Regular positive ternary quadratic forms

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ABSTRACT

A positive definite quadratic form f is said to be regular if it globally represents all integers that are represented by the genus of f. In 1997, Jagy, Kaplansky and Schiemann provided a list of 913 candidates of primitive positive definite regular ternary quadratic forms, and all but 22 of them are verified to be regular. In this talk we show that 8 forms among 22 candidates are, in fact, regular. At the end of the talk, we show some finiteness result on ternary forms that represent every eligible integer in some arithmetic progression.

[1] Quadratic Forms

(1.1) Definition of quadratic forms

(i) $S_m(\mathbb{Z}) = \{$Symmetric matrices of rank m over $\mathbb{Z}\}$

(ii) $L = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_m$ with a non-degenerate symmetric bilinear form

$$B(x_i, x_j) = m_{ij} \in \mathbb{Z} \text{ and } Q(x) := B(x, x).$$

(iii) $f_L(x_1, x_2 \ldots, x_m) = \sum_{i,j} m_{ij} x_i x_j$

$$L \longleftrightarrow M_L := (B(x_i, x_j)) \longleftrightarrow f_L$$

(1.2) Representation of quadratic forms

(i) For $M \in S_m(\mathbb{Z})$ and $N \in S_n(\mathbb{Z})$,

$$N \longrightarrow M \text{ iff } \exists T \in M_{m,n}(\mathbb{Z}) \text{ such that } T^t M T = N.$$

(ii) For two $\mathbb{Z}$-lattices ℓ and L,

$$\ell \longrightarrow L \text{ iff } \sigma : \ell \hookrightarrow L \text{ such that}$$

$$B(x, y) = B(\sigma(x), \sigma(y)) \text{ for all } x, y \in \ell.$$

(iii) For $f(x_1, \ldots, x_m)$, $g(y_1, \ldots, y_n)$,

$$g \longrightarrow f \text{ iff } \exists T \in M_{m,n}(\mathbb{Z}) \text{ such that } f(T\mathbf{y}) = g(\mathbf{y}).$$

(1.3) We define

$$R(\ell, L) = \{\sigma : \ell \longrightarrow L\} \quad \text{and} \quad O(L) = R(L, L)$$

If ℓ and L is positive definite, then $R(\ell, L)$ is a finite set.

(1.4) If $\ell \longrightarrow L$ and $L \longrightarrow \ell$, we write $\ell \simeq L$.

(1.5) A $\mathbb{Z}$-lattice L is called *even* if $Q(x) \in 2\mathbb{Z}$ for every $x \in L$.

(1.6) Throughout this talk, we always assume that

every $\mathbb{Z}$-lattice L is *primitive* and *positive definite*.

[2] Well Known Theorems

L : $\mathbb{Z}$-lattice, $\mathbb{Z}_p$: p-adic integer ring

(2.1) Scalar extension of a $\mathbb{Z}$-lattice L

We define $L_p = L \otimes \mathbb{Z}_p$ for every prime p.

If $\exists x \in L_p$ such that $Q(x) = a$, then we write $a \longrightarrow L_p$.

Note that $a \longrightarrow L_p$ if and only if there is an $x_n \in L$ such that

$$Q(x_n) \equiv a \pmod{p^n},$$

for every non-negative integer n.

(2.2) Genus and class number

$$\text{gen}(L) := \{L' \mid L'_p \simeq L_p \text{ for every prime } p\}, \quad h(L) := \mid \text{gen}(L)/\sim\mid$$

(2.3) Local information

If $a \longrightarrow L_p$ for every prime p (we write $a \longrightarrow \text{gen}(L)$), then

$$a \longrightarrow L' \qquad \text{for some } L' \in \text{gen}(L).$$

Hence if $h(L) = 1$, then

$$a \longrightarrow L \quad \text{if and only if} \quad a \longrightarrow \text{gen}(L).$$

- We define

$Q(L) = \{a \in \mathbb{Z} \mid a \longrightarrow L\}$ and $Q(\text{gen}(L)) = \{a \in \mathbb{Z} \mid a \longrightarrow \text{gen}(L)\}$.

Every integer $a \in Q(\text{gen}(L))$ is called *eligible* and $a \in Q(\text{gen}(L)) - Q(L)$ is called *exceptional*.

(2.4) Definition of regular lattices

• A $\mathbb{Z}$-lattice L is called **regular** if it represents every integer a that is represented by the genus of L.

Note that

$$L \text{ is regular} \Longleftrightarrow Q(L) = Q(\text{gen}(L))$$

• If $h(L) = 1$, L is regular. The converse is not true in general. For example,

$$\text{gen}(h = 2x^2 + 2xy + 2y^2 + 18z^2) = \{h, 2x^2 + 6y^2 + 6z^2 + 6yz\}.$$

Both are regular forms (Hsia).

[3] Brief History

(3.1) (Kitaoka) If L is a binary $\mathbb{Z}$-lattice, then

$$L \text{ is regular if and only if } \quad h(L) = 1.$$

(3.2) (Earnest) There are infinitely many quaternary regular $\mathbb{Z}$-lattices.

(3.3) (Kim) classified all *diagonal* quaternary regular $\mathbb{Z}$-lattices:

$$\begin{array}{ll} x^2 + y^2 + z^2 + dt^2 : & \text{for } d = 1, 3, 5, 7 \text{ or } d = 2^{2r+1}x \ (x = 1, 2, 3) \\ & \vdots \\ 3x^2 + 16y^2 + 48z^2 + dt^2 : & \text{for } d = 16 \cdot 3^{2r+1}x \ (x = 1, 2) \\ 3x^2 + 4y^2 + 8z^2 + dt^2 : & \text{for } d = 8, 12, 16, 20. \end{array}$$

(3.4) There is no known example of a ternary $\mathbb{Z}$-lattice L such that $Q(L)$ is completely determined and

$$|Q(\text{gen}(L)) - Q(L)| \geq 2,$$

except the following result:

(3.5) (Ono-Soundararajan) (Assuming that GRH is true)

For the Ramanujan form $f_L = x^2 + y^2 + 10z^2$,

$$\begin{aligned} Q(\text{gen}(L)) - Q(L) = \ & 3, 7, 21, 31, 33, 43, 67, \\ & 79, 87, 133, 217, 219, 223, 253, 307, 391, \\ & 679\text{(Jones and Pall)},\ 2719\text{(Gupta)}. \end{aligned}$$

(3.4) (Legendre, Gauss and Dirichlet)

$x^2 + y^2 + z^2 = n$ has a solution if and only if $n \neq 4^k(8s + 7)$.

(3.5) (Lebesque, Dirichlet, Liouville)

$x^2 + y^2 + az^2$ is regular for $a = 2, 3, 5$.

(3.6) ($\sim$ Jones-Pall)

There are exactly 102 *diagonal* regular ternary $\mathbb{Z}$-lattices (up to equivalence). Among them, 20 lattices have class number bigger than 1, for example, $h(x^2 + 48y^2 + 144z^2) = 4$.

(3.7) (Watson)

There are only finitely many regular ternary $\mathbb{Z}$-lattices.

(3.8) ($\sim$ Hsia, Kaplansky, Jagy, Jagy-Kaplansky-Schiemann)

There are at most 913 regular ternary $\mathbb{Z}$-lattices.

• They remained the following 22 ternary $\mathbb{Z}$-lattices as candidates:

$$L(1)=\begin{pmatrix}2&1&1\\1&10&2\\1&2&26\end{pmatrix},\quad L(2)=\begin{pmatrix}2&0&1\\0&12&3\\1&3&26\end{pmatrix},\quad L(3)=\begin{pmatrix}4&1&2\\1&10&2\\2&2&22\end{pmatrix},$$

$$L(4)=\begin{pmatrix}6&3&3\\3&10&3\\3&3&30\end{pmatrix},\quad L(5)=\begin{pmatrix}2&0&1\\0&20&5\\1&5&58\end{pmatrix},\quad \mathbf{L(6)}=\begin{pmatrix}4&1&1\\1&14&-6\\1&-6&44\end{pmatrix},$$

$$L(7)=\begin{pmatrix}10&2&1\\2&16&-4\\1&-4&22\end{pmatrix},\quad L(8)=\begin{pmatrix}10&3&3\\3&18&9\\3&9&30\end{pmatrix},\quad L(9)=\begin{pmatrix}10&3&5\\3&18&6\\5&6&34\end{pmatrix},$$

$$L(10)=\begin{pmatrix}4&0&1\\0&30&15\\1&15&64\end{pmatrix},\quad \mathbf{L(11)}=\begin{pmatrix}14&2&7\\2&16&6\\7&6&46\end{pmatrix},\quad L(12)=\begin{pmatrix}10&3&3\\3&18&0\\3&0&54\end{pmatrix},$$

$$L(13)=\begin{pmatrix}10&1&3\\1&26&-6\\3&-6&66\end{pmatrix},\quad L(14)=\begin{pmatrix}18&6&3\\6&22&-4\\3&-4&58\end{pmatrix},\quad L(15)=\begin{pmatrix}22&3&6\\3&30&-3\\6&-3&78\end{pmatrix},$$

$$L(16)=\begin{pmatrix}3&1&1\\1&6&2\\1&2&14\end{pmatrix},\quad \mathbf{L(17)}=\begin{pmatrix}7&2&2\\2&8&0\\2&0&20\end{pmatrix},\quad \mathbf{L(18)}=\begin{pmatrix}7&3&1\\3&15&-3\\1&-3&23\end{pmatrix},$$

$$\mathbf{L(19)}=\begin{pmatrix}11&4&1\\4&16&4\\1&4&19\end{pmatrix},\quad \mathbf{L(20)}=\begin{pmatrix}5&2&2\\2&20&-4\\2&-4&68\end{pmatrix},\quad \mathbf{L(21)}=\begin{pmatrix}11&4&1\\4&16&4\\1&4&51\end{pmatrix},$$

$$\mathbf{L(22)}=\begin{pmatrix}7&1&2\\1&23&6\\2&6&92\end{pmatrix}.$$

• Exactly 794 lattices have class number 1 among 913.

(3.9) (O) The following 8 $\mathbb{Z}$-lattices among 22 candidates are, in fact, regular:

$$\mathbf{L(6)} = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 14 & -6 \\ 1 & -6 & 44 \end{pmatrix}, \quad \mathbf{L(11)} = \begin{pmatrix} 14 & 2 & 7 \\ 2 & 16 & 6 \\ 7 & 6 & 46 \end{pmatrix}, \quad \mathbf{L(17)} = \begin{pmatrix} 7 & 2 & 2 \\ 2 & 8 & 0 \\ 2 & 0 & 20 \end{pmatrix},$$

$$\mathbf{L(18)} = \begin{pmatrix} 7 & 3 & 1 \\ 3 & 15 & -3 \\ 1 & -3 & 23 \end{pmatrix}, \quad \mathbf{L(19)} = \begin{pmatrix} 11 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 19 \end{pmatrix}, \quad \mathbf{L(20)} = \begin{pmatrix} 5 & 2 & 2 \\ 2 & 20 & -4 \\ 2 & -4 & 68 \end{pmatrix},$$

$$\mathbf{L(21)} = \begin{pmatrix} 11 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 51 \end{pmatrix}, \quad \mathbf{L(22)} = \begin{pmatrix} 7 & 1 & 2 \\ 1 & 23 & 6 \\ 2 & 6 & 92 \end{pmatrix}.$$

[4] Watson Transformation

Let L be a $\mathbb{Z}$-lattice.

(4.1) Gamma transformation

Let $\epsilon = 2$ if L is even and $p = 2$, $\epsilon = 1$ otherwise.

$$\Gamma_p(L) = \{x \in L \mid Q(x+z) \equiv Q(z) \mod \epsilon p,\ \forall z \in L\}.$$

$\gamma_p(L)$: primitive $\mathbb{Z}$-lattice obtained from $\Gamma_p(L)$ by a suitable scaling.

(4.2) A $\mathbb{Z}_p$-lattice L_p is called *isotropic* if $\exists x \in L - \{0\}$ such that $Q(x) = 0$, otherwise L is called *anisotropic*.

- $L_p = \mathfrak{L}_{p,0} \perp \cdots \perp \mathfrak{L}_{p,t}$: Jordan decomposition such that

$$\mathfrak{s}(\mathfrak{L}_{p,i}) = p^i \mathbb{Z}_p \quad \text{or} \quad \mathfrak{L}_{p,i} = 0.$$

(4.3) Lemma

Assume that $\mathfrak{L}_{p,0}$ is anisotropic and additionally, $\mathfrak{L}_{2,1} = 0$ only when $p = 2$ and L_2 is even. Then

$$Q(L) \cap \epsilon p\mathbb{Z} = Q(\Gamma_p(L)) \quad \text{and} \quad Q(\mathrm{gen}(L)) \cap \epsilon p\mathbb{Z} = Q(\mathrm{gen}(\Gamma_p(L))).$$

Hence if L is regular, then so is $\gamma_p(L)$. Conversely, if $\gamma_p(L)$ is regular (that is, $\Gamma_p(L)$ is regular), then

$$(Q(\mathrm{gen}(L)) - Q(L)) \cap \epsilon p\mathbb{Z} = \emptyset.$$

• If $\mathfrak{L}_{p,0}$ is isotropic, γ_p-transformation does not preserve the regularity in general. For example $f_L = x^2 + y^2 + z^2 + 7t^2$ is regular, but $f_{\gamma_7(L)} = x^2 + 7y^2 + 7z^2 + 7t^2$ is not regular.

(4.4) Note that

$$\gamma_3(L(18)) = L(20), \quad \gamma_5(L(19)) = L(22).$$

(4.5) Sketch of proof

Let $L = L(17)$ with $f_L = 7x^2 + 8y^2 + 20z^2 + 4xy + 4xz$.

(i) $\ell = \gamma_3(L)$ with $f_\ell = 3x^2 + 7y^2 + 7z^2 + 2xy + 2xz + 6yz$ is regular. Hence

$$(Q(\text{gen}(L)) - Q(L)) \cap 3\mathbb{Z} = \emptyset.$$

(ii) Since $3\ell \longrightarrow L \longrightarrow \ell$,

$$Q(\text{gen}(L)) \subset Q(\text{gen}(\ell)) = Q(\ell).$$

Assume that $a \in Q(\text{gen}(L))$ and $3 \nmid a$. Then there is an $x \in \ell$ such that $Q(x) = a$. Define

$$S^{\pm} = \{y \in \ell/3\ell \mid Q(y) \equiv \pm 1 \pmod 3\}.$$

Assume that $a \equiv 1 \pmod 3$. Then $x \pmod 3 \in S^{+}$.

(iii) Show that there is an isometry $\sigma_y \in R(3\ell, L)$ such that

$$\sigma_y(3y) \in 3L, \quad \text{for every } y \in S^+.$$

Hence $a \in Q(L)$ for every $a \equiv 1 \pmod 3$.

If $a \equiv -1 \pmod 3$, there are some vectors $y \in S^-$ such that $\sigma(3y) \notin 3L$ for every $\sigma \in R(3\ell, L)$. Let S_1^- be the set of such vectors and $S_2^- = S^- - S_1^-$. Assume that $x \pmod 3 \in S_1^-$.

(iv) Find isometries $\tau_1, \tau_2 \in R(3\ell, \ell)$ such that $\frac{1}{3}\tau_i(3x) \in \ell$ for $i = 1, 2$ and $\tau_1 \circ \tau_2 \in O(\mathbb{Q}\ell)$ has an infinite order. Note that $Q(\frac{1}{3}\tau_i(3x)) = a$. We may assume that $\frac{1}{3}\tau_i(3x) \in S_1^-$.

(v) Define $x_0 = \frac{1}{3}\tau_2(3x)$ and $x_n = \frac{1}{9}(\tau_1 \circ \tau_2)^n(9x)$. Show that if $x_n \in S_1^-$, then $x_{n+1} \in \ell$.

(vi) Show that $x_n \neq x_m$ for every $n \neq m$. If $x_n \in S_1^-$ for every n, then there are infinitely many vectors $x_n \in \ell$ such that $Q(x_n) = a$. This is a contradiction to the fact that $R(a, \ell) < \infty$. Therefore there is an integer m such that x_m (mod 3) $\in S_2^-$ and hence $a \in Q(L)$. This completes the proof.

• We may replace 3 and ℓ in the above with other integers and $\mathbb{Z}$-lattices, respectively, to prove the regularity for some other $\mathbb{Z}$-lattices.

(4.6) Numerical data for $L = L(17)$

$$R(3\ell, L) = \left\{ \begin{pmatrix} -1 & -3 & -1 \\ 1 & 0 & -2 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -3 \\ 1 & -2 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 3 \\ -1 & 2 & 0 \\ -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 1 \\ -1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix} \right\},$$

$$S^{+} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\},$$

$$S^{-} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

$$S_1^{-} = \{(0, \pm 1, \pm 1), (0, \pm 1, \mp 1)\}.$$

If $x \equiv (0, \pm 1, \pm 1) \pmod 3$,

$$\tau_1 = \begin{pmatrix} 1 & -4 & -2 \\ -2 & -1 & -2 \\ 0 & 0 & 3 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & -2 & -4 \\ 0 & 3 & 0 \\ -2 & -2 & -1 \end{pmatrix} \in R(3\ell, \ell),$$

and if $x \equiv (0, \pm 1, \mp 1) \pmod 3$,

$$\tau_1 = \begin{pmatrix} 3 & 2 & 2 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 4 & 4 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix} \in R(3\ell, \ell).$$

(4.7) Remaining candidates

- $\gamma_3(L(8)) = L(4)$ and $\gamma_3(L(4)) = L(1)$, $\gamma_3(L(1)) = L(4)$.

- $\gamma_3(L(13)) = L(15)$, $\gamma_3(L(15)) = L(13)$.

- There are exactly 14 candidates of regular ternary $\mathbb{Z}$-lattices:

Candidates of regular ternary $\mathbb{Z}$-lattices

$$L(2) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 12 & 3 \\ 1 & 3 & 26 \end{pmatrix}, \quad L(3) = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 10 & 2 \\ 2 & 2 & 22 \end{pmatrix}, \quad L(5) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 20 & 5 \\ 1 & 5 & 58 \end{pmatrix},$$

$$L(7) = \begin{pmatrix} 10 & 2 & 1 \\ 2 & 16 & -4 \\ 1 & -4 & 22 \end{pmatrix}, \quad L(8) = \begin{pmatrix} 10 & 3 & 3 \\ 3 & 18 & 9 \\ 3 & 9 & 30 \end{pmatrix}, \quad L(9) = \begin{pmatrix} 10 & 3 & 5 \\ 3 & 18 & 6 \\ 5 & 6 & 34 \end{pmatrix},$$

$$L(10) = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 30 & 15 \\ 1 & 15 & 64 \end{pmatrix}, \quad L(12) = \begin{pmatrix} 10 & 3 & 3 \\ 3 & 18 & 0 \\ 3 & 0 & 54 \end{pmatrix}, \quad L(13) = \begin{pmatrix} 10 & 1 & 3 \\ 1 & 26 & -6 \\ 3 & -6 & 66 \end{pmatrix},$$

$$L(14) = \begin{pmatrix} 18 & 6 & 3 \\ 6 & 22 & -4 \\ 3 & -4 & 58 \end{pmatrix}, \quad L(16) = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 6 & 2 \\ 1 & 2 & 14 \end{pmatrix}.$$

$$L(1) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 10 & 2 \\ 1 & 2 & 26 \end{pmatrix}, \quad L(4) = \begin{pmatrix} 6 & 3 & 3 \\ 3 & 10 & 3 \\ 3 & 3 & 30 \end{pmatrix}, \quad L(15) = \begin{pmatrix} 22 & 3 & 6 \\ 3 & 30 & -3 \\ 6 & -3 & 78 \end{pmatrix}.$$

[5] Representations of an Arithmetic Progression

$S_{d,a} = \{dn + a \mid n = 0, 1, \ldots\}$ $(d \in \mathbb{Z}^+, a \in \mathbb{Z}^+ \cup \{0\})$

(5.1) Definitions

(i) A $\mathbb{Z}$-lattice L is called $S_{d,a}$-**universal** if $S_{d,a} \subset Q(L)$.

(ii) A $\mathbb{Z}$-lattice L is called $S_{d,a}$-**regular** if

$$S_{d,a} \cap Q(L) = S_{d,a} \cap Q(\text{gen}(L)) \neq \emptyset.$$

• If L is $S_{d,a}$-universal, then L is $S_{d,a}$-regular.

Assume that $\text{rank}(L) \geq 3$.

• If L is $S_{d,a}$-regular, then for any m dividing $8 \prod_{p \mid dL, \text{odd}} p^2$, there is an integer a_0 such that L is S_{md,a_0}-universal.

(5.2) $S_{d,a}$-universal $\mathbb{Z}$-lattices

• Unary case (Fermat's four squares theorem)

There are no four distinct squares in arithmetic progression.

• Binary case (Alaca-Alaca-Williams)

There is no binary $S_{d,a}$-universal $\mathbb{Z}$-lattice for any d, a.

• Ternary case

There is no $S_{d,0}$-universal ternary $\mathbb{Z}$-lattice for every positive integer d.

(i) (Kaplansky)

∘ There are exactly 5 ternary $S_{2,1}$-universal $\mathbb{Z}$-lattices.

∘ There are at most 18 ternary even $S_{4,2}$-universal $\mathbb{Z}$-lattices including 4 as candidates (one was confirmed by Jagy).

(ii) (Lemma) If L is an $S_{d,a}$-universal ternary $\mathbb{Z}$-lattice, then

$$dL \leq 16d^3(3d + a)^2.$$

Hence there are only finitely many ternary $S_{d,a}$-universal $\mathbb{Z}$-lattices for any d, a.

(5.3) Ternary $S_{d,a}$-regular $\mathbb{Z}$-lattices

• The Ramanujan form $x^2 + y^2 + 10z^2$ is $S_{3,2}$-regular and $S_{10,5}$-regular though it is not regular.

• $L(1)$ is $S_{4,0}, S_{16,6}$ and $S_{16,10}-$regular.

• (Main Theorem) For any d, a, there is a polynomial $f(d, a)$ satisfying the following: For any $S_{d,a}$-regular ternary $\mathbb{Z}$-lattice L,

$$\det(L) \leq f(d, a).$$

Therefore there are only finitely many $S_{d,a}$-regular $\mathbb{Z}$-lattices.

(5.4) Sketch of proof

L : $S_{d,a}$-regular lattice.

(i) Find d_1, a_1 with $d_1 \equiv 0 \pmod 2$ such that L is S_{d_1,a_1}-regular and for any $p \mid d_1$, $S_{d_1,a_1} \subset Q(L_p)$.

(ii) Show that for any prime p ($\gcd(p, d_1) = 1$) such that the unimodular component of L_p is anisotropic, $\gamma_p(L)$ is also S_{d_1,a_2}-regular for some integer a_2.

(iii) By using (ii), show that L_p is isotropic for every large prime p.

(iv) If p is large enough, show that $\det(L)$ is not divisible by p.

(v) For any $p \mid \det(L)$, by taking γ_q-transformation to L, if necessary repeatedly, for any prime $q \neq p$, show that $\mathrm{ord}_p(\det(L))$ is bounded.

(vi) In each step, show that every constant has a polynomial growth to d, a.

• (Duke and Schulze-Pillot) For any ternary $\mathbb{Z}$-lattice L, there is a constant C satisfying the following property: If an integer a satisfies

(i) a is primitively represented by the genus of L;

(ii) a is represented by the spinor genus of L;

(iii) $a > C$;

then a is represented by L.

• (Corollary) For any integers S and d, a with $\gcd(d, a) = 1$, there is a constant C satisfying the following property: For any $\mathbb{Z}$-lattice L with $\det(L) > C$,

$$|(Q^*(\text{gen}(L)) - Q(L)) \cap \{m \mid m \equiv a \pmod{d}\}| \geq S.$$