n})$ have determinant ±1. We partition $\varLambda$ as $(I_n \mid \varLambda_\star)$, where I_n is identity $(n\times n)$-matrix. Thus $\varLambda_\star$ is $n\times(m-n)$-matrix, and refer to it as the *refined submatrix.*

Definition 1.5. The *combinatorial quasitoric data* $(P,\varLambda)$ consist of an oriented combinatorial simple polytope P and integer $(n\times m)$-matrix $\varLambda$ with the properties above.

We may specify P by a matrix inequality $A_Px+b_P\geqslant 0$, where A_P is the $(m\times n)$-matrix of row vectors $a_i\in\mathbb{R}^n$, and $b_P\in\mathbb{R}^m$ is the column vector of scalars b_i. We may interpret the matrix A_P as a linear transformation $\mathbb{R}^n\to\mathbb{R}^m$. Since the points of P are specified by the constraint $A_Px+b_P\geqslant 0$, the intersection of the affine subspace $A_P(\mathbb{R}^n)+b_P$ with the positive cone $\mathbb{R}^m_{\geqslant}$ is a copy of P in $\mathbb{R}^m$. The formula $i_P(x)=A_Px+b_P$ defines an affine injection

$$i_P\colon \mathbb{R}^n\longrightarrow\mathbb{R}^m,$$

which embeds P as a submanifold with corners of the positive cone.

2. Sign and weights of an isolated fixed point

We consider smooth $2n$-dimensional manifolds M, equipped with a smooth action α of a k-dimensional torus $\mathbb{T}^k$. We may choose an action of $\mathbb{T}^k$ on $\mathbb{C}^l$ and a $\mathbb{T}^k$-equivariant embedding $i\colon M\to\mathbb{C}^l$ for suitably large l. Let we can choose a $\mathbb{T}^k$-equivariant unitary structure c_ν on the $\mathbb{T}^k$-equivariant normal bundle $\nu(i)$ of i.

Definition 2.1. *Normally complex* $\mathbb{T}^k$*-manifold* is a triple element (M,α,c_ν).

We interpret M as a *tangentially stably complex* $\mathbb{T}^k$*-manifold* (M,α,c_τ) whenever an equivariant complex structure c_τ is chosen for its stable tangent bundle. So

$$c_\tau\colon \tau(M)\oplus\mathbb{R}^{2(l-n)}\longrightarrow\xi$$

is a real isomorphism for some complex vector bundle ξ, and the composition

$$r(t)\colon \xi \xrightarrow{c_\tau^{-1}} \tau(M)\oplus\mathbb{R}^{2(l-n)} \xrightarrow{d\alpha(t)\oplus I} \tau(M)\oplus\mathbb{R}^{2(l-n)} \xrightarrow{c_\tau} \xi$$

is a *complex transformation* for any $t\in\mathbb{T}^k$, where $d\alpha(t)$ is the differential of the action by $\alpha(t)$. So $r(t)$ corresponds to a representation

$$\rho\colon \mathbb{T}^k \to \mathrm{Hom}_{\mathbb{C}}(\xi,\xi).$$

Let $x\in M$ be an isolated fixed point of the $\mathbb{T}^k$-action α on a tangentially stably complex $\mathbb{T}^k$-manifold (M,α,c_τ). The representation

$$\rho_x\colon \mathbb{T}^k \longrightarrow GL(l,\mathbb{C})$$

associated with c_τ decomposes the fibre $\xi_x\cong\mathbb{C}^l$ as $\mathbb{C}^n\oplus\mathbb{C}^{l-n}$, where ρ_x acts without trivial summands on $\mathbb{C}^n$, and trivially on $\mathbb{C}^{l-n}$.

Moreover, the isomorphism $c_{\tau,x}$ induces an orientation of the tangent space $\tau_x(M)$.

Definition 2.2. The *sign* $\sigma(x)$ of an isolated fixed point x is $+1$ if the map

$$\tau_x(M)\xrightarrow{I\oplus 0}\tau_x(M)\oplus\mathbb{R}^{2(l-n)}\xrightarrow{c_{\tau,x}}\xi_x\cong\mathbb{C}^n\oplus\mathbb{C}^{l-n}\xrightarrow{\pi}\mathbb{C}^n$$

preserves the orientation, and -1 otherwise, where π is projection onto the first summand.

The representation $\rho_x\colon\mathbb{T}^k\to GL(n,\mathbb{C})$ decomposes as

$$\rho_{x,1}\oplus\cdots\oplus\rho_{x,n},$$

where $\rho_{x,j}$ is a non-trivial one-dimensional representation of $\mathbb{T}^k$ given by

$$\rho_{x,j}(e^{2\pi\sqrt{-1}\varphi_1},\dots,e^{2\pi\sqrt{-1}\varphi_k})v = e^{2\pi\sqrt{-1}\langle \boldsymbol{w}_j(x),\,\phi\rangle}v$$

where $\phi=(\varphi_1,\dots,\varphi_k)\in\mathbb{R}^k$, $\boldsymbol{w}_j(x)=(w_{1j}(x),\dots,w_{kj}(x))\in\mathbb{Z}^k$ and

$$\langle \boldsymbol{w}_j(x),\,\phi\rangle=\sum_{q=1}^k w_{qj}(x)\varphi_q.$$

Definition 2.3. The sequence $\{\boldsymbol{w}_1(x),\dots,\boldsymbol{w}_n(x)\}$ is said to be the sequence of *weight vectors* of an isolated fixed point x.

3. Quasitoric manifolds

The *moment-angle manifold* $\mathscr{Z}_P$ (see Lecture IV) is defined by the pull-back diagram

$$\begin{array}{ccc} \mathscr{Z}_P & \xrightarrow{i_{\mathscr{Z}}} & \mathbb{C}^m \\ {\scriptstyle \varrho_P}\big\downarrow & & \big\downarrow{\scriptstyle \varrho} \\ P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant} \end{array}$$

Here $\varrho(z_1,\dots,z_m)$ is given by $(|z_1|^2,\dots,|z_m|^2)$, the vertical maps are projections onto the quotients by the $\mathbb{T}^m$-actions, and $i_{\mathscr{Z}}$ is a $\mathbb{T}^m$-equivariant embedding. There is a $\mathbb{T}^m$-equivariant decomposition

$$\tau(\mathscr{Z}_P) \oplus \nu(i_{\mathscr{Z}}) \cong \mathscr{Z}_P \times \mathbb{C}^m,$$

where $\tau(\mathscr{Z}_P)$ is the tangent bundle of $\mathscr{Z}_P$ and $\nu(i_{\mathscr{Z}})$ is the normal bundle of the embedding $i_{\mathscr{Z}}$.

Matrix Λ defines a surjective homomorphism $\ell\colon \mathbb{T}^m \to \mathbb{T}^n$. The kernel of ℓ (which we denote $K(\Lambda)$) is isomorphic to $\mathbb{T}^{m-n}$. The action of $K(\Lambda)$ on $\mathscr{Z}_P$ is free due to the condition on the minors of Λ. So its quotient $M = \mathscr{Z}_P/K(\Lambda)$ is a $2n$-dimensional smooth manifold with an action of the n-dimensional torus $\mathbb{T}^m/K(\Lambda)$. We denote this action by α. It satisfies the *Davis–Januszkiewicz' conditions*:

(1) α is locally isomorphic to the standard coordinatewise representation of $\mathbb{T}^n$ in $\mathbb{C}^n$,

(2) there is a projection $\pi\colon M \to P$ whose fibres are orbits of α.

We refer to $M = M(P,\Lambda)$ as the *quasitoric manifold associated with the combinatorial data* (P,Λ).

Additional structure on quasitoric manifold M is associated to the *facial submanifolds* M_i, defined as the inverse images of the facet F_i under π, for $1 \leqslant i \leqslant m$. Every M_i has codimension 2, and its isotropy subgroup is the subcircle $\ell(\mathbb{T}_i) \subset \mathbb{T}^n$, where $\mathbb{T}_i \subset \mathbb{T}^m$ is the i-th coordinate subcircle. The quotient map

$$\mathscr{Z}_P \times_K \mathbb{C}_i \longrightarrow M$$

defines a canonical *complex line bundle* ρ_i, whose restriction to M_i is isomorphic to the *normal bundle* ν_i of its embedding in M. The submanifolds M_i are mutually *transverse.*

Definition 3.1. An *omniorientation* of a quasitoric manifold M consists of a choice of an *orientation* for M and for *every* facial normal bundle ν_i, $i = 1,\dots,m$.

Theorem 3.2. *Every pair (P,Λ) determines a $2n$-dimensional omnioriented quasitoric manifold.*

Proof. An interior point of the quotient polytope P admits an open neighborhood U, whose inverse image under the projection π is canonically diffeomorphic to $U\times\mathbb{T}^n$ as a subspace of M. Since $\mathbb{T}^n$ is oriented by the standard choice of basis, orientations of M correspond bijectively to orientations of P. Since the homomorphism $\ell\colon \mathbb{T}^m \to \mathbb{T}^n$ determines a complex structure on each ρ_i, it encodes equivalent information. □

Theorem 3.3. *Any omnioriented toric manifold admits a canonical stably complex structure, which is invariant under the $\mathbb{T}^n$-action.*

Proof. We obtained an embedding

$$i_P : P \longrightarrow \mathbb{R}^m_{\geqslant}$$

which respects facial codimensions and gives a pullback diagram

$$\begin{array}{ccc} \mathscr{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m \\ {\scriptstyle\varrho_P}\big\downarrow & & \big\downarrow{\scriptstyle\varrho} \\ P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant} \end{array}$$

of identification spaces. Here i_Z is a $\mathbb{T}^m$-equivariant *embedding.* So, there is a K-*equivariant* decomposition ($K = \ker \ell$)

$$\tau(\mathscr{Z}_P) \oplus \nu(i_Z) \simeq \mathscr{Z}_P \times \mathbb{C}^m,$$

obtained by restricting the tangent bundle $\tau(\mathbb{C}^m)$ to $\mathscr{Z}_P$. Factoring out K yields

$$\tau(M) \oplus (\xi/K) \oplus (\nu(i_Z)/K) \simeq \mathscr{Z}_P \times_K \mathbb{C}^m,$$

where ξ denotes the $(m-n)$–plane bundle of tangents along the fibres of

$$\pi_\ell : \mathscr{Z}_P \longrightarrow M.$$

$\mathscr{Z}_P \times_K \mathbb{C}^m$ is isomorphic to $\bigoplus\limits_{i=1}^{m} \rho_i$ as $GL(m,\mathbb{C})$-bundles. This is an example of Szczarba's Theorem. The embedding $i_{\mathbb{Z}} : \mathscr{Z}_P \longrightarrow C^m \simeq R^{2m}$ is $\mathbb{T}^m$-equivariantly framed, so $\nu(i_{\mathbb{Z}})/K$ is trivial. The bundle ξ/K canonically isomorphic to the adjoint bundle of the principal K-bundle, which is trivial, because K is an abelian group. So, we obtain an isomorphism

$$\tau(M) \oplus \mathbb{R}^{2(m-n)} \simeq \rho_1 \oplus \ldots \oplus \rho_m,$$

although different choices of trivialisations may lead to different isomorphisms.

Since M is connected and $GL(2(m-n),\mathbb{R})$ has two connected components, such isomorphisms are equivalent when and only when the induced orientations agree on $\mathbb{R}^{2(m-n)}$. We choose the orientation which is compatible with those on $\tau(M)$ and

$\rho_1 \oplus \ldots \oplus \rho_m$, as given by the omniorientation. The induced structure is *invariant* under the action of $\mathbb{T}^n$, because i_Z is $\mathbb{T}^m$-equivariant. □

Every $\mathbb{T}^n$-fixed point $x \in M = M(P, \Lambda)$ can be obtained as the intersection $M_{j_1} \cap \cdots \cap M_{j_n}$ of n facial submanifolds. The tangent space to M at x therefore decomposes into the sum of normal subspaces to M_{j_k} for $1 \leqslant k \leqslant n$:

$$\tau_x(M) = \nu_{j_1}|_x \oplus \ldots \oplus \nu_{j_n}|_x.$$

Lemma 3.4. *Let $x = M_{j_1} \cap \ldots \cap M_{j_n}$ be a fixed point.*

(1) *We have $\sigma(x) = 1$ if the orientation of $\tau_x(M)$ determined by the orientation of M coincides with the orientation of $\nu_{j_1}|_x \oplus \ldots \oplus \nu_{j_n}|_x$ determined by the orientations of ν_{j_k} for $1 \leqslant k \leqslant n$, and $\sigma(x) = -1$ otherwise.*

(2) *In terms of combinatorial data (P, Λ), we have*

$$\sigma(x) = \operatorname{sign}\bigl(\det(\lambda_{j_1}, \ldots, \lambda_{j_n}) \det(a_{j_1}, \ldots, a_{j_n})\bigr).$$

Let $\mathbb{T}^n$-fixed point $x \in M = M(P, \Lambda)$ be the intersection $M_{j_1} \cap \cdots \cap M_{j_n}$ of n facial submanifolds. Denote by Λ_x the $(n \times n)$-submatrix of Λ formed by the columns $\lambda_{j_1}, \ldots, \lambda_{j_n}$ (note $\det \Lambda_x = \pm 1$).

Lemma 3.5. *The weight vectors $\{\mathbf{w}_1(x), \ldots, \mathbf{w}_n(x)\}$ of the tangential $\mathbb{T}^n$-represent ation in $\tau_x(M)$ are given by the column vectors $\mu_1, \ldots, \mu_n$ of the matrix $\mathcal{M}_x$ satisfying*

$$\mathcal{M}_x^t \Lambda_x = I_n.$$

In other words, $\{\mathbf{w}_1(x), \ldots, \mathbf{w}_n(x)\}$ is the basis of $\mathbb{R}^n$ conjugated to $\{\lambda_{j_1}, \ldots, \lambda_{j_n}\}$.

4. Toric manifolds

Every nonsingular projective toric variety (*toric manifold*) M is determined by the normal fan of a simple polytope $Q \subset \mathbb{R}^n$; it is *integral*, insofar as its vertices lie in the lattice $\mathbb{Z}^n$.

We may assume that the origin is a distinguished vertex, that its incident facets lie in the respective coordinate hyperplanes and the remaining facets $F_{n+1}, \ldots, F_m$ are ordered.

For any such M we let P be the oriented combinatorial type of Q and the columns of Λ be the primitive integral inward pointing normal vectors to $F_1, \ldots, F_m$ respectively. So $\Lambda = A_Q^t$.

We can identify the stably complex structure associated to the combinatorial data (P, Λ) with the canonical complex structure on M.

Corollary 4.1. *For any fixed point x of the canonical $\mathbb{T}^n$-action on toric manifold M:*

- *$\sigma(x) = 1$;*
- *the weight vectors are given by the column vectors of the matrix $(A_{Q,x})^{-1}$.*

Definition 4.2. A polytope $P^n \subset \mathbb{R}^n$ is called *Delzant* if and only if at each vertex the normal vectors of the facets through the vertex may be chosen to be $\mathbb{Z}$-basis for $\mathbb{R}^n$.

Thus, Delzant polytope P^n is a simple polytope.

Using our description of the Stasheff polytope K_{n+1} as the simple polytope in $\mathbb{R}^{n-1}$ one can obtain immediately that K_{n+1} is the Delzant polytope.

Let $L(\mathbb{T}^n) = \mathbb{R}^n$ be the Lie algebra of $\mathbb{T}^n$.

Definition 4.3. A Hamiltonian $\mathbb{T}^n$-manifold is a triple $(M^{2n}, \omega, \mathscr{H})$, where (M^{2n}, ω) is a symplectic manifold and $\mathscr{H}\colon M^{2n} \to L(\mathbb{T}^n)^*$ – the moment map.

A Hamiltonian toric manifold is a compact connected Hamiltonian $\mathbb{T}^n$-manifold M^{2n} such that $\mathbb{T}^n$ action is effective.

Theorem 4.4 (see [19])**.** *There is a bijective correspondence between Delzant polytopes and Hamiltonian toric manifolds.*

Let Γ be a connected graph on the vertex set $[n] = \{1, \ldots, n\}$ and without loops or multiple edges.

Consider the graphical building set $B(\Gamma)$.

Theorem 4.5. *The mapping*

$$\mathbb{R}^n \longrightarrow \mathbb{R}^{n-1} \;:\; e_i \to e_i - e_1,\ i = 1, \ldots, n,$$

transform the simple polytope $P_{B(\Gamma)}$ in the Delzant polytope.

Proof (see [22]) use that any facet of $P_{B(\Gamma)}$ in the hyperplane $\{x \in \mathbb{R}^n \;:\; \sum x_i = \mu([1,n])\}$ can be described by the equation

$$\sum_{i \in I_k} x_i = \mu(I_k)$$

for some $I_k \in B(\Gamma)$ (see Lemma in Lecture III). $\square$

Appendix A. Cohomological rigidity for polytopes and manifolds by Taras E. Panov

We fix a coefficient ring $\mathbf{k}$ (usually $\mathbb{Z}$ or a field).

Definition A.6. We say that a family of closed manifolds is *cohomologically rigid* over $\mathbf{k}$ if the manifolds in the family are distinguished up to homeomorphism by their cohomology rings with coefficients in $\mathbf{k}$. That is, a family is cohomologically rigid if the graded ring isomorphism $H^*(M_1;\mathbf{k}) \cong H^*(M_2;\mathbf{k})$ implies a homeomorphism $M_1 \cong M_2$ whenever M_1 and M_2 are in the family.

A manifold M in the given family is said to be *cohomologically rigid* if for any other manifold M' in the family the ring isomorphism $H^*(M;\mathbf{k}) \cong H^*(M';\mathbf{k})$ implies a homeomorphism $M \cong M'$. Obviously a family is cohomologically rigid whenever every its element is rigid.

There is a smooth version of cohomological rigidity for families of smooth manifolds, with homeomorphisms replaced by diffeomorphisms.

The cohomological rigidity property for families of manifolds arising in toric topology has been studied by several authors, whose results we briefly review below.

In general, cohomological rigidity remains open for the class of *quasitoric manifolds* (see [18], [3] or Lecture V for definition):

Problem A.7. Assume that M_1 and M_2 are quasitoric manifolds with isomorphic integral cohomology rings. Are they homeomorphic (or even diffeomorphic)?

Bott towers constitute an important family of quasitoric manifolds, for which a machinery has been developed to attack the cohomological rigidity problem. A Bott tower B_n is the total space of a tower of fibre bundles

$$B_n \to B_{n-1} \to \cdots \to B_2 \to B_1 \tag{1}$$

with base $B_1 = \mathbb{C}P^1$ and fibres $\mathbb{C}P^1$, where each bundle in the tower is obtained as the projectivisation of a sum of two line bundles over the previous stage. Every Bott tower supports an action of the torus T^n turning it into a quasitoric manifold, or even into a complete non-singular *toric variety*, see [26] and [17].

Theorem A.8 ([31, Th. 5.1])**.** *Assume that the integral cohomology ring of a Bott tower B_n is isomorphic to $H^*((\mathbb{C}P^1)^n;\mathbb{Z})$, where $(\mathbb{C}P^1)^n$ denotes the product of n copies of $\mathbb{C}P^1$. Then every fibration in* (1) *is topologically trivial, i.e. B_n is diffeomorphic to $(\mathbb{C}P^1)^n$.*

In other words, $(\mathbb{C}P^1)^n$ is cohomologically rigid in the family of Bott towers (in the smooth category). Another result of [31] states that $(\mathbb{C}P^1)^n$ is cohomologically rigid in the wider family of quasitoric manifolds, but only in the topological category:

Theorem A.9 ([31, Th. 5.7])**.** *If the integral cohomology ring of a quasitoric manifold is isomorphic to that of $(\mathbb{C}P^1)^n$, then the two are homeomorphic.*

The orbit space of the T^n-action on a Bott tower is a combinatorial cube I^n. Theorem A.9 reduces to theorem A.8 by establishing the following two facts:

(i) A cohomology isomorphism $H^*(M;\mathbb{Q}) \cong H^*((\mathbb{C}P^1)^n;\mathbb{Q})$ for a quasitoric manifold over a cube I^n implies that M is homeomorphic to a Bott tower.

(ii) If M is a quasitoric manifold, and N a quasitoric manifold over a cube, then a cohomology ring isomorphism $H^*(M;\mathbb{Z}) \cong H^*(N;\mathbb{Z})$ implies that M is also a quasitoric manifold over a cube.

Statement (i) above has been generalised in [14] to quasitoric manifolds over arbitrary products of simplices (note that the cube is a product of segments) and *generalised Bott towers*. The latter means the total space of a tower of fibrations (1) with base $B_1 = \mathbb{C}P^k$, where each bundle in the tower is obtained as the projectivisation of a sum of line bundles over the previous stage. These towers where considered in [21], and the main result there is a criterion of decomposability of a quasitoric manifold over a product of simplices into such a tower of fibrations. Theorem A.8 has been also generalised in [15] as follows:

Theorem A.10 ([15, Th. 1.1])**.** *Assume that the integral cohomology ring of a generalised Bott tower* (1) *is isomorphic to $H^*(\prod_{i=1}^n \mathbb{C}P^{k_i};\mathbb{Z})$. Then every fibration in* (1) *is trivial, i.e. B_n is diffeomorphic to the product of complex projective spaces.*

In other words, the trivial generalised Bott tower is cohomologically rigid in the family of generalised Bott towers (in the smooth category). Another result of [15] states that the families of 2- and 3-stage Bott towers are cohomologically rigid.

Statement (ii) above leads to the following combinatorial counterpart for the notion of cohomological rigidity for manifolds.

Definition A.11. A simple polytope P is *cohomologically rigid* (over $\mathbb{Z}$) if its combinatorial type is determined by the integral cohomology ring of any quasitoric manifold over P. In more detail, P is cohomologically rigid if there exists a quasitoric manifold M over P, and whenever there exists a quasitoric manifold N over another polytope Q with a graded ring isomorphism $H^*(M;\mathbb{Z}) \cong H^*(N;\mathbb{Z})$, there is a combinatorial equivalence $P \approx Q$.

The above mentioned result of [31] implies that the cube I^n is cohomologically rigid. The rigidity of an arbitrary product of simplices has been established in [16], alongside with the rigidity of a number of low-dimensional polytopes. The proofs use the bigraded cohomology and Betti numbers of moment-angle complexes (see Theorem 7.5 in Lecture IV).

Although no examples of cohomologically non-rigid quasitoric manifolds are known, there are non-rigid simple polytopes. The first examples of these were constructed in [32, Ex. 4.3] by applying a "vertex cut" operation to a 3-simplex tree times. There are three combinatorially non-equivalent ways to do so, and each of the resulting 3-dimensional polytopes P_1, P_2, P_3 arises as the orbit space of a quasitoric manifold homeomorphic to the connected sum of three copies of $\mathbb{C}P^3$. Actually, all known cohomologically non-rigid polytopes are obtained as multiple vertex-cuts.

The above considerations have an $\mathbb{R}$-analogue, with quasitoric manifolds replaced by *small covers* [18], Bott towers replaced by *real Bott towers*, and the cohomology rings taken with $\mathbb{Z}/2$-coefficients. A small cover over an n-dimensional simple polytope P is a manifold M with an action of the "$\mathbb{R}$-torus" $(\mathbb{Z}/2)^n$ whose orbit space is P. The quotient projection $M \to P$ is a ramified covering of P by a manifold, with smallest possible number of leaves. A real Bott tower is defined similarly to the complex tower, with complex projectivisations replaced by the real ones.

Cohomological rigidity over $\mathbb{Z}/2$ holds for the family of real Bott towers [28], but fails for generalised real Bott towers [30] (it is open for complex Bott towers). Also, every small cover over a product of simplices is a generalised real Bott tower [14] (this fails for quasitoric manifolds).

Finally, moment-angle manifolds (see [3] and Section 7 of Lecture V) constitute another important family for testing the cohomological rigidity property. A related question is stated as Problem 7.6 in Lecture IV. We note that there exist combinatorially non-equivalent simple polytopes for which the corresponding moment-angle manifolds $\mathscr{Z}_P$ are diffeomorphic (in particular, their bigraded cohomology algebras are isomorphic). This was first observed in [1]: according to [1, Th. 6.3], the moment-angle manifolds corresponding to the above described 3-dimensional polytopes obtained by applying a vertex cut to a 3-simplex 3 times are all diffeomorphic.

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