Vahlen's involution and q-series identities

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Contents

1. Introduction

Basic notations

- λ=$(\lambda_1, \lambda_2, \cdots, \lambda_l)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0$ and $\lambda_i \in \mathbb{Z}$
- For $\sum_i \lambda_i = n$, we denote $|\lambda| = n$.
- If there is not any part in λ, define $\lambda = \emptyset$.
- $l(\lambda) = l$, $a(\lambda) = \lambda_1$ and $s(\lambda) = \lambda_l$

- For q-series identities

$$\lambda \leftrightarrow (a_1 q^{\lambda_1},\, a_2 q^{\lambda_2},\, \ldots,\, a_l q^{\lambda_l}),$$

  where $a_1,\, a_2,\, \ldots,\, a_l$ are coefficients which imply some properties of λ

- $\lambda \oplus \mu$: a partition with parts $\{\lambda_i + \mu_i\}$
  $\lambda \uplus \mu$: a partition with parts $\{\lambda_i, \mu_j\}$

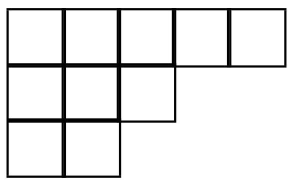
- P : the set of partitions
  D : the set of partitions whose parts are all distinct
  For $S \subseteq P$

$$S^k = \{\lambda \in S \mid \lambda_i \equiv k \pmod 2\}\ (k = 1, 2)$$

- $$\begin{aligned}
&(a)_0 = 1,\\
&(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})\ (n \geq 1),\\
&(a;q^k)_n = (1-a)(1-aq^k)\cdots(1-aq^{k(n-1)})\ (n \geq 1),\\
&(a)_\infty = (a;q)_\infty = \prod_{n\geq 0}(1-aq^n),\\
&(a;q^k)_\infty = \prod_{n\geq 0}(1-aq^{kn}).
\end{aligned}$$

Young diagram $[\lambda]$ for $|\lambda| = n$

- A collection of n 1×1 squares (i, j) on a square grid $\mathbb{Z}^2$, with $1 \le i \le l(\lambda)$, $1 \le j \le \lambda_i$
- The first coordinate i increasing downward, while the second coordinate j increases from left to right
- We assume $\lambda = [\lambda]$.



(5,3,2)

2-modular diagram $[\lambda]_2$

A Young diagram with the integers 1 or 2 written in squares, such that 1 can appear only in the last square of a row, and no 1 can appear above 2

2	2	2	2	2	2	1
2	2	2	2	1		
2	2	2				
2	2	1				
2						

Vahlen's involution

- $$\frac{(q)_\infty}{(q)_\infty} = 1$$

$$(\lambda,\, \mu) \leftrightarrow ((q)_\infty,\, \frac{1}{(q)_\infty}) \Longrightarrow (\lambda,\, \mu) \in D \times P$$

$sign(\lambda,\, \mu) = (-1)^{l(\lambda)}$

- Combinatorial interpretation

  (i) $s(\lambda) \le s(\mu)$: move $s(\lambda)$ to μ

  (ii) $s(\lambda) > s(\mu)$: move $s(\mu)$ to λ.

- $$\frac{(-q)_\infty}{(-q)_\infty} = 1$$

2. Basic identities

First identities

- $$\sum_{n=0}^{\infty}(-1)^n q^{n^2} = \sum_{n=0}^{\infty} \frac{q^{2n}}{(q)_{2n}}(q)_\infty \tag{1}$$

- $$\sum_{n=0}^{\infty}(-1)^n q^{(n+1)^2} = \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q)_{2n+1}}(q)_\infty \tag{2}$$

Identity (1)

- Equivalent form

$$\sum_{n=0}^{\infty}(-1)^n q^{n^2} = (q;q^2)_\infty (q^2;q^2)_\infty - \sum_{n=1}^{\infty}(-q^{2n})(q^{2n+1};q^2)_\infty (q^{2n+2};q^2)_\infty$$

- The right hand side

$$\lambda^1 \leftrightarrow (q;q^2)_\infty \in D^1$$

$$\lambda^2 \leftrightarrow (q^2;q^2)_\infty \in D^2$$

where $s(\lambda^1) < s(\lambda^2)$

- The left hand side

$$(-1)^n q^{n^2} \leftrightarrow (-q^{2(n-1)+1},\ -q^{2(n-2)+1}, \ldots,\ -q^1)$$

- For λ^1 with $l = l(\lambda^1)$
$$\lambda^1 = \lambda^{11} \oplus \lambda^{12}$$
where
$$\lambda^{11} = (-q^{2(l-1)+1}, -q^{2(l-2)+1}, \ldots, -q^1) \in D^1$$
$$\lambda^{12} = (q^{\lambda^1_1 - (2(l-1)+1)}, q^{\lambda^1_2 - (2(l-2)+1)}, \ldots, q^{\lambda^1_l - 1}) \in P^2$$
- $(\lambda^{12}, \lambda^2) \in P^2 \times D^2$, $sign(\lambda^{12}, \lambda^2) = (-1)^{l(\lambda^2)}$
- Apply Vahlen's involution to $(\lambda^{12}, \lambda^2)$
- Exceptional case : $|\lambda^{12}| = 0$, $\lambda^2 = \emptyset$

Identity (2)

- Equivalent form

$$\sum_{n=0}^{\infty}(-1)^n q^{(n+1)^2} = \sum_{n=0}^{\infty} q^{2n+1}(q^{2n+3};q^2)_\infty(q^{2n+2};q^2)_\infty$$

- The right hand side

$$\lambda^1 \leftrightarrow q^{2n+1}(q^{2n+3};q^2)_\infty \in D^1$$
$$\lambda^2 \leftrightarrow (q^{2n+2};q^2)_\infty \in D^2$$

where $s(\lambda^1) < s(\lambda^2)$

- The left hand side

$$(-1)^n q^{(n+1)^2} \leftrightarrow (-q^{2n+1},\ -q^{2(n-1)+1},\ \ldots,\ -q^{2\cdot 1+1},\ q^1)$$

General forms

Counting the number of parts

- $$\sum_{n=0}^{\infty}(-1)^n z^n q^{n^2}$$
$$=\sum_{n=0}^{\infty}\frac{q^{2n}}{(zq;q^2)_n(q^2;q^2)_n}(zq;q^2)_\infty(q^2;q^2)_\infty$$

- $$\sum_{n=0}^{\infty}(-1)^n z^{n+1} q^{(n+1)^2}$$
$$=\sum_{n=0}^{\infty}\frac{zq^{2n+1}}{(zq;q^2)_{n+1}(q^2;q^2)_n}(zq;q^2)_\infty(q^2;q^2)_\infty$$

Second Identities

- $$\sum_{n=0}^{\infty}(-1)^n q^{n^2} = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q)_{2n}}(q;q^2)_\infty \qquad (3)$$

- $$\sum_{n=0}^{\infty}(-1)^n q^{(n+1)^2} = \sum_{n=0}^{\infty} \frac{q^{(n+1)(2n+1)}}{(q)_{2n+1}}(q;q^2)_\infty \qquad (4)$$

Identity (3)

- Equivalent form

$$\sum_{n=0}^{\infty}(-1)^n q^{n^2} = \sum_{n=0}^{\infty}\frac{q^{n(2n+1)}}{(q^2;q^2)_n}(q^{2n+1};q^2)_\infty$$

- The right hand side

$$\lambda^1 \leftrightarrow q^{n^2}(q^{2n+1};q^2)_\infty \in D^1$$

$$\lambda^2 \leftrightarrow \frac{q^{n(n+1)}}{(q^2;q^2)_n} \in D^2$$

- The left hand side

$$(-1)^n q^{n^2} \leftrightarrow (-q^{2(n-1)+1}, -q^{2(n-2)+1}, \ldots, -q^1)$$

- For λ^1 with $l = l(\lambda^1)$

$$\lambda^1 = \lambda^{11} \oplus \lambda^{12}$$

where

$$\lambda^{11} = (q^{2l-1}, q^{2l-3}, \ldots, q^1) \in D^1$$

$$\lambda^{12} = (-q^{\lambda^1_1-(2l-1)}, -q^{\lambda^1_2-(2l-3)}, \ldots, -q^{\lambda^1_{l-n}-(2n+1)}) \in P^2$$

- $(\lambda^{12}, \lambda^2) \in P^2 \times D^2$, $sign(\lambda^{12}, \lambda^2) = (-1)^{l(\lambda^{12})}$
- Apply Vahlen's involution to $(\lambda^{12}, \lambda^2)$
- Exceptional case : $|\lambda^{12}| = 0$, $\lambda^2 = \emptyset$

Identity (4)

- Equivalent form

$$\sum_{n=0}^{\infty}(-1)^n q^{(n+1)^2} = \sum_{n=0}^{\infty} \frac{q^{(n+1)(2n+1)}}{(q^2;q^2)_n}(q^{2n+3};q^2)_\infty$$

- The right hand side

$$\lambda^1 \leftrightarrow q^{(n+1)^2}(q^{2n+3};q^2)_\infty \in D^1$$

$$\lambda^2 \leftrightarrow \frac{q^{n(n+1)}}{(q^2;q^2)_n} \in D^2$$

- The left hand side

$$(-1)^n q^{(n+1)^2} \leftrightarrow (-q^{2n+1},\ -q^{2(n-1)+1},\ \ldots,\ -q^{2\cdot 1+1},\ q^1)$$

Derived identity

- Combination of (1) with (3)

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(q)_{2n}} (q^2; q^2)_\infty = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q)_{2n}}$$

- Equivalent form

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(q)_{2n}} (-q)_\infty = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q)_{2n}} \frac{1}{(q)_\infty}$$

- Since

$$(-q)_\infty = \sum_{m=0}^{\infty} \frac{q^{m(2m+1)}}{(q)_{2m}} + \sum_{m=0}^{\infty} \frac{q^{(m+1)(2m+1)}}{(q)_{2m+1}}$$

$$\frac{1}{(q)_\infty} = \sum_{m=0}^{\infty} \frac{q^{2m}}{(q)_{2m}} + \sum_{m=0}^{\infty} \frac{q^{2m+1}}{(q)_{2m+1}}$$

we have

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(q)_{2n}} \left(\sum_{m=0}^{\infty} \frac{q^{m(2m+1)}}{(q)_{2m}} + \sum_{m=0}^{\infty} \frac{q^{(m+1)(2m+1)}}{(q)_{2m+1}} \right)$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q)_{2n}} \left(\sum_{m=0}^{\infty} \frac{q^{2m}}{(q)_{2m}} + \sum_{m=0}^{\infty} \frac{q^{2m+1}}{(q)_{2m+1}} \right)$$

- Cancelling common terms

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(q)_{2n}} \sum_{m=0}^{\infty} \frac{q^{(m+1)(2m+1)}}{(q)_{2m+1}} = \sum_{m=0}^{\infty} \frac{q^{2m+1}}{(q)_{2m+1}} \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q)_{2n}} \tag{5}$$

- L and R : the l.h.s. and the r.h.s. of (5) respectively

$$L = \{(\lambda^1, \mu^1) \mid (\lambda^1, \mu^1) \in P \times D\}$$
$$R = \{(\lambda^2, \mu^2) \mid (\lambda^2, \mu^2) \in P \times D\}$$

where

$$(l(\lambda^1), l(\mu^1)) \equiv (0, 1) \pmod 2$$
$$(l(\lambda^2), l(\mu^2)) \equiv (1, 0) \pmod 2$$

- Vahlen's involution to $(\lambda^1, \mu^1) \longrightarrow (\lambda^2, \mu^2)$

General forms

- Counting the number of parts

$$\sum_{n=0}^{\infty} \frac{z^{2n} q^{2n}}{(q)_{2n}} \sum_{m=0}^{\infty} \frac{z^{2m+1} q^{(m+1)(2m+1)}}{(q)_{2m+1}}$$

$$= \sum_{m=0}^{\infty} \frac{z^{2m+1} q^{2m+1}}{(q)_{2m+1}} \sum_{n=0}^{\infty} \frac{z^{2n} q^{n(2n+1)}}{(q)_{2n}}$$

equivalently

$$\sum_{n=0}^{\infty} \frac{z^{2n} q^{2n}}{(q)_{2n}} (-zq)_{\infty} = \sum_{n=0}^{\infty} \frac{z^{2n} q^{n(2n+1)}}{(q)_{2n}} \frac{1}{(zq)_{\infty}}$$

$$\sum_{n=0}^{\infty} \frac{z^{2n+1} q^{2n+1}}{(q)_{2n+1}} (-zq)_{\infty} = \sum_{n=0}^{\infty} \frac{z^{2n+1} q^{(n+1)(2n+1)}}{(q)_{2n+1}} \frac{1}{(zq)_{\infty}}$$

3. Ramanujan's identities

The first identity in Ramanujan's lost notebook

- $$\sum_{n=0}^{\infty} \frac{q^n}{(-aq)_n(-bq)_n}$$
$$=(1+a^{-1})\sum_{n=0}^{\infty} \frac{(-b/a)^n q^{\frac{n(n+1)}{2}}}{(-bq)_n} - a^{-1}\frac{\sum_{n=0}^{\infty}(-b/a)^n q^{\frac{n(n+1)}{2}}}{(-aq)_\infty(-bq)_\infty}$$

- Equivalent form
$$\sum_{n=0}^{\infty} aq^n(-aq^{n+1})_\infty(-bq^{n+1})_\infty$$
$$=(-a)_\infty\sum_{n=0}^{\infty}(-b/a)^n q^{\frac{n(n+1)}{2}}(-bq^{n+1})_\infty - \sum_{n=0}^{\infty}(-b/a)^n q^{\frac{n(n+1)}{2}}$$

- $$\lambda^a \leftrightarrow (-a)_\infty,\ \lambda^{b/a} \leftrightarrow (-b/a)^n q^{\frac{n(n+1)}{2}},\ \lambda^b \leftrightarrow (-bq^{n+1})_\infty$$
- The right hand side

  $$(-b/a)^n q^{\frac{n(n+1)}{2}} \leftrightarrow ((-b/a)q^n, (-b/a)q^{n-1}, \ldots, (-b/a)q^1)$$

  Either $\lambda^a \neq \emptyset$ or $\lambda^b \neq \emptyset$

  $a(\lambda^{b/a}) < s(\lambda^b)$
- The left hand side

  $$\lambda^{b/a} = \emptyset,\ s(\lambda^a) < s(\lambda^b)$$

- For $\lambda^{b/a} \uplus \lambda^b$ with $l_b = l(\lambda^{b/a} \uplus \lambda^b)$

$$\lambda^{b/a} \uplus \lambda^b = \lambda^{b1} \oplus \lambda^{b2}$$

where

$$\lambda^{b1} = ((-b/a)q^{l_b}, (-b/a)q^{l_b-1}, \ldots, (-b/a)q^1)$$

$$\lambda^{b2} = (-aq^{\lambda^b_1 - l_b}, -aq^{\lambda^b_2-(l_b-1)}, \ldots, -aq^{\lambda^b_{l_b-n}-(n+1)})$$

- $(\lambda^a, \lambda^{b2}) \in D \times P$, $sign(\lambda^a, \lambda^{b2}) = (-1)^{l(\lambda^{b2})}$
- Apply Vahlen's involution to $(\lambda^a, \lambda^{b2})$
- Exceptional case : $l(\lambda^{b1}) = l(\lambda^{b2})$, $s(\lambda^a) < s(\lambda^{b1} \oplus \lambda^{b2})$
  Equivalently

$$\lambda^{b/a} = \emptyset,\ s(\lambda^a) < s(\lambda^b)$$

Other identities in Ramanujan's lost notebook

- $$\sum_{n=0}^{\infty} \frac{q^n}{(-bq)_n(-b^{-1}q)_n}$$
  $$=(1+b)\sum_{n=0}^{\infty} \frac{(-b^2)^n q^{\frac{n(n+1)}{2}}}{(-bq)_n} - b\frac{\sum_{n=0}^{\infty}(-b^2)^n q^{\frac{n(n+1)}{2}}}{(-bq)_\infty(-b^{-1}q)_\infty}$$

  Equivalent form

  $$\sum_{n=0}^{\infty} b^{-1}q^n(-b^{-1}q^{n+1})_\infty(-bq^{n+1})_\infty$$
  $$=(-b^{-1})_\infty \sum_{n=0}^{\infty}(-b/b^{-1})^n q^{\frac{n(n+1)}{2}}(-bq^{n+1})_\infty - \sum_{n=0}^{\infty}(-b/b^{-1})^n q^{\frac{n(n+1)}{2}}$$

-

$$(1+b^{-1})\sum_{n=0}^{\infty}\frac{(-a/b)^n q^{\frac{n(n+1)}{2}}}{(-aq)_n}-(1+a^{-1})\sum_{n=0}^{\infty}\frac{(-b/a)^n q^{\frac{n(n+1)}{2}}}{(-bq)_n}$$

$$=(b^{-1}-a^{-1})\frac{(aq/b)_\infty(bq/a)_\infty(q)_\infty}{(-aq)_\infty(-bq)_\infty}$$

Equivalent form

$$b\{(-a)_\infty\sum_{n=0}^{\infty}(-b/a)^n q^{\frac{n(n+1)}{2}}(-bq^{n+1})_\infty-\sum_{n=0}^{\infty}(-b/a)^n q^{\frac{n(n+1)}{2}}\}$$

$$=a\{(-b)_\infty\sum_{n=0}^{\infty}(-a/b)^n q^{\frac{n(n+1)}{2}}(-aq^{n+1})_\infty-\sum_{n=0}^{\infty}(-a/b)^n q^{\frac{n(n+1)}{2}}\}$$

$$=\sum_{n=0}^{\infty}baq^n(-aq^{n+1})_\infty(-bq^{n+1})_\infty$$

•

$$\sum_{n=0}^{\infty} \frac{q^{2n+1}}{(-aq;q^2)_{n+1}(-a^{-1}q;q^2)_{n+1}}$$

$$=\sum_{n=0}^{\infty} a^{3n+1}q^{3n^2+2n}(1-aq^{2n+1}) - \frac{\sum_{n=0}^{\infty}(-1)^n a^{2n+1}q^{n^2+n}}{(-aq;q^2)_\infty(-a^{-1}q;q^2)_\infty}$$

Equivalent form

$$\sum_{n=0}^{\infty} a^{-1}q^{2n+1}(-aq^{2n+3};q^2)_\infty(-a^{-1}q^{2n+3};q^2)_\infty$$

$$=(-a^{-1}q;q^2)_\infty \sum_{n=0}^{\infty}(-1)^n a^{2n}q^{n(n+1)}(-aq^{2n+3};q^2)_\infty$$

$$-\sum_{n=0}^{\infty}(-1)^n a^{2n}q^{n(n+1)}$$

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